MATHEMATICAL MODELS AND METHODS FOR SMART MATERIALS

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Cortona, Italy 25 – 29 June 2001

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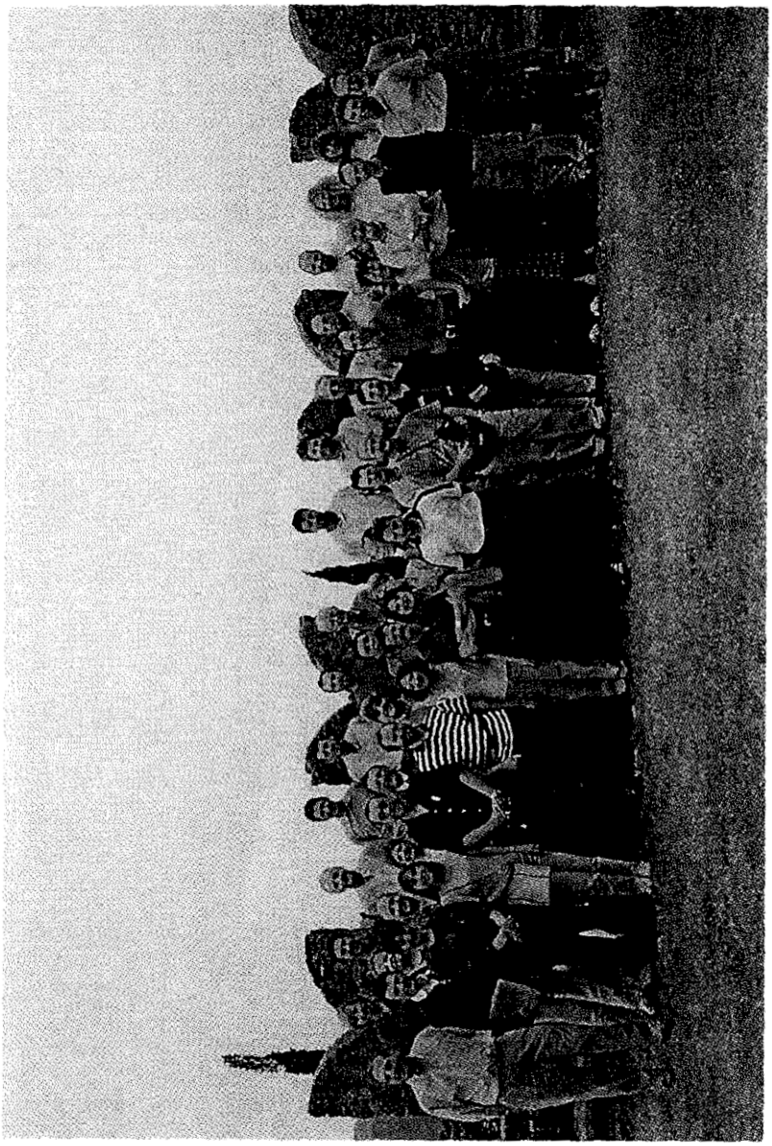

Palazzone (Cortona) - June 25-29, 2001

Preface

This book contains the papers presented at the Conference on "Mathematical Models and Methods for Smart Materials" which was held in Cortona, June 25-29, 2001, organized by M. Fabrizio, G.A. Hanyga e A. Morro in memory of Giorgio Gentili. The topics of the Conference were much influenced by the research developed by Gentili within the framework of mathematical problems connected with the modelling of materials.

The Conference was organized within the program of the Italian INdAM. The Editors are grateful to INdAM for accepting the proposal of the Conference and the general support. The hospitality at the Palazzone has made the stay an easy, friendly and stimulating occasion for scientific contacts among the participants. Indeed, it is worth remarking that similar Conferences on mathematical models and methods for continuous media were organized at the Palazzone and Gentili was one of the most active participants. The authors of the present papers witness their friendship and admiration for the scientific activity of Gentili.

The papers are gathered in four parts. First, "Methods in materials science" deals mainly with mathematical techniques for the investigation of physical systems such as liquid crystals, materials with internal variables, amorphous materials, thermoelastic materials. Also, techniques are exhibited for the analysis of stability and controllability of classical models of continuum mechanics and of dynamical systems.

"Modelling of smart materials" is devoted to models of superfluids, superconductors, materials with memory, nonlinear elastic solids, damaged materials. In the elaboration of the models, thermodynamic aspects play a central role in the characterization of the constitutive properties.

"Well-posedness in materials with memory" involves existence, uniqueness and stability for the solution to problems, most often expressed by integro-differential equations, which involve materials with fading memory. Also, attention is addressed to exponential decay in viscoelasticity, inverse problems in heat conduction with memory, automatic control for parabolic equations.

"Analytic problems in phase transitions" deals with nonlinear partial differential equations associated with phase transitions, hysteresis, possibly involving fading memory effects. Particular applications are developed for the phase-field model with memory, the Stefan problem with a Cattaneo-type equation, the hysteresis in thermo-visco-plasticity, the solid-solid phase transition.

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Giorgio Gentili

OBITUARY

It is a deeply sorrow task to trace the activity of a young collaborator at the University, a community where the time spent is certainly more than that spent in the family or in any other community. That is why, quite naturally, strict collaborators are also close friends. This aspect is especially true for Giorgio Gentili, a very generous and kind man.

He was born on 1961, in Fossombrone. After the "Maturità" from the Liceo Scientifico of Fermo on 1980, he went to the University of Bologna and received the degree in Physics on 1985, maximum cum laude. He developed a thesis on "A macroscopic non-local theory of superfluidity" with G. Turchetti and M. Fabrizio as supervisors.

I knew him through C. Giorgi. They both were active members of the catholic movement "Comunione e Liberazione". Next, at the Department of Mathematics, especially B. Lazzari joined them thus constituting a group of friends both at the University and in everyday life.

After the degree, Gentili received a fellowship from CNR and next pursued the doctoral studies in Mathematics at the University of Bologna.

He doesn't attained the PhD degree because he got a permanent position as a researcher of Mathematical Physics on March 1990 and this was incompatible with the position of PhD student. Since 1990 he was very active in tutoring students of Rational Mechanics and has given the course of Mechanics of space flight within the DU of Aerospace Engineering at Forli'.

Since November 1999, he was Associate professor of Rational Mechanics, still at the Faculty of Engineering at the University of Bologna.

He delivered the course of Rational Mechanics for Environment Engineering and, in the first semester of 2000/01, Applied Mathematics for Electronic Engineering.

He couldn't complete the semester course because of his sudden death on December 3, 2000, at the age of 39 years.

His teaching was always open to any questions from the students; he was very handy for interviews and generous during the examinations. The students used to approach him so frequently for questions about the course and the solution of problems; he was especially skillful in calculations and in relating results and modelling. He was also an efficient member of the governing Counsel (Giunta) of the Department often performing burdensome tasks.

Since the beginning, I had the impression of a deeply correct and reserved man. Next I realised that he had a strong character supported by a well-grounded faith.

He used to view his and other people's inconveniences with a remarkable sense of humour but also to face problems with skill and firmness. That is why he was able to perform successfully many different tasks with an uncommon efficiency. In particular, while he was engaged with the academic work and the family (with two little children) he was much involved in the Centro Manfredini which is renowned, at least in Bologna, for the organisation of a cultural activity.

After the degree in 1985, he began the scientific research while the mathematical physics was undergoing a strong development, both qualitative and quantitative, since the early seventies.

Incidentally, since 1970 to 2000 the number of papers of the Italian community grew over ten times while the papers on international journals passed from a few exceptions to almost all. Sure, Gentili took advantage of this rapid positive evolution.

In 1990 he stayed some months at the Carnegie-Mellon Institute of Pittsburgh, a well-renowned center for the research on continuum thermomechanics. That stay allowed him to improve the rigorous approach and the understanding of advanced topics of research.

The academic career was quite difficult at the time mainly because, after the generous availability of positions in the seventies and eighties, there was a consequent shortage in the nineties. Nevertheless he was able to succeed in the competition of 1997 and become associate professor of Rational Mechanics.

The first two papers deal with superfluidity, the subject investigated in the thesis. A subject similar to superconductivity, though more involved, superfluidity is getting new modelling improvements (cf. the joint paper with A. Morro). The idea in Gentili's papers is that helium II is a mixture of two fluids, one is a normal Navier-Stokes fluid, the other one is a fluid with non-local constitutive properties.

Next he began the research on materials with memory, a deeply-investigated subject in the Italian community.

Materials with memory traces back to V. Volterra and received an outstanding improvement by D. Graffi. Really, V. Volterra gave the subject a firm mathematical basis along with the understanding of the physical aspects of the model and a connection with problems of the modern functional analysis.

Next D. Graffi provided a twofold contribution. First, the analysis of questions related to the pertinent partial differential equations were answered through theorems about uniqueness and continuous dependence of the solution. Secondly, physical or constitutive aspects have been related to mathematical consequences. Perhaps the main result is the restriction placed by the second law of thermodynamics on the memory kernel (in 1928) and next the expression of the free energy which is now termed after him.

This digression on materials with memory is relevant to the work of Gentili. Even through various joint papers, Gentili investigated the thermodynamic restrictions for heat conductors, viscoelastic fluids, electromagnetic solids and some examples of nonlocal materials. A proper mention is in order for the results on free energies and the extension of Golden's minimal free energy to anisotropic solids. He also investigated the well posedness of problems both in the quasi-static approximation and for materials with singular kernels. Further, he examined the inversion of the constitutive equation in viscoelasticity.

The last part of his research was addressed to hysteresis and phase transitions. Even in joint works with C. Giorgi, he elaborated models of hysteretic materials where the cicles are compatible with a residual magnetization and a coercive field. He also provided a way to circumvent the paradox of the vibrational instability. The last works exhibit a more refined model which allows for the existence of universal processes of demagnetization and for the smoothness of the hysteresis functional relative to BV function spaces.

Mauro Fabrizio

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Contents

Temperance for order/disorder transition in nematics

P. Biscari* G. Capriz†

1. – Introduction

Loss of mesoscopic order may occur, in continua with substructure, for purely geometric reasons: strong anchoring conditions at the boundary may be incompatible with smooth, perfectly ordered fields within the body (some examples in the theory of nematic liquid crystals are proffered in [2, 3]); even the curvature of the bounding surface may induce disorder in the bulk [4]. Partial order or lack of it can be described in terms of purely geometric parameters: the degrees of prolation and of (optical) biaxiality [1]. However, thermodynamic concepts could also be stretched to apply here; for instance, *temperance* (a sort of reciprocal of absolute temperature) and also a *tensorial temperance* can be introduced and be suggestive. Negative values of temperance may occur under very simple conditions; they do not appear to be at all exceptional. A rather gauche example was ventured already in [5], but the implied expression of energy was outlandish. Here developments follow strictly standard lines.

2. – Nematic temperance tensor

The microscopic configuration of a single nematic molecule can be described by means of the second order tensor $\mathbf{Q_n} := \mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathbf{I}$, where $\mathbf{n}\in\mathbb{S}^2$ is a unit vector parallel to the molecular direction, and $\mathbf{I}$ is the identity tensor: thus, the head-and-tail symmetry which characterizes nematic molecules is automatically enforced. For a population of molecules, let $f_x(\mathbf{n}) : \mathbb{S}^2 \to \mathbb{R}^+$ be the probability distribution of molecular directions near the point $x \in \mathcal{B}$. Then, the mesoscopic nematic order tensor at x is defined as the mean value of $\mathbf{Q_n}$ on the distribution f_x:

$$\mathbf{Q}(x) := \langle\, \mathbf{Q_n} \,\rangle_{f_x} = \int_{\mathbb{S}^2} f_x(\mathbf{n}) \left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathbf{I}\right)\, da\ ; \tag{1}$$

thus, $\mathbf{Q}$ is automatically a symmetric traceless tensor. The nematic is optically *isotropic* at x, if $\mathbf{Q}(x) = \mathbf{0}$; it is optically *uniaxial* if two of the eigenvalues of $\mathbf{Q}$ coincide therein; otherwise, it is optically *biaxial* [6]. The optical properties of the

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nematic are usually described quantitatively in terms of the measures of *prolation* s and *biaxiality* β of the molecular distribution as follows [1]. Let μ_1, μ_2, and μ_3 be the eigenvalues of $\mathbf{Q}$; then,

$$s := \left(\frac{27}{2}\prod_{i=1}^{3}\mu_i\right)^{\frac{1}{3}} \qquad \text{and} \qquad \beta := \left(6\sqrt{3}\prod_{i<j=1}^{3}\left|\mu_i - \mu_j\right|\right)^{\frac{1}{3}}$$

$\beta \in [0,1]$ vanishes whenever two of the eigenvalues of $\mathbf{Q}$ coincide, and it is positive otherwise; s ranges from $-\frac{1}{2}$ (corresponding to the planar distribution in which all the directions in the plane are equiprobable for the molecules) to 1 (completely ordered distribution in which all the molecules are parallel); s and β vanish together when and only when the order tensor is isotropic.

To generalize the concept of temperature to the mesoscopic disorder described by f_x, we now focus on situations in which, at equilibrium, the molecular distribution approaches a local Maxwellian:

$$(2) \qquad f_x(\mathbf{Q_n}) = \mathrm{k}\,\exp\left(\mathbf{B}(x)\cdot\mathbf{Q_n}\right), \quad \text{with} \quad \mathrm{k} = \left[\int_{\mathbb{S}^2}\exp\left(\mathbf{B}(x)\cdot\mathbf{Q_n}\right)\,da\right]^{-1}.$$

In this case, the properties of the local order tensor $\mathbf{Q}(x)$, defined in (1), are determined by the *nematic temperance tensor* $\mathbf{B}(x)$, that without loss of generality can be assumed to be traceless and symmetric too, since its isotropic and skew parts do not contribute to the scalar product in (2). In particular, the proper directions of the order tensor $\mathbf{Q}$ coincide with those of the temperance tensor $\mathbf{B}$, as it can be proved by direct computation.

To draw a parallel with the classical kinetic theory, we remark that $\mathbf{B}$ plays the rôle of the inverse of a temperature tensor [7]. In the remaining part of this section, we will consider separately the cases in which the temperance tensor is isotropic, uniaxial, and biaxial, in order to study how the optical properties of $\mathbf{Q}$ descend from $\mathbf{B}$.

Isotropic temperance

When $\mathbf{B} = \mathbf{0}$, f_x is uniform, and (1) immediately shows that also $\mathbf{Q}$ vanishes: the nematic is microscopically disordered, in accordance with the interpretation of $\mathbf{B}$ as the inverse of a temperature: $\mathbf{B} = \mathbf{0}$ corresponds to an infinite temperature, and thus to a null degree of order.

Uniaxial temperance

When two eigenvalues of $\mathbf{B}$ coincide, we can write the temperance tensor as

$$(3) \qquad \mathbf{B} = \gamma\left(\mathbf{e}\otimes\mathbf{e} - \frac{1}{3}\mathbf{I}\right),$$

where $\mathbf{e}$ is the symmetry direction of $\mathbf{B}$, and the *scalar nematic temperance* $\gamma \in \mathbb{R}$ plays the rôle of inverse of a nematic temperature. Substituting (3) in (1), we obtain for the order tensor the expression

$$\mathbf{Q} = s(\gamma)\left(\mathbf{e}\otimes\mathbf{e} - \frac{1}{3}\mathbf{I}\right),$$

where

$$s(\gamma)=\frac{3}{4\gamma}\left[\frac{2e^{\gamma}\sqrt{\gamma}}{\sqrt{\pi}\,\mathrm{Erfi}\left(\sqrt{\gamma}\right)}-1\right]-\frac{1}{2}\quad\text{if } \gamma\neq 0\,,\quad\text{with } s(0)=0 \tag{4}$$

and $\mathrm{Erfi}(x)=-\mathrm{i}\,\mathrm{Erf}(\mathrm{i}x)$ denotes the imaginary error function.

Figure 1 illustrates how s depends on γ: a positive temperance implies that the nematic molecules tend to be oriented along the direction of $\mathbf{e}$, with the limiting case $s=1$ of perfect orientation being reachable only when $\gamma\to\infty$ (corresponding to a vanishing positive nematic temperature). On the contrary, when gamma is negative the nematic molecules tend to lie in the plane orthogonal to $\mathbf{e}$; the limiting case $s=-\frac{1}{2}$ is approached when $\gamma\to-\infty$, which corresponds to vanishing, but negative, nematic temperature. Thus, within the class of uniaxial distributions, it is not possible to "cross" over smoothly from positive to negative values of absolute temperature through the value zero.

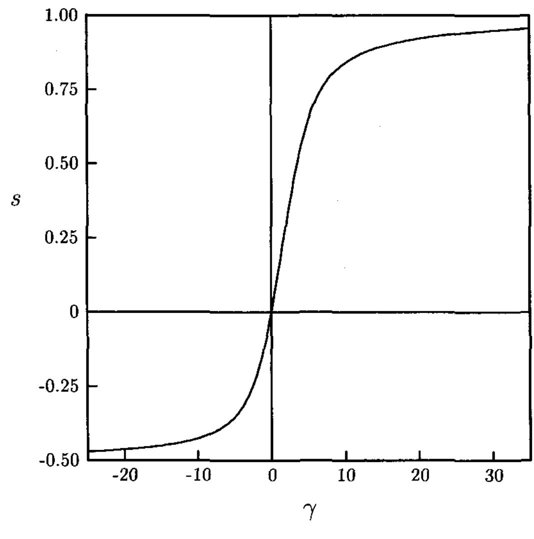

Figure 1: Degree of prolation s of a uniaxial nematic as a function of the scalar temperance γ.

Biaxial temperance

When the eigenvalues of $\mathbf{B}$ are all different, also $\mathbf{Q}$ has a biaxial structure. To prove this fact, we introduce an orthonormal basis $\{\mathbf{e}_x\,,\ \mathbf{e}_y\,,\ \mathbf{e}_z\}$, made up of eigenvectors of $\mathbf{B}$, and without loss of generality we write $\mathbf{B}$ as

$$\mathbf{B}=\gamma\left(\mathbf{e}_z\otimes\mathbf{e}_z-\frac{1}{3}\mathbf{I}\right)+\Delta\left(\mathbf{e}_y\otimes\mathbf{e}_y-\mathbf{e}_x\otimes\mathbf{e}_x\right) \tag{5}$$

and thus $\mathbf{Q} = Q_x\,\mathbf{e}_x\otimes\mathbf{e}_x + Q_y\,\mathbf{e}_y\otimes\mathbf{e}_y + Q_z\,\mathbf{e}_z\otimes\mathbf{e}_z$. If we insert (5) in (1), we obtain:

$$
\begin{aligned}
Q_z &= \frac{\displaystyle\int_0^1 \mathrm{e}^{\gamma u^2}\left(u^2-\frac{1}{3}\right) I_0\left(\Delta(1-u^2)\right)\mathrm{d}u}{\displaystyle\int_0^1 \mathrm{e}^{\gamma u^2} I_0\left(\Delta(1-u^2)\right)\mathrm{d}u}\,,\\
Q_y &= -\frac{Q_z}{2} + \frac{\displaystyle\int_0^1 \mathrm{e}^{\gamma u^2}(1-u^2)\, I_1\left(\Delta(1-u^2)\right)\mathrm{d}u}{\displaystyle 2\int_0^1 \mathrm{e}^{\gamma u^2} I_0\left(\Delta(1-u^2)\right)\mathrm{d}u}\,,\\
Q_x &= -\frac{Q_z}{2} - \frac{\displaystyle\int_0^1 \mathrm{e}^{\gamma u^2}(1-u^2)\, I_1\left(\Delta(1-u^2)\right)\mathrm{d}u}{\displaystyle 2\int_0^1 \mathrm{e}^{\gamma u^2} I_0\left(\Delta(1-u^2)\right)\mathrm{d}u}\,,
\end{aligned}
$$

where $I_n(z)$ denotes the n-th hyperbolic Bessel function, that is, the solution of the differential equation $z^2\,y'' + z\,y' - (z^2+n^2)\,y = 0$ which is regular at $z=0$. Thus, in particular, $Q_x \neq Q_y$ whenever $B_{xx} \neq B_{yy}$ (that is, $\Delta \neq 0$), since the difference

$$
Q_y - Q_x = \frac{\displaystyle\int_0^1 \mathrm{e}^{\gamma u^2}(1-u^2)\, I_1\left(\Delta(1-u^2)\right)\mathrm{d}u}{\displaystyle\int_0^1 \mathrm{e}^{\gamma u^2} I_0\left(\Delta(1-u^2)\right)\mathrm{d}u}
$$

has the same sign as Δ, because I_0 is always positive and $I_1(z)$ has the same sign as z. Clearly, a similar reasoning allows us to prove that any couple of eigenvalues of $\mathbf{Q}$ are different whenever the corresponding eigenvalues of $\mathbf{B}$ are so.

In most nematic liquid crystals, the mesoscopic state is almost uniaxial, that is $\Delta \ll 1$. If this is the case, and considering that

$$
I_0(z) = 1 + o(z) \quad \text{and} \quad I_0(z) = \frac{z}{2} + o(z) \quad \text{when } z \to 0,
$$

we obtain:

$$
\begin{aligned}
Q_z &= \frac{1}{2\gamma}\left[\frac{2\mathrm{e}^{\gamma}\sqrt{\gamma}}{\sqrt{\pi}\,\mathrm{Erfi}\left(\sqrt{\gamma}\right)} - 1\right] - \frac{1}{3} + o(\Delta)\\
Q_y &= -\frac{Q_z}{2} + \frac{g(\gamma)}{2}\,\Delta + o(\Delta)\\
Q_x &= -\frac{Q_z}{2} - \frac{g(\gamma)}{2}\,\Delta + o(\Delta)\,,
\end{aligned}
$$

so that $Q_y - Q_x = g(\gamma)\,\Delta + o(\Delta)$, with

$$
g(\gamma) := \frac{\displaystyle\int_0^1 \mathrm{e}^{\gamma u^2}(1-u^2)^2\,\mathrm{d}u}{\displaystyle 2\int_0^1 \mathrm{e}^{\gamma u^2}\,\mathrm{d}u} = \frac{4\gamma^2+4\gamma+3}{8\gamma^2} - \frac{(2\gamma+3)\,\mathrm{e}^{\gamma}}{4\sqrt{\pi}\gamma^{\frac{3}{2}}\,\mathrm{Erfi}\left(\sqrt{\gamma}\right)} \quad \text{if } \gamma \neq 0
$$

and $g(0) = \frac{4}{15}$. Figure 2 illustrates how the coefficient g depends on γ: it shows that it is easier to induce some biaxiality in an oblate nematic (that is when γ, and thus s, is negative) than in a prolate one.

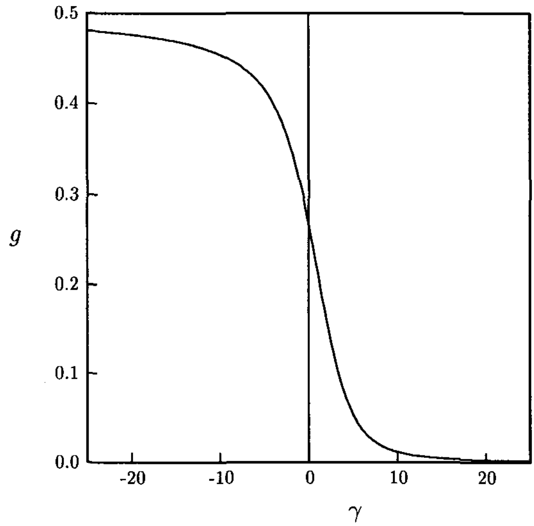

Figure 2: Coefficient of Δ in the difference $Q_x - Q_y$, as a function of γ.

3. – Decay of negative nematic temperances

The equilibrium configuration in a nematic liquid crystal can be obtained by minimizing the free energy density functional

$$\mathcal{F}[\mathbf{Q}, \nabla\mathbf{Q}] = \mathcal{F}_{\rm el}[\mathbf{Q}, \nabla\mathbf{Q}] + \mathcal{F}_{\rm int}[\mathbf{Q}] + \mathcal{F}_{\rm ext}[\mathbf{Q}]\,, \tag{6}$$

where $\mathcal{F}_{\rm el}$, $\mathcal{F}_{\rm int}$, and $\mathcal{F}_{\rm ext}$, respectively denote the elastic part of the free energy, the internal free energy (which depends only on the scalar invariants associated with $\mathbf{Q}$), and the external free energy which takes into account the effects of any electric or magnetic field acting on the nematic liquid crystal. Using the 1-constant approximation for the elastic part of the free energy, the Landau-de Gennes' expression for the internal potential, and involving only an external electric field, we have

$$\mathcal{F}[\mathbf{Q}, \nabla\mathbf{Q}] = \int_{\mathcal{B}} \Big(\frac{\kappa}{2}\left|\nabla\mathbf{Q}\right|^2 + a\,{\rm tr}\,\mathbf{Q}^2 - b\,{\rm tr}\,\mathbf{Q}^3 + c\,{\rm tr}\,\mathbf{Q}^4 - \chi_{\rm a}\,\mathbf{E}\cdot\mathbf{Q}\mathbf{E}\Big)\,dv\,, \tag{7}$$

where κ is an elastic constant; $a < 0$ and $b, c > 0$ are constitutive parameters, and $\chi_{\rm a} > 0$ is the anisotropic electric susceptibility (the sign of $\chi_{\rm a}$ is chosen in order to describe nematic molecules which tend to lie parallel to the electric field $\mathbf{E}$). The choice of the signs of a, b, and c, ensures that the internal potential is minimized when the nematic is uniaxial with positive degree of prolation (that is, positive nematic temperance); nevertheless, negative values of s can be induced by imposing on the system suitable boundary conditions. In this section we show with a simple example how boundary uniaxial configurations with negative degree of prolation decay in the bulk towards positive temperance configurations.

Let us consider a nematic liquid crystal confined within the half-space $\mathcal{B} = \{z \geq 0\}$, subject to a constant external electric field $\mathbf{E} = E\,\mathbf{e}_z$, and subject to the boundary condition $\mathbf{Q}\big|_{z=0} = \mathbf{Q}_0 := -\frac{1}{2}\left(\mathbf{e}_z \otimes \mathbf{e}_z - \frac{1}{3}\mathbf{I}\right)$, which requires the molecules to lie in the boundary plane, without any preferred direction on it. In such a case, a unique stationary distribution of the free energy functional can be found in the class of axisymmetric nematic distributions, that is, uniaxial distributions with director $\mathbf{e} = \mathbf{e}_z$: indeed, if we take $\mathbf{Q}(z) = s(z)\left(\mathbf{e}_z \otimes \mathbf{e}_z - \frac{1}{3}\mathbf{I}\right)$ in (7), and we insert a normalization factor to ensure that the energy be finite, we obtain:

$$\mathcal{F}[s] = \mathcal{F}_0 \int_0^{\infty} \left(s'^2 + \phi(s) - \phi(1)\right) \, dx\,, \tag{8}$$

where $\mathcal{F}_0 := \sqrt{\dfrac{2c\kappa}{27}}$, a prime denotes derivative with respect to $x := \dfrac{z}{\xi}$, $\xi := \sqrt{\dfrac{3\kappa}{2c}}$, $\phi(s) := -\zeta^2 s + \tilde{a}\, s^2 - \tilde{b}\, s^3 + s^4$, and

$$\zeta := \frac{E}{E_\xi}\,, \quad E_\xi := \sqrt{\frac{\kappa}{2\chi_a \xi}}\,; \qquad \tilde{a} := \frac{3a}{c}\,, \quad \tilde{b} := \frac{b}{c}\,.$$

In particular, when $\tilde{a} = \frac{3}{2}\tilde{b} - 2$, with $\tilde{b} \in \left(\frac{2}{3}, \frac{4}{3}\right)$, the Landau-de Gennes' potential possesses an absolute minimum at $s = 1$, a relative maximum at $s = 0$, and a relative minimum at $s = \frac{3}{4}\tilde{b} - 1 \in \left(-\frac{1}{2}, 0\right)$.

The Euler-Lagrange equations for the functional (8), with the boundary conditions $s(0) = -\frac{1}{2}$, $\lim\limits_{x\to\infty} s(x) = 1$, can be directly integrated to give

$$\begin{aligned} &\int_{-\frac{1}{2}}^{s(x)} \frac{ds}{\sqrt{\phi(s) - \phi(1)}} = x\,, && \text{for} \quad 0 \leq x \leq x_1 := \int_{-\frac{1}{2}}^{1} \frac{ds}{\sqrt{\phi(s) - \phi(1)}}\,, \\ &s(x) \equiv 1 && \text{for} \quad x \geq x_1. \end{aligned} \tag{9}$$

Using (9) and (4) we can also determine how the nematic temperance γ depends on the position x. Figure 3 illustrates both $s(x)$ and $\gamma(x)$ in the particular case $\zeta = 1$, $\tilde{b} = 1$, $\tilde{a} = -\frac{1}{2}$.

Figure 3 shows that negative nematic temperances can be induced inside a uniaxial nematic liquid crystal by imposing on it suitable boundary conditions. However, we will now show that the anchoring must be sufficiently strong in order to achieve this goal. To analyze better this problem, we now replace the strong anchoring condition with a condition of minimality involving an anchoring potential which still favours the planar distribution $\mathbf{Q}_0$ at the boundary $z = 0$, without enforcing it

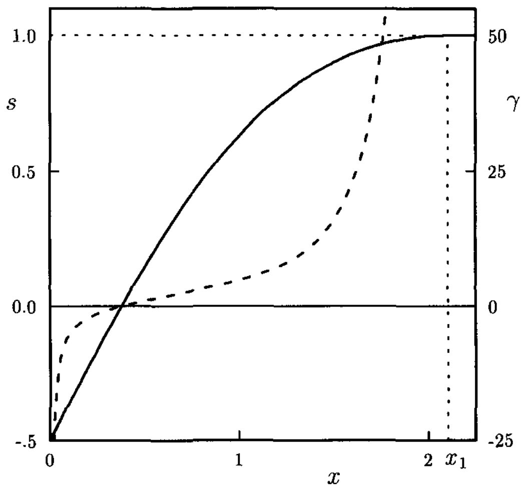

Figure 3: Degree of prolation s (continuous line) and nematic temperance γ (dashed line) as a function of position in a nematic liquid crystal confined in a half-space, with planar anchoring on the boundary, subject to a uniform electric field.

necessarily. Thus, we add to the free energy (6) the usual boundary potential and we minimize the functional

$$\begin{aligned}\mathcal{F}_w[\mathbf{Q},\nabla\mathbf{Q}] = \int_{\mathcal{B}}\Big(\frac{\kappa}{2}\left|\nabla\mathbf{Q}\right|^2 + a\,\mathrm{tr}\,\mathbf{Q}^2 - b\,\mathrm{tr}\,\mathbf{Q}^3 + c\,\mathrm{tr}\,\mathbf{Q}^4 - \chi_{\mathrm{a}}\,\mathbf{E}\cdot\mathbf{Q}\mathbf{E}\Big)\,dv \\ + w\,\left|\mathbf{Q}(0) - \mathbf{Q}_0\right|^2 ,\end{aligned} \tag{10}$$

where w is the *anchoring strength*. The minimizer of (10) in the class of axisymmetric distributions is still of the form (9), with the only difference that now the initial condition $s(0)$, which is also the lower bound of the integrals in (9), is to be determined from the natural boundary condition

$$s'(0) = \frac{w}{3\mathcal{F}_0}\,\left(2s(0)+1\right) ,$$

which implies that

$$\sqrt{\phi\left(s(0)\right) - \phi(1)} = \frac{w}{3\mathcal{F}_0}\left(2s(0)+1\right) . \tag{11}$$

Equation (11) admits only one real solution in the interval $s(0) \in \left[-\frac{1}{2}, 1\right]$. Figure 4 illustrates how $s(0)$ depends on the anchoring strength, again for $\zeta = 1$, $\tilde{b} = 1$, and $\tilde{a} = -\frac{1}{2}$. In general, (11) shows that the boundary degree of prolation (and thus the boundary nematic temperance) will be negative if and only if the anchoring strength w exceeds the critical value

$$w_{\mathrm{cr}} := 3\,\mathcal{F}_0\,\sqrt{\phi(0) - \phi(1)} = 3\,\mathcal{F}_0\,\sqrt{\zeta^2 - \tilde{a} + \tilde{b} - 1} ,$$

which is always positive in the nematic range defined by $\tilde{a} = \frac{3}{2}\,\tilde{b} - 2$, with $\tilde{b} \in \left(\frac{2}{3}, \frac{4}{3}\right)$.

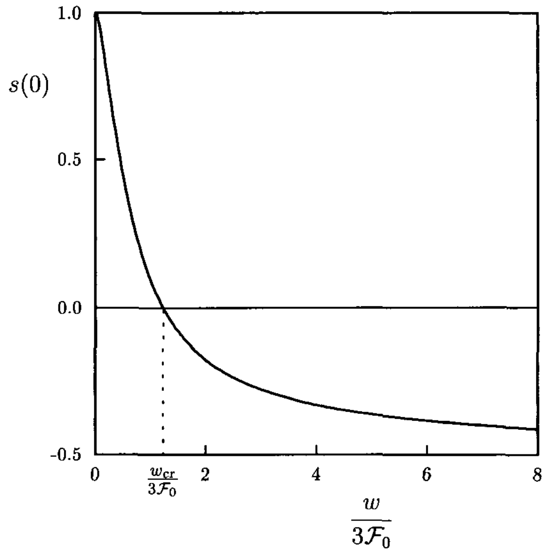

Figure 4: Boundary degree of prolation s as a function of the anchoring strength w: the boundary succeeds in inducing a negative boundary degree of prolation only if $w > w_{cr}$.

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Null Lagrangians and Surface Interaction Potentials in Nonlinear Elasticity

S. Carillo *

1. – Introduction

A problem in the framework of nonlinear elasticity is the subject of this study. Specifically, an elastic body immersed into a surrounding media is considered and, in particular, the interest is focussed on the nonlinear interaction between the body and the environment. The reasons why the classical approach [6] turns out to be inadequate to model the interaction body-environment has been already pointed out [17] and, subsequently, [3] and specifically reconsidered in [16] which is devoted to this subject. In particular, the classical assumption prescribes that the surface potential density at each point of the body boundary is a function which depends on two variables: the position on the body boundary and the deformation at that position on the body boundary. Here, according to the results comprised in [17], [3] and [4], the surface potential density is assumed to depend also on the deformation gradients, respectively, the first one alone and, afterwards, on the second one too.

The first two questions which can be addressed to are:

- how can be mathematically modeled these body-environment interactions?
- how do they affect the equilibrium problem?

An answer to these questions has been proposed to by Podio-Guidugli and Vergara Caffarelli [17] and, subsequently, by Carillo, Podio-Guidugli and Vergara Caffarelli [3] on introduction of two different body-environment interaction potentials. These potentials have been termed, in turn, First-Order and Second-Order Surface Interaction Potentials.

Here, on the basis of the results obtained in [3] and [4], some remarks in connection with Second-Order Surface Interaction Potentials, on one side, and a comparison among the classical case, the First-Order and Second-Order Surface Interaction Potentials, on the other side, are comprised.

Indeed, both the body as well as the environment, which is surrounding it, can be considered as elastic media. Thus, to model the interaction between the two

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of them, it is required to prescribe suitable *transmission conditions* between them. Hence, it turns out to be convenient to recall the definition of elastic *materials of grade N* [21]. An elastic material is termed to be of grade N when the corresponding stress response to an admissible deformation history depends on the first N gradients of the deformation. Accordingly, the grade N is related to the nonlocality of the stress response, namely, increasing N also the nonlocality increases. Simple materials, or of grade 1 [21], are those whose volume energy density, here denoted by σ, is characterized by the dependence on the deformation only through the first deformation gradient. In general, the two bodies, the elastic body, say of grade M, and the surrounding media, say of grade N, following the terminology in [3], are termed to form an *elastic body-environment pair of grade* (N, M).

Here, only the case of an environment represented by a simple material is considered; a discussion concerning more general cases and further generalizations is comprised in [3]. Accordingly, the cases of body-environment pairs of grade, in turn, $(1,1)$ [17] and $(1,2)$, [3] and [4], that is, respectively, the case of First-Order as well as Second-Order Surface Interaction Potentials are considered. In particular, the interconnection [3] between tangency conditions and Null Lagrangians is pointed out. Indeed, there is a connection between the two different problems of stationary solutions, on one side, and of existence of Null Lagrangians, on the other one; indeed, stationary solutions s well as Null Lagrangians satisfy some extra conditions, *tangency conditions* according to [3].

This relation motivates to briefly recall, in the introductory Section 2, the variational setting of the equilibrium problem in the cases of body-environment pairs of grade, in turn, $(1,1)$, studied in [17], and $(1,2)$, considered in [3] and [4]. The subsequent Section 3 is focused on Null Lagrangians and, in particular, some results obtained in [17] and, subsequently, in [3] and [4] are discussed. The aim is, again, to point out where the differencies and where the analogies are when the Null Lagrangians which correspond to the two cases of First-Order Surface interaction potentials, [17], and of Second-Order ones, [3] [4], are considered.

2. – Variational Equilibrium Problem

This Section is devoted to a brief survey of the results obtained in [17] and [3] wherein, respectively, the equilibrium problem in the two cases of body-environment pairs of grade (1,1) and (1,2) has been investigated.

According to the standard approach, the equilibrium problem of interest, namely the equilibrium problem concerning body-environment pairs of grade (1,1)[17] and (1, 2) [3][1], admits the variational formulation

$$\delta E = 0$$

as soon as the energy E has been specified together with the suitable function space wherein the equilibrium solutions are looked for.

[1] these cases are also termed, respectively, First-Order and Second-Order Surface Interaction Potential case to point out that they are different only as far as the potential density τ is concerned while the environment is, in both of them, assumed to be a simple material.

The environmental body is assumed of grade 1 and the reference configuration of the elastic body immersed in the surrounding media is denoted by $\Omega \in \boldsymbol{R}^3$. The total energy can be written in the form

$$\mathrm{E}_i = \Sigma - \mathrm{T}_i \quad , \quad i = 1, 2 \, ; \tag{1}$$

where T_1 and T_2 represent, respectively, the First-Order and Second-Order Surface Potentials and Σ the total stored energy. Indeed, let the energy density per unit volume be $\sigma(x, F)$ at $x \in \Omega$ and $F \in \boldsymbol{R}^{3\times3}$, then, given a deformation f whose gradient is $F := \nabla f$, the total stored energy is given by

$$\Sigma = \int_\Omega \sigma(x, \nabla f(x)) \; dV \quad ; \tag{2}$$

In addition, the elastic body is supposed to admit a reference configuration represented by a closed regular set $\overline{\Omega} \subset \boldsymbol{R}^3$ whose boundary is regular[2] (here $\partial\Omega \in C^4$).

The potential energy density due to the body-environment interaction, denoted by τ, and termed *surface potential density* [17] and [3], is, respectively, assumed of the form

$$\tau_1 = \tau_1(x, f, F) \quad , \quad F := \nabla f(x) \quad ; \tag{3}$$

where

$$(x, f, F) \in \Omega \times \boldsymbol{R}^3 \times \boldsymbol{R}^{3\times3}$$

termed First-Order Surface Potential density and

$$\tau_2 = \tau_2(x, f, F, \mathcal{F}) \quad , \quad F := \nabla f(x) \quad , \quad \mathcal{F} := \nabla^2 f(x) \; ; \tag{4}$$

where

$$(x, f, F, \mathcal{F}) \in \Omega \times \boldsymbol{R}^3 \times \boldsymbol{R}^{3\times3} \times \{\mathcal{G} \in \boldsymbol{R}^{3\times3\times3} | \mathcal{G}_{ijh} = \mathcal{G}_{ihj}\}$$

termed Second-Order Surface Potential density[3].

Thus, the corresponding First-Order and Second-Order Surface Potentials are, respectively, given by

$$\mathrm{T}_1 = \int_{\partial\Omega} \tau_1(x, f(x), \nabla f(x)) \, dA \tag{6}$$

and

$$\mathrm{T}_2 = \int_{\partial\Omega} \tau_2(x, f(x), \nabla f(x), \nabla^2 f(x)) \, dA \; , \tag{7}$$

which allow to write the total energy under the form (1) respectively, in the two cases of First-Order and Second-Order Surface Potentials. In addition, surface interactions

[2]extra conditions are obtained if the boundary (here $\partial\Omega$) exhibits lines of discontinuity of the outer normal unit vector $\mathbf{n}$, see [17] and [3], where these problems have been developed, respectively, in the case of first order and second order potentials.

[3]classically [6], the function τ is assumed to depend on the position $x \in \partial\Omega$ and on the deformation $f(x)$, namely the most common assumption is $\tau(x, f(x)) = \mathbf{b}(x) \cdot (f(x) - x)$. Thus, a *dead* surface load corresponds to a surface load which does not depend on the deformation;that is

$$\mathbf{b}(x) = \tau_f(x, f(x)) \quad , \quad S\mathbf{n} = \mathbf{b}(x) \quad on \quad \partial\Omega \, . \tag{5}$$

are supposed to be **conservative**, and this hypothesis, as it will be shown in the next Section, implies that

$$\delta T_i\{f\}[\mathbf{v}] = \int_{\partial\Omega} s\{f\} \cdot \mathbf{v}\, dA \quad , \quad i = 1,2 \tag{8}$$

together with further conditions, termed "*tangency*" conditions in [17] and [3], which, also, follow. It should also be remarked that the body-environment contact interaction is accounted for by [4]

$$S\,\mathbf{n} = s \quad , \quad S = \sigma_F \ . \tag{9}$$

To find equilibrium solutions, written the total energy (1), when the corresponding variational derivative is imposed to be zero for all variations $\mathbf{v}$ in the space of admissible variations, namely

$$\delta \mathrm{E}_i = 0 \quad , \quad i = 1,2 \tag{10}$$

where, corresponding to $i = 1$, namely a body-environment pairs of grade (1,1),

$$\mathrm{E}_1 = \Sigma - \mathrm{T}_1 == \int_{\Omega} \sigma(x, \nabla f(x))\, dV - \int_{\partial\Omega} \tau_1(x, f(x), \nabla f(x)) \ dA \tag{11}$$

and, corresponding to $i = 2$, namely a body-environment pairs of grade (1,2),

$$\mathrm{E}_2 = \Sigma - \mathrm{T}_2 == \int_{\Omega} \sigma(x, \nabla f(x))\, dV - \int_{\partial\Omega} \tau_2(x, f(x), \nabla f(x), \nabla^2 f(x)) \ dA \ . \tag{12}$$

The (standard) field equation

$$\mathrm{Div}\ \sigma_F\,(x, \nabla f(x)) = 0 \quad , \quad x \in \Omega \ , \tag{13}$$

follows within the open set Ω. The subsequent application of the divergence theorem, allows to find further conditions on $\partial\Omega$ which involve volume energy density σ as well as the surface interaction potential density. On the boundary $\partial\Omega$, $S\mathbf{n} = s$ now reads

$$\sigma_F \mathbf{n} = \tau_f - {}^s\mathrm{Div}\ \tau_F \ . \tag{14}$$

The further condition [17] follows

$$\tau_F\,[\mathbf{n}] = 0 \ . \tag{15}$$

The latter is termed [17], *Tangency Condition* since it is identically satisfied when the First-Order Surface Potential is required T **to coincide** with its tangential part. Indeed, (15) is obtained on decomposition of all the quantities of interest into two parts which are, respectively, tangential to the body boundary (denoted as ${}^s\cdot$) and orthogonal to the same boundary [5]. In the case of Second-Order Surface Potentials, [3], the condition $S\mathbf{n} = s$ on $\partial\Omega$ can be written in the form

$$\sigma_F\ \mathbf{n} = \tau_f - {}^s\mathrm{Div}\ \tau_F + {}^s\mathrm{Div}\ ({}^s\mathrm{Div}\ \tau_{\mathcal{F}}) - \tau_{\mathcal{F}}\,[\mathbf{n} \otimes {}^s\nabla H] + {}^s\mathrm{Div}\ \ (\tau_{\mathcal{F}}[\mathbf{n}]^s\nabla \mathbf{n}) \ . \tag{16}$$

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[4] Petrowski ellipticity [18] guarantees, on $\partial\Omega$, the local invertibility of $S\mathbf{n}$ in (9) with respect to the normal derivative of the deformation f.

[5] i.e. ${}^s\mathrm{T}_i[f]$, $i = 1,2$ denotes the tangential part of $\mathrm{T}_i[f]$, $i = 1,2$ the detailed computations are comprised in [17], [3] and also in [5]

In addition, **Two Tangency Conditions**

$$\tau_{\mathcal{F}}\left[\mathbf{n}\otimes\mathbf{n}\right]=0\ , \tag{17}$$

$$\tau_{F}\left[\mathbf{n}\right]-2\,{}^{s}\mathrm{Div}\ \left(\tau_{\mathcal{F}}\left[\mathbf{n}\right]\right)+\tau_{\mathcal{F}}\left[{}^{s}\nabla\mathbf{n}\right]=0 \tag{18}$$

are obtained. Again, as in the case of First-Order Surface Potential, they are identically satisfied when a Second-Order Surface Potential T **coincides** with its tangential part, i.e., when $\mathrm{T}_2[f]\equiv{}^{s}\mathrm{T}_2[f]$.

3. – Null Lagrangians

This section is concerned about Null Lagrangians and, in particular, their connection with surface potentials. Indeed, the introduction of the concept of Null Lagrangians goes back to the 60's , see to Ericksen [8] and Edelen [7], and, furthermore, has a much wider application which is not restricted to the case of problems in elasticity. Subsequently, the concept has been exploited following both a constitutive viewpoint, on one side, and an analytic one related to symmetry properties, on the other side. A wide account on recent advances on the more general subject of Null Lagrangians in connection to the general case of Lagrange equations has been discussed by Olver [14] and [15]. To the same author and co-workers, see ref.s [1], [11], [12], [9], [13], are also due investigations on the symmetry structure of Null Lagrangians in various cases.

Here, a brief idea of the concept of Null Lagrangians is given and, afterwards, the connection between the stationary conditions and Null Lagrangians is reconsidered on the basis of the investigations comprised in [17], [3] and [4].

The connection between the two different problems of finding stationary solutions, on one side, and of Null Lagrangians, on the other one, is related to tangency conditions; indeed, to find stationary solutions as well as Null Lagrangians some extra conditions, and precisely, according to [3], *tangency conditions.*

Here, to start with, the definition of Null Lagrangian is recalled. The idea, which stands behind such a concept, is to consider the divergence theorem to establish the equivalence between two different potentials. In particular, when a volume potential density ρ is introduced, whenever it can be represented in the form:

$$\rho=\mathrm{Div}\ \mathbf{w} \tag{19}$$

i.e. $\exists$ $\mathbf{w}$ s.t. (19) holds $\forall P\in\Omega$, where, respectively:

$$\rho=\rho(x,f,\nabla f)\quad ,\quad \rho=\rho(x,f,\nabla f,\nabla^2 f) \tag{20}$$

namely, there exists a vector field $\mathbf{w}\in\boldsymbol{R}^3$ such that (19) is verified at each point $P\in\partial\Omega$, then ρ can be represented under the divergence form. Indeed, on introduction of the surface potential density τ defined by

$$\tau:=\mathbf{w}\cdot\mathbf{n}\ , \tag{21}$$

where $\mathbf{n}$ denotes the outer unit vector normal to $\partial\Omega$, at the point $P\in\partial\Omega$, then, the divergence theorem reads

$$\int_{\Omega}\rho(\ldots\ldots)\ dV=\int_{\partial\Omega}\tau(\ldots\ldots)\ dA \tag{22}$$

and, accordingly, τ represents the surface potential density

$$\tau = \tau(x, f, \nabla f) \quad , \quad \tau = \tau(x, f, \nabla f, \nabla^2 f) \ . \tag{23}$$

Hence, [4] the following definition can be introduced.

DEFINITION 1 A functional $\int_\Omega \rho\{f\}\ dV$ is called a Null Lagrangian if there exists a vector $\mathbf{w}\{f\}$, such that

$$\int_\Omega \rho(\ldots\ldots)\ dV = \int_{\partial\Omega} \tau(\ldots\ldots)\ dA \quad , \quad \rho = \mathrm{Div}\ \mathbf{w} \quad , \quad \tau = \mathbf{w} \cdot \mathbf{n} \tag{24}$$

for all admissible deformations f.

3.1. – *First-Order Null Lagrangians.* – According to the definition, in particular, a First-Order Null Lagrangian [17] is a functional of the form

$$\int_\Omega \rho\{f\}\ dV$$

such that, correspondingly, there exists a vector $\mathbf{w}\{f\}$, which satisfies the condition that the functional

$$\mathcal{N}\{f\} := \int_\Omega \rho(x, f, \nabla f)\ dV \tag{25}$$

admits a corresponding functional which can be written under the form

$$\mathcal{T}_\mathcal{N}\{f\} := \int_{\partial\Omega} \tau(x, f, \nabla f)\ dA \tag{26}$$

and, in addition, the two functionals coincide for all admissible deformations f, namely

$$\mathcal{N}\{f\} = \mathcal{T}_\mathcal{N}\{f\} \quad \forall f \quad admissible\ . \tag{27}$$

When First-Order Null Lagrangian [17] are considered, on imposition of the previous conditions, it follows that the vector function $\mathbf{w}$ is required to depend on the position, the deformation and, in addition with respect to the classical case, also on the first deformation gradient, that is

$$\mathbf{w} = \mathbf{w}(x, f, F)\ .$$

Correspondingly, also ρ and the surface potential density which is related to it, τ, must depend on the same set of variables, that is:

$$\rho(x, f, \nabla f) = \mathrm{Div}\ \mathbf{w}(x, f, \nabla f),$$

$$\tau(x, f, \nabla f) = \mathbf{n} \cdot \mathbf{w}(x, f, \nabla f)\ .$$

Thus, the unknown vector function $\mathbf{w}$ is subject to satisfy the *skew-symmetry condition*[6]

$$\mathbf{w}_{iF_{\bullet j}} = -\mathbf{w}_{jF_{\bullet i}}\ .$$

Notably, First-Order Null Lagrangians identically satisfy the *tangency condition* $\tau_F[\mathbf{n}] = 0$.

[6]the details are comprised in [17] where the set of First-Order Null Lagrangians has been characterized.

3.2. – *Second-Order Null Lagrangians.* – When Second-Order Surface Interaction Potentials are considered [3][4], the definition can be re-written in the following form.

DEFINITION 2 A Second-Order Surface Potential density $\tau = \tau(x, \mathbf{n}, f, F, \mathcal{F})$ is termed Null Lagrangian if there exists a volume potential density

$$\sigma = \sigma(x, f, F, \mathcal{F})$$

such that

$$\int_\Omega \sigma(x, f, \nabla f, \nabla^2 f)\; dV = \int_{\partial\Omega} \tau(x, f, \mathbf{n}, \nabla f, \nabla^2 f)\; dA \tag{28}$$

for all admissible deformations f.

According to the definition, in particular, a Second-Order Null Lagrangian [3] is a functional of the form

$$\int_\Omega \rho\{f\}\; dV$$

such that, correspondingly, there exists a vector

$$\mathbf{w} = \mathbf{w}(x, f, F, \mathcal{F})$$

which satisfies the condition that the functional

$$\mathcal{N}\{f\} := \int_\Omega \rho(x, f, \nabla f, \nabla^2 f)\; dV \tag{29}$$

admits a corresponding functional which can be written under the form

$$\mathcal{T}_\mathcal{N}\{f\} := \int_{\partial\Omega} \tau(x, f, \nabla f, \nabla^2 f)\; dA \tag{30}$$

and, furthermore, the two functional coincide for all admissible deformations f, namely

$$\mathcal{N}\{f\} = \mathcal{T}_\mathcal{N}\{f\} \quad \forall f \quad \textit{admissible}\ . \tag{31}$$

When Second-Order Null Lagrangian [3][4] are considered, on imposition of the previous conditions, it follows that the vector function $\mathbf{w}$ is required to depend on the position, the deformation and, in addition, also on the first and, possibly, on the second deformation gradients, that is

$$\mathbf{w} = \mathbf{w}(x, f, F, \mathcal{F})\ .$$

Correspondingly, also ρ and the surface potential density which is related to it,τ, depends on the same set of variables, that is:

$$\begin{aligned} \rho(x, f, \nabla f, \nabla^2 f) &= \mathrm{Div}\ \mathbf{w}(x, f, \nabla f, \nabla^2 f), \\ \tau(x, f, \nabla f, \nabla^2 f) &= \mathbf{n} \cdot \mathbf{w}(x, f, \nabla f, \nabla^2 f)\ . \end{aligned} \tag{32}$$

Thus, the unknown vector function $\mathbf{w}$ is subject to satisfy the *cyclic-symmetry condition*[7]

$$\mathbf{w}_{i,\mathcal{F}_{\bullet hk}} + \mathbf{w}_{k,\mathcal{F}_{\bullet ih}} + \mathbf{w}_{h,\mathcal{F}_{\bullet ki}} = 0 \;\; \forall i, h, k\ . \tag{33}$$

[7] the details are comprised in [4].

Notably, when a Second-Order Null Lagrangian is considered, it is such that it identically satisfies the tangency condition $\tau_{\mathcal{F}}[\mathbf{n} \otimes \mathbf{n}] = 0$. As a final remark, when it is considered the particular case of a Second-Order Null Lagrangian, which is represented [4] by a functional of the form (29), in which ρ is supposed to dependent on the deformation up to the first order gradient, then, the equilibrium conditions coincide with the tangency conditions.

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Automatic control problems for integrodifferential parabolic equations

Cecilia Cavaterra *

1. – Introduction

Our aim is to study some automatic control problems related to the following integrodifferential parabolic equations

$$(u + k * u)_t = \Delta u + h * \Delta u + f \qquad \text{in } \Omega_T \tag{1}$$

and

$$u_t = \Delta u + au - bu^2 - u(h * u) \qquad \text{in } \Omega_T \tag{2}$$

where $\Omega_T := (0, T) \times \Omega$, Ω being a bounded domain in $\mathbb{R}^n (n \geq 1)$ with a smooth boundary Γ and $T > 0$. Here z_t denotes the partial derivative of a function z with respect to time, and Δ indicates the spatial Laplace operator. Moreover, $(p * q)(t)$ stands for the usual time convolution product over $(0, t)$. The convolution kernels h and k are time dependent functions, while a and b some given constants.

Equation (1) describes for example the evolution of the temperature u in a material with thermal memory. The equation can be easily derived from the energy balance assuming that both the internal energy and the heat flux also depend on the past history of the temperature (see [5] and its references). Here the source term f accounts for the heat supply and the past history of u, up to the time $t = 0$.

Equation (2) is a reaction-diffusion equation that models a number of phenomena in various field of applied sciences like, e.g., biology, ecology and biochemistry as well as the classical fields of physical and engineering sciences. Among them, we recall population dynamics (see, e.g., [9], [10], [13], [4] and their references), chemical reactions (see, e.g., [9] and references therein), and nuclear reactor dynamics (cf., e.g., [9] and [14]).

In many applications it may arise the problem of keeping the range of the state variable u within a given interval of values, on any prescribed finite time interval. This goal can be achieved in several ways. In particular, on account of [3] and [6] (see also their references), we are interested in formulating and studying some automatic control problems for equation (1) or (2) based on feedback devices located on the boundary Γ. For this reason, we are forced to allow interactions with the external environment through Γ. Therefore, the first step is to formulate suitable initial and boundary value problems associated with the equations (1) and (2). These problems

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are characterized by nonhomogeneous boundary conditions of the third type. More precisely, we introduce

Problem (P1) *Find* $u : \Omega_T \to \mathbb{R}$ *such that*

$$(u + k * u)_t = \Delta u + h * \Delta u + f \qquad \text{in } \Omega_T \tag{3}$$

$$u(0, \cdot) = u_0 \qquad \text{in } \Omega \tag{4}$$

$$u_{\mathbf{n}} + h * u_{\mathbf{n}} = \alpha(u_e - u_\Gamma) + g \qquad \text{on } \Gamma_T \tag{5}$$

and

Problem (P2) *Find* $u : \Omega_T \to \mathbb{R}$ *such that*

$$u_t = \Delta u + au - bu^2 - u(h * u) \qquad \text{in } \Omega_T \tag{6}$$

$$u(0, \cdot) = u_0 \qquad \text{in } \Omega \tag{7}$$

$$u_{\mathbf{n}} = \alpha(u_e - u_\Gamma) \qquad \text{on } \Gamma_T. \tag{8}$$

Here $\Gamma_T := (0, T) \times \Gamma$ and α is a positive constant. Moreover, $u_{\mathbf{n}}$ indicates the outward normal derivative of u on Γ, u_Γ denotes the trace of u on Γ and u_e represents the value of u in the external environment.

Observe that in problem **(P1)** the term g in the boundary condition (5) accounts for the past history of u on the boundary up to $t = 0$.

Concerning problem **(P2)**, in equation (6) the nonlinear terms bu^2 and $u(h * u)$ occur. Therefore, since our aim consists on keeping the range of the state variable u within a given interval of values, we need to make some assumptions on the sign of the coefficients b and h in order to prevent blow-up phenomena in finite time (see, e.g., [9]). In particular, we suppose

$$b \geq 0 \tag{9}$$

$$h : [0, T] \to [0, +\infty). \tag{10}$$

As already stated above, we want to control u acting on u_e. Accordingly, we need a feedback device based on measurements of u. For the sake of brevity, here we consider only the case in which the measurement is the values of u at a fixed point $x_1 \in \overline{\Omega}$; that is,

$$\mathcal{M}(u)(t) := u(t, x_1) \quad t \in [0, T]. \tag{11}$$

Observe that, from the mathematical viewpoint, this pointwise control usually is the hardest case since it requires continuous solutions. Now, we introduce a thermostat-like device that modify u_e on account of $\mathcal{M}(u)$. A reasonable description of how this device acts on u_e is detailed, for instance, in [6] and [3] (cf. also their references). On account of it, we arrive at the following relationship

$$u_e = \mathcal{F}(\mathcal{W}(\mathcal{M}(u))) \quad \text{on } \Gamma_T \tag{12}$$

where

$$\mathcal{F}(r)(t, y) := \int_0^t E(t, \tau, y) r(\tau) d\tau + E_0(t, y) \tag{13}$$

$$E(t, \tau, y) := e^{-(t-\tau)/\varepsilon} u_A(t, y) \tag{14}$$

$$E_0(t, y) := \left(\int_0^t e^{-(t-\tau)/\varepsilon} u_C(\tau) d\tau + \varphi_0 e^{-t/\varepsilon} \right) u_A(t, y) + u_B(t, y). \tag{15}$$

Here $u_B : \Gamma_T \to \mathbb{R}$ is a given reference boundary value (e.g., the temperature of the external environment); while $u_A : \Gamma_T \to \mathbb{R}$ is the fraction of u_e that can be controlled by our device. Moreover, $u_C : [0,T] \to \mathbb{R}$ is a given function, ε is a positive parameter and $\varphi_0 \in \mathbb{R}$.

Possible choices of the nonlinear operator $\mathcal{W}$ will be discussed in the following section. Summing up, on account of (12), the feedback control problem reduces to a system with a nonlinear and nonlocal boundary condition. More precisely, recalling **(P1)** and **(P2)**, we shall deal with the following problems

Problem (CP1) *Find u such that*

$$(u + k * u)_t = \Delta u + h * \Delta u + f \qquad \text{in } \Omega_T \tag{16}$$

$$u(0,\cdot) = u_0 \qquad \text{in } \Omega \tag{17}$$

$$Bu = \mathcal{F}(\mathcal{W}(\mathcal{M}(u))) - \alpha^{-1} h * u_{\mathbf{n}} + \alpha^{-1} g \qquad \text{on } \Gamma_T \tag{18}$$

and

Problem (CP2) *Find u such that*

$$u_t = \Delta u + au - bu^2 - u(h * u) \qquad \text{in } \Omega_T \tag{19}$$

$$u(0,\cdot) = u_0 \qquad \text{in } \Omega \tag{20}$$

$$Bu = \mathcal{F}(\mathcal{W}(\mathcal{M}(u))) \qquad \text{on } \Gamma_T. \tag{21}$$

Here B is the linear boundary operator defined by

$$Bz := z_\Gamma + \alpha^{-1} z_{\mathbf{n}}. \tag{22}$$

Our goal is to present results of existence and uniqueness of solutions to the nonlinear problems **(CP1)** and **(CP2)** in a suitable functional framework, provided that $\mathcal{W}$ is a relay switch or a hysteresis operator of Preisach type.

2. – The operator $\mathcal{W}$

Here we complete the mathematical description of the feedback action by introducing two possible choices for the operator $\mathcal{W}$. We report the details for the reader's convenience (cf. also [3] and [6]).

(A) The relay switch operator. Assume that the state variable $r \in C^0([0,T])$ has to be kept under control on account of two critical time-dependent thresholds ρ_L, ρ_U such that

$$\rho_L,\ \rho_U \in C^0([0,T]) \tag{23}$$

$$0 < \delta \le \rho_L(t) < \rho_U(t) \qquad \text{on } [0,T]. \tag{24}$$

Without loss of generality, we suppose that $r(0) \le \rho_L(0) < \rho_U(0)$. Since r, ρ_L, ρ_U are continuous functions, then there exists a time interval $[0,t_1) \subset [0,T]$ in which $r(t) < \rho_U(t)$. In this time interval, the control device is switched on, that is we can think at it as a *heating* device. Indicating by $\mathcal{W}$ the associated operator, we have

$\mathcal{W}(r(t)) = 1$, for any $t \in [0,t_1)$. This situation holds up to a possible instant in which r reaches the upper threshold ρ_U. Thus, if at time $t = t_1$, $t_1 \in (0,T)$, we have $r(t_1) = \rho_U(t_1)$, then the device instantaneously switches off, that is $\mathcal{W}(r(t_1)) = -1$, and it remains in this position, acting like a *cooling* device, till before a possible time in which r reaches again the lower threshold ρ_L.

Proceeding formally, we set $t_1 := \inf\{\{t \in (0,T] \,:\, r(t) = \rho_U(t)\} \cup \{T\}\}$. If $t_1 = T$, this means that $r(t) < \rho_U(t)$ on $[0,T)$ and we have $\mathcal{W}(r(t)) = 1$ on $[0,T)$. If $t_1 \in (0,T)$, then, at time $t = t_1$, $\mathcal{W}$ switches its value from $+1$ to -1; namely,

$$\mathcal{W}(r(t)) = 1 \quad \text{in } [0,t_1), \qquad \mathcal{W}(r(t)) = -1 \quad \text{in } [t_1,t_2)$$

where $t_2 = \inf\{\{t \in (t_1,T] \,:\, r(t) = \rho_L(t)\} \cup \{T\}\}$. Observe that in this case we have

$$\begin{aligned} &r(0) < \rho_L(0), &&r(t) < \rho_U(t) \quad \text{in } [0,t_1) \\ &r(t_1) = \rho_U(t_1), &&r(t) > \rho_L(t) \quad \text{in } [t_1,t_2). \end{aligned}$$

If $t_2 \in (t_1,T)$, we iterate the procedure. Thanks to (24) and to the continuity of r, we can find a finite sequence $\{t_h\}_{h=0}^m$ of switching times satisfying

$$0 =: t_0 < t_1 < \cdots < t_m := T \tag{25}$$

$$\mathcal{W}(r(t)) = (-1)^h \quad \text{in } [t_h,t_{h+1}) \tag{26}$$

where

$$t_{h+1} := \inf\left\{\left\{t \in (t_h,T] \,:\, r(t) = \begin{cases} \rho_U(t) & \text{if } h \text{ is even} \\ \rho_L(t) & \text{if } h \text{ is odd} \end{cases}\right\} \cup \{T\}\right\} \tag{27}$$

for $h = 0,\dots,m-1$. Observe that the value of $W(r(T))$ is not assigned by (25-27) but it can be easily calculated on account of m and $r(T)$. Finally, note that

$$\mathcal{W} : C^0([0,T]) \to L^\infty(0,T). \tag{28}$$

(B) The Preisach operator. The hysteresis operator of Preisach type is basically obtained by a parallel coupling of a distribution of relay switches, each one being characterized by a pair of constant thresholds. Here it is introduced following the standard reference [12] (cf. also [11]).

As in the previous case, we indicate by $r \in C^0([0,T])$ the state variable that we want to keep under control. Let us first introduce the *delayed relay operator* $\mathcal{W}_{(\rho_1,\rho_2)}(r,\xi) : [0,T] \to \{-1,+1\}$ as follows. For any pair $(\rho_1,\rho_2) \in \mathbb{R}^2$ satisfying $\rho_1 < \rho_2$, we set

$$\mathcal{W}_{(\rho_1,\rho_2)}(r,\xi)(0) := \begin{cases} +1 & \text{if } r(0) \le \rho_1 \\ \xi(\rho_1,\rho_2) & \text{if } \rho_1 < r(0) < \rho_2 \\ -1 & \text{if } r(0) \ge \rho_2 \end{cases} \tag{29}$$

$\xi := \mathcal{P} \to \{-1,+1\}$ being a given Borel measurable function. Here $\mathcal{P}$ is the so called Preisach plane defined as $\mathcal{P} := \{(\rho_1,\rho_2) \in \mathbb{R}^2 \,:\, \rho_1 < \rho_2\}$.

Further, for any $t \in (0,T]$, we set

$$X_t := \{\tau \in (0,t] \,:\, r(\tau) = \rho_1 \text{ or } r(\tau) = \rho_2\} \tag{30}$$

$$\mathcal{W}_{(\rho_1,\rho_2)}(r,\xi)(t) := \begin{cases} \mathcal{W}_{(\rho_1,\rho_2)}(r,\xi)(0) & \text{if } X_t = \emptyset \\ +1 & \text{if } X_t \neq \emptyset \text{ and } r(\max X_t) = \rho_1 \\ -1 & \text{if } X_t \neq \emptyset \text{ and } r(\max X_t) = \rho_2. \end{cases} \tag{31}$$

It can be easily checked that the mapping $\mathcal{W}_{(\rho_1,\rho_2)}(r,\xi)$ is uniquely defined.

Now, if μ is a nonnegative Borel measure on the plane $\mathcal{P}$, the associated Preisach operator is represented in this way

$$\mathcal{W}(r)(t) := \int_{\mathcal{P}} \mathcal{W}_{(\rho_1,\rho_2)}(r,\xi)(t)\,d\mu(\rho_1,\rho_2). \tag{32}$$

We recall that, thanks to [11] (cf. also [12]), there holds

PROPOSITION 1 *Assume*

$$\mu \textit{ is a nonnegative Borel measure with bounded density} \tag{33}$$

$$\mu(\{\rho_1\} \times (\rho_1,+\infty)) = \mu((-\infty,\rho_2) \times \{\rho_2\}) = 0 \qquad \forall (\rho_1,\rho_2) \in \overline{\mathcal{P}}. \tag{34}$$

Then we have

$$\|\mathcal{W}(r)\|_{L^\infty(0,T)} \leq \mu(\mathcal{P}) < +\infty \qquad \forall\, r \in C^0([0,T]) \tag{35}$$

$$\mathcal{W} \textit{ is strongly continuous from } C^0([0,T]) \textit{ to } C^0([0,T]). \tag{36}$$

Moreover, suppose that there exists a positive constant Λ such that

$$\mu(A) \leq \Lambda \mathcal{L}(A) \qquad \textit{for all Lebesgue measurable sets } A \subset \mathcal{P} \tag{37}$$

$\mathcal{L}$ denoting the Lebesgue measure in $\mathbb{R}^2$. Then, there exists a positive constant Λ_1, depending on $\mu(\mathcal{P})$ and Λ, such that, for any $r_1, r_2 \in C^0([0,T])$,

$$|(\mathcal{W}(r_1) - \mathcal{W}(r_2))(t)| \leq \Lambda_1 \|r_1 - r_2\|_{C^0([0,t])} \qquad \forall\, t \in [0,T]. \tag{38}$$

3. – Notation

This section is devoted to introduce some useful notation and, in particular, the Banach spaces we need to formulate our results. The reader is referred to chapter 5 of [7] for the details. First, we introduce spaces of continuous functions that are Hölder continuous or continuously differentiable either with respect to time or with respect to the space variables. We set, for any $\alpha > 0$,

$$C^{\alpha,0}([0,T]\times\overline{\Omega}) := \Big\{ f \in C^0(\overline{\Omega}_T) \,:\, f(\cdot,x) \in C^\alpha([0,T]),\ \forall\, x \in \overline{\Omega},$$

$$\|f\|_{C^{\alpha,0}([0,T]\times\overline{\Omega})} := \sup_{x\in\overline{\Omega}} \|f(\cdot,x)\|_{C^\alpha([0,T])} < +\infty \Big\}$$

$$C^{0,\alpha}([0,T]\times\overline{\Omega}) := \Big\{ f \in C^0(\overline{\Omega}_T) \,:\, f(t,\cdot) \in C^\alpha(\overline{\Omega}),\ \forall\, t \in [0,T],$$

$$\|f\|_{C^{\alpha,0}([0,T]\times\overline{\Omega})} := \sup_{t\in[0,T]} \|f(t,\cdot)\|_{C^\alpha(\overline{\Omega})} < +\infty\Big\}.$$

Further, we define the space $C^{1,2}([0,T]\times\overline{\Omega})$ and its norm as

$$C^{1,2}([0,T]\times\overline{\Omega}) := \{f \in C^0(\overline{\Omega}_T) \;:\; D_t f, D_{ij} f \in C^0(\overline{\Omega}_T),\ i,j=1,...,n\}$$

$$\|f\|_{C^{1,2}([0,T]\times\overline{\Omega})} := \|f\|_{C^0(\overline{\Omega}_T)} + \|f_t\|_{C^0(\overline{\Omega}_T)} + \sum_{i=1}^n \|D_i f\|_{C^0(\overline{\Omega}_T)} + \sum_{i,j=1}^n \|D_{ij} f\|_{C^0(\overline{\Omega}_T)}$$

where $D_i f$ and $D_{ij} f$ are the derivatives of f with respect to x_i and x_i, x_j, respectively. Now we recall the definition of the parabolic Hölder spaces. For $0<\alpha<2$ we set

$$C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega}) := C^{\alpha/2,0}([0,T]\times\overline{\Omega}) \cap C^{0,\alpha}([0,T]\times\overline{\Omega})$$

endowed with the norm

$$\|f\|_{C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})} = \|f\|_{C^{\alpha/2,0}([0,T]\times\overline{\Omega})} + \|f\|_{C^{0,\alpha}([0,T]\times\overline{\Omega})}$$

and

$$C^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})$$
$$:= \{f \in C^{1,2}([0,T]\times\overline{\Omega}) \;:\; f_t,\ D_{ij} f \in C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega}),\ i,j=1,...,n\}$$

normed by

$$\|f\|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega})} := \|f\|_{C^{1,2}([0,T]\times\overline{\Omega})} + \|f_t\|_{C^{\frac{\alpha}{2},\alpha}([0,T]\times\overline{\Omega})} + \sum_{i,j=1}^n \|D_{ij} f\|_{C^{\frac{\alpha}{2},\alpha}([0,T]\times\overline{\Omega})}.$$

In a similar way it is possible to define the functional spaces $C^{\alpha/2,\alpha}([0,T]\times\Gamma)$, $C^{0,\alpha}([0,T]\times\Gamma)$ and $C^{1+\alpha/2,2+\alpha}([0,T]\times\Gamma)$

Finally, let X be a real Banach space with norm $\|\cdot\|_X$ and $k \in \mathbf{N}\cup\{0\}$. The space of all functions $u : [0,T] \to X$ which are continuous along with their first k time derivatives is denoted by $C^k([0,T];X)$ and normed as usual. Then, for any $\alpha \in (0,1)$ and $k \in \mathbf{N}\cup\{0\}$, we define

$$C^{k+\alpha}([0,T];X) := \{u \in C^k([0,T];X) \;:$$

$$|u^{(k)}|_{\alpha,T,X} := \sup_{0\le s<t\le T} (t-s)^{-\alpha}\|u^{(k)}(t) - u^{(k)}(s)\|_X < +\infty\}$$

and we endow it with the norm

$$\|u\|_{C^{k+\alpha}([0,T];X)} := \|u\|_{C^k([0,T];X)} + |u^{(k)}|_{\alpha,T,X}.$$

We recall that all the functional spaces here introduced are Banach spaces.

4. – Preliminary results on problems (P1) and (P2)

In this section we assume that u_e, representing the value of u in the external environment, is a prescribed function. Hence we show that problems **(P1)** and **(P2)** are well posed in suitable functional spaces. In particular, since we want to deal with continuous functions, we shall use the results on parabolic equations with nonhomogeneous boundary conditions contained in chapter 5 of [7].

Let $\alpha \in (0, 1/2)$. Here and below, we suppose that Ω is a bounded domain in $\mathbb{R}^n$ with boundary Γ uniformly $C^{2+2\alpha}$ (see, e.g., chapter 0 of [7]). Moreover, for the sake of simplicity, we indicate by Λ a generic positive constant depending only on known quantities.

Consider problem **(P1)** first. We have an existence and uniqueness result.

THEOREM **1** *Assume*

$$
\begin{aligned}
&(39) \qquad & k &\in C^1([0,T]) \\
&(40) \qquad & h &\in C^{(1+2\alpha)/2}([0,T]) \\
&(41) \qquad & f &\in C^{\alpha,2\alpha}([0,T]\times\overline{\Omega}) \\
&(42) \qquad & u_0 &\in C^{2+2\alpha}(\overline{\Omega}) \\
&(43) \qquad & g &\in C^{(1+2\alpha)/2,1+2\alpha}([0,T]\times\Gamma) \\
&(44) \qquad & u_e &\in C^{(1+2\alpha)/2,1+2\alpha}([0,T]\times\Gamma) \\
&(45) \qquad & Bu_0 &= u_e(0,\cdot)+\alpha^{-1}g(0,\cdot) \qquad \text{on } \Gamma.
\end{aligned}
$$

Then problem **(P1)** *admits a unique solution* $u \in C^{1+\alpha,2+2\alpha}([0,T]\times\overline{\Omega})$ *and*

$$
\begin{aligned}
(46) \qquad \|u\|_{C^{\alpha}([0,T];C^0(\overline{\Omega}))} \le\ & \Lambda\Big(\|f\|_{C^{\alpha,2\alpha}([0,T]\times\overline{\Omega})} + \|g\|_{C^{(1+2\alpha)/2,1+2\alpha}([0,T]\times\Gamma)} \\
&+ \|u_0\|_{C^{2\alpha}(\overline{\Omega})} + \|u_e\|_{C^{(1+2\alpha)/2,1+2\alpha}([0,T]\times\Gamma)}\Big).
\end{aligned}
$$

Further, the following continuous dependence result holds.

THEOREM **2** *Let the assumptions of theorem 1 hold and let u be the solution to problem* **(P1)** *corresponding to the boundary data u_e. Then the map $u_e \mapsto u$ is continuous from $C^{(1+2\alpha)/2,1+2\alpha}([0,T]\times\Gamma)$ to $C^{1+\alpha,2+2\alpha}([0,T]\times\overline{\Omega})$; that is,*

$$
(47) \qquad \|u_2-u_1\|_{C^{1+\alpha,2+2\alpha}([0,T]\times\overline{\Omega})} \le \Lambda\|u_{e2}-u_{e1}\|_{C^{(1+2\alpha)/2,1+2\alpha}([0,T]\times\Gamma)}.
$$

In order to prove theorems 3 and 4, we adapt an argument of [8] and we use the Banach contraction theorem in a suitable functional framework (see [2] for details).

Let us consider now problem **(P2)**. Then we have

THEOREM **3** *Assume (9-10) and, moreover,*

$$
\begin{aligned}
&(48) \qquad & h &\in C^0([0,T]) \\
&(49) \qquad & u_0 &\in C^{2+2\alpha}(\overline{\Omega}), \quad u_0 \ge 0 \quad \text{in } \overline{\Omega} \\
&(50) \qquad & u_e &\in C^{(1+2\alpha)/2,1+2\alpha}([0,T]\times\Gamma), \quad u_e \ge 0 \quad \text{on } [0,T]\times\Gamma \\
&(51) \qquad & Bu_0 &= u_e(0,\cdot) \qquad \text{on } \Gamma.
\end{aligned}
$$

Then problem **(P2)** *admits a unique solution* $u \in C^{1+\alpha,2+2\alpha}([0,T] \times \overline{\Omega})$ *and*

$$\|u\|_{C^\alpha([0,T];C^0(\overline{\Omega}))} \leq \Lambda\Big(\|u_0\|_{C^{2\alpha}(\overline{\Omega})} + \|u_e\|_{C^{(1+2\alpha)/2,1+2\alpha}([0,T]\times\Gamma)}\Big). \tag{52}$$

Concerning the continuous dependence on the data for problem **(P2)**, we can prove

THEOREM **4** *Let the assumptions of theorem 3 hold and let u be the solution to problem* **(P2)** *corresponding to the boundary data u_e. Then the map $u_e \mapsto u$ is continuous from $C^0(\overline{\Gamma}_T)$ to $C^0(\overline{\Omega}_T)$; that is,*

$$\|u_2 - u_1\|_{C^0(\overline{\Omega}_T)} \leq \Lambda\|u_{e2} - u_{e1}\|_{C^0(\overline{\Gamma}_T)}. \tag{53}$$

Proofs of Theorems 3 and 4 can be found in [1]. Observe that, due to the nonlinearity of equation (6), an a priori bound for the solution u is needed. To obtain that, assumptions (9-10) and (49-50) play a crucial role.

5. – Main results on the control problems (CP1) and (CP2)

On account of the preliminary results on **(P1)** and **(P2)**, we are now in a position to state the main theorems related to the control problems formulated in the introduction. The reader is referred to [1] and [2] for the details of their proofs.

(A) The relay switch case. Let $\mathcal{W}$ be a relay switch operator and ρ_L and ρ_U the associated pair of lower and upper thresholds (see section 2). Further, assume

$$\mathcal{M}(u_0) \leq \rho_L(0) \tag{54}$$

$$\mathcal{W}(\mathcal{M}(u_0)) = +1 \tag{55}$$

and (cf. (12-15))

$$u_A,\ u_B \in W^{1,\infty}(0,T;C^{1+2\alpha}(\Gamma)) \tag{56}$$

$$u_C \in L^\infty(0,T). \tag{57}$$

Then we have the following existence and uniqueness results.

THEOREM **5** *Let (23-24), (39-44) and (54-57) hold. Moreover, assume*

$$Bu_0(y) = \varphi_0 u_A(0,y) + u_B(0,y) + \alpha^{-1} g(0,y) \qquad y \in \Gamma. \tag{58}$$

Then problem **(CP1)** *admits one and only one solution* $u \in C^{1+\alpha,2+2\alpha}([0,T] \times \overline{\Omega})$.

THEOREM **6** *Let (9-10), (23-24), (49-50) and (54-57) hold. Moreover, assume*

$$\left(\int_0^t e^{-\frac{(t-\tau)}{\epsilon}}[u_C(\tau) - 1]d\tau + \varphi_0 e^{-\frac{t}{\epsilon}}\right) u_A(t,y) + u_B(t,y) \geq 0 \quad (t,y) \in \overline{\Gamma}_T \tag{59}$$

$$Bu_0(y) = \varphi_0 u_A(0,y) + u_B(0,y) \qquad y \in \Gamma. \tag{60}$$

Then problem **(CP2)** *admits one and only one solution* $u \in C^{1+\alpha,2+2\alpha}([0,T] \times \overline{\Omega})$.

On account of the preliminary results of Section 4, the proofs of Theorems 5 and 6 can be achieved via an inductive argument introduced in [6] (see also [3]).

(B) The Preisach operator case. Let ξ and μ be the Borel measurable functions and the Borel measures on the Preisach plane $\mathcal{P}$ defining the Preisach operator $\mathcal{W}$ (see section 2). The first two results are concerned with the existence of a solution for problems **(CP1)** and **(CP2)**. More precisely, we have

THEOREM **7** *Let (33-34), (39-44), and (56-58) hold. Then problem* **(CP1)** *admits at least a solution.*

THEOREM **8** *Let (9-10), (33-34), (49-50), (56-57) and (60) hold. Moreover, assume*

$$(61)\quad \left(\int_0^t e^{-\frac{(t-\tau)}{\epsilon}}[u_C(\tau)-\mu(\mathcal{P})]d\tau+\varphi_0 e^{-\frac{t}{\epsilon}}\right)u_A(t,y)+u_B(t,y)\geq 0\ \ (t,y)\in\overline{\Gamma}_T.$$

Then problem **(CP2)** *admits at least a solution.*

The proofs of Theorems 7 and 8 follow from an application of the Schauder fixed point theorem which is made possible owing to Proposition 1 combined with the results of Section 4.

Stronger assumptions on the data allow us to obtain the uniqueness of the solution for both problems **(CP1)** and **(CP2)**; that is,

THEOREM **9** *Let the assumptions of Theorem 7 hold and assume μ satisfying (37). Then the solution to problem* **(CP1)** *is unique.*

THEOREM **10** *Let the assumptions of Theorem 8 hold and assume μ satisfying (37). Then the solution to problem* **(CP2)** *is unique.*

We recall that uniqueness bears strongly upon property (38) which is entailed by (37).

REMARK **1** As we mentioned in the introduction, it is quite easy to check that our results still hold when the measurement functional $\mathcal{M}$ has one of the following forms

$$(62)\qquad \mathcal{M}(u)(t):=\sum_{i=1}^{p}\omega_i u(t,x_i)\quad t\in[0,T]$$

or

$$(63)\qquad \mathcal{M}(u)(t):=\int_{\Omega_0}u(t,x)\omega_I(x)dx+\int_{\Gamma_0}z(u,y)\omega_S(y)d\Gamma\quad t\in[0,T].$$

Here, ω_i, $i=1,\ldots,p$, are positive constants and the points x_i, $i=1,\ldots,p$, belong to $\overline{\Omega}$. As far as (5.10) is concerned, ω_I and ω_S are suitable nonnegative summable weight functions, $\Omega_0\subset\Omega$, and $\Gamma_0\subset\Gamma$. It is worth observing that the action of real sensors can be more likely represented by functionals like (5.9) or (5.10).

REMARK **2** Referring, e.g., to **(CP1)**, an interesting inverse problem may arise when the convolution kernels h and/or k are supposed to be unknown. In order to solve this identification problem we need an additional information on the solution u like, for instance,

$$\int_{\Omega_0} \phi(x)u(t,x)dx = m(t) \qquad t \in [0,T], \qquad \Omega_0 \subset \Omega$$

where ϕ and m are given functions. It would be interesting to investigate this problem on the basis of the presented results and the related techniques.

REMARK **3** For the sake of simplicity, here we have considered a single reaction-diffusion equation (cf. **(P2)** and **(CP2)**). Nevertheless, analogous results hold for quite a large class of reaction-diffusion systems (see [1]).

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Asymptotic partition in the linear thermoelasticity backward in time

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1. – Introduction

The study of the equations of dynamical linear thermoelasticity backward in time was initiated by Ames and Payne [1] in order to obtain stabilizing criteria for solutions of the boundary-final value problem. It is well known that this type of problem is ill posed. We recall that the backward in time problems have been initially considered by Serrin [2] who established uniqueness results for the Navier-Stokes equations. Explicit uniqueness and stability criteria for classical Navier-Stokes equations backward in time have further established by Knops and Payne [3] and Galdi and Straughan [4] (see also Payne and Straughan [5] for a class of improperly posed problems for parabolic partial differential equations).

The boundary-final value problems associated with the linear theory of thermoelasticity have also been considered by Ciarletta [6] for establishing uniqueness and continuous dependence results upon mild requirements concerning the thermoelastic coefficients.

The spatial behaviour of the thermoelastic processes backward in time has been studied by Ciarletta and Chiriţă [7]. A time-weighted volume measure is used for establishing a first-order partial differential inequality which implies the spatial estimate describing the spatial exponential decay of the thermoelastic process backward in time.

In this paper we consider the boundary-final value problem associated with the linear theory of thermoelasticity. The final data are given at $t = 0$ and then we are interested in extrapolating to previous all times. We study the temporal behaviour of the thermoelastic processes backward in time. In this aim we introduce the Cesàro means of various parts of the total energy and then, by means of some auxiliary Lagrange-Brun identities [8,9], we establish the relations describing the asymptotic behaviour of the mean energies, provided some mild restrictions are imposed on the backward in time process.

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2. – Linear thermoelasticity backward in time

Throughout this paper we shall denote by B a bounded regular region of the physical space $\mathcal{E}^3$, whose boundary surface is ∂B. Identified $\mathcal{E}^3$ with the associated vector space, an orthonormal system of reference is introduced, so that vectors and tensors will have components denoted by Latin subscripts ranging over 1,2,3. Summation over repeated subscripts and other typical conventions for differential operations are implied such as a superposed dot or a comma followed by a subscript to denote partial derivative with respect to time or the corresponding cartesian coordinate.

Throughout this paper we suppose that B is filled by an anisotropic and inhomogeneous thermoelastic material. We consider the boundary–final value problems associated with the linear theory of thermoelasticity on the time interval $(-\infty, 0]$. Thus, in the absence of the supply terms, the fundamental system of field equations consists [10] of the strain–displacement relation

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \overline{\mathrm{B}} \times (-\infty, 0], \tag{1}$$

the thermal gradient–temperature relation

$$g_i = \theta_{,i} \quad \text{in } \overline{\mathrm{B}} \times (-\infty, 0], \tag{2}$$

the stress–strain-temperature relation

$$S_{ij} = C_{ijkl}e_{kl} + M_{ij}\theta \quad \text{in } \overline{\mathrm{B}} \times (-\infty, 0], \tag{3}$$

the heat conduction equation

$$q_i = -K_{ij}g_j \quad \text{in } \overline{\mathrm{B}} \times (-\infty, 0], \tag{4}$$

the equations of motion

$$S_{ji,j} = \rho\ddot{u}_i \quad \text{in } \mathrm{B} \times (-\infty, 0], \tag{5}$$

and the energy equation

$$-q_{i,i} + \theta_0 M_{ij}\dot{e}_{ij} = c\dot{\theta} \quad \text{in } \mathrm{B} \times (-\infty, 0]. \tag{6}$$

In the above relations we have used the following notations: u_i are the components of the displacement vector, θ is the temperature variation from the uniform reference temperature $\theta_0 > 0$, e_{ij} are the components of the strain tensor, g_i are the components of the thermal gradient vector, S_{ij} are the components of the stress tensor and q_i are the components of the heat flux vector. Further, ρ is the mass density, C_{ijkl} are the components of the elasticity tensor, M_{ij} are the components of the stress–temperature tensor, c is the specific heat and K_{ij} are the components of the conductivity tensor. In what follows we assume that the density ρ and the specific heat c are continuous functions of $\mathbf{x}$ on $\bar{B}$. We also assume that the elasticity tensor, the stress-temperature tensor and the conductivity tensor are continuous differentiable functions of $\mathbf{x}$ on $\bar{B}$ and they satisfy the symmetry relations

$$C_{ijkl} = C_{klij} = C_{jikl}, \tag{7}$$

$$M_{ij} = M_{ji}, \tag{8}$$

$$K_{ij} = K_{ji}. \tag{9}$$

In what follows we consider the boundary–final value problem $(\mathcal{P})$ defined by the relations (1) to (6), the final conditions

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \qquad \dot{u}_i(\mathbf{x}, 0) = \dot{u}_i^0(\mathbf{x}), \qquad \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \qquad \mathbf{x} \in \bar{B}, \tag{10}$$

and the homogeneous boundary conditions

$$\begin{aligned} &u_i(\mathbf{x}, t) = 0 \ \text{ on } \ \Sigma_1 \times (-\infty, 0], \qquad &&s_i(\mathbf{x}, t) = 0 \ \text{ on } \ \Sigma_2 \times (-\infty, 0], \\ &\theta(\mathbf{x}, t) = 0 \ \text{ on } \ \Sigma_3 \times (-\infty, 0], \qquad &&q(\mathbf{x}, t) = 0 \ \text{ on } \ \Sigma_4 \times (-\infty, 0], \end{aligned} \tag{11}$$

where u_i^0, $\dot{u}_i^0$ and θ^0 are prescribed functions and

$$s_i(\mathbf{x}, t) = S_{ji}(\mathbf{x}, t) n_j(\mathbf{x}), \qquad q(\mathbf{x}, t) = q_i(\mathbf{x}, t) n_i(\mathbf{x}), \tag{12}$$

n_i are the components of the outward unit normal vector to the boundary surface and Σ_1, Σ_2, Σ_3, Σ_4 are subsurfaces of ∂B so that $\Sigma_1 \cup \bar{\Sigma}_2 = \Sigma_3 \cup \bar{\Sigma}_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$.

3. – The transformed boundary-initial value problem. Some auxiliary identities

We use an appropriate change of variables and notations convenably chosen in order to transforme the boundary–final value problem $(\mathcal{P})$ into the boundary–initial value problem $(\mathcal{P}^*)$ defined by the following equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{13}$$

$$g_i = \theta_{,i}, \tag{14}$$

$$S_{ij} = C_{ijkl} e_{kl} + M_{ij} \theta, \tag{15}$$

$$q_i = -K_{ij} g_j, \tag{16}$$

in $\bar{B} \times [0, \infty)$,

$$S_{ji,j} = \rho \ddot{u}_i, \tag{17}$$

$$q_{i,i} + \theta_0 M_{ij} \dot{e}_{ij} = c\dot{\theta}, \tag{18}$$

in $B \times (0, \infty)$, with the initial conditions

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = \dot{u}_i^0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \quad \mathbf{x} \in \bar{B}, \tag{19}$$

and the boundary conditions

$$\begin{aligned} &u_i(\mathbf{x}, t) = 0 \ \text{ on } \ \Sigma_1 \times [0, \infty), \qquad &&s_i(\mathbf{x}, t) = 0 \ \text{ on } \ \Sigma_2 \times [0, \infty), \\ &\theta(\mathbf{x}, t) = 0 \ \text{ on } \ \Sigma_3 \times [0, \infty), \qquad &&q(\mathbf{x}, t) = 0 \ \text{ on } \ \Sigma_4 \times [0, \infty). \end{aligned} \tag{20}$$

By a solution of the boundary–initial value problem $(\mathcal{P}^*)$ we mean an ordered array $\pi = [u_i, e_{ij}, S_{ij}, \theta, g_i, q_i]$ with the following properties:

(i) $u_i, \dot{u}_i, \ddot{u}_i$, $(u_{i,j}+u_{j,i})$ and $(\dot{u}_{i,j}+\dot{u}_{j,i})$ are continuous on $\bar{B}\times[0,\infty)$;

(ii) e_{ij} is a continuous symmetric tensor field on $\bar{B}\times[0,\infty)$;

(iii) S_{ij} and $S_{ji,j}$ are continuous on $\bar{B}\times[0,\infty)$;

(iv) θ, $\theta_{,i}$, $\dot{\theta}$ are continuous on $\bar{B}\times[0,\infty)$;

(v) g_i are continuous on $\bar{B}\times[0,\infty)$;

(vi) q_i and $q_{i,i}$ are continuous on $\bar{B}\times[0,\infty)$,

and which meets the equations (13) to (20).

We proceed now to establish some auxiliary identities concerning the solutions of the boundary-initial value problem $(\mathcal{P}^*)$. These identities constitute the essential ingredients in our analysis concerning the temporal behaviour of the solutions of the boundary-initial value problem $(\mathcal{P}^*)$. .

LEMMA 1 *Let* $\pi = [u_i, e_{ij}, S_{ij}, \theta, g_i, q_i]$ *be a solution of the boundary-initial value problem* $(\mathcal{P}^*)$. *Then, for all* $t\in[0,\infty)$, *we have*

$$\begin{aligned}&\frac{1}{2}\int_B[\rho\dot{u}_i(t)\dot{u}_i(t)+C_{ijkl}e_{ij}(t)e_{kl}(t)+\frac{c}{\theta_0}\theta(t)^2]dv\\ &\qquad=\frac{1}{2}\int_B[\rho\dot{u}_i(0)\dot{u}_i(0)+C_{ijkl}e_{ij}(0)e_{kl}(0)+\frac{c}{\theta_0}\theta(0)^2]dv\\ &\qquad\quad+\textstyle\int_0^t\int_B\frac{1}{\theta_0}K_{ij}g_i(s)g_j(s)dvds.\end{aligned}\tag{21}$$

PROOF. The relations (7), (8), (13) and (17) imply that

$$\rho\dot{u}_i(s)\ddot{u}_i(s)=[S_{ji}(s)\dot{u}_i(s)]_{,j}-S_{ij}(s)\dot{e}_{ij}(s),\tag{22}$$

so that, by means of the relations (15) and (18), we get

$$\begin{aligned}&\frac{\partial}{\partial s}\Big\{\frac{1}{2}[\rho\dot{u}_i(s)\dot{u}_i(s)+C_{ijkl}e_{ij}(s)e_{kl}(s)+\frac{c}{\theta_0}\theta(s)^2]\Big\}\\ &\qquad=[S_{ji}(s)\dot{u}_i(s)+\frac{1}{\theta_0}q_j(s)\theta(s)]_{,j}-\frac{1}{\theta_0}q_j(s)g_j(s).\end{aligned}\tag{23}$$

Finally, we substitute the relation (16) into (23) and then integrate the result over $B\times[0,t]$. Thus, we get the identity (21) and the proof is complete. ∎

LEMMA 2 *Let* $\pi = [u_i, e_{ij}, S_{ij}, \theta, g_i, q_i]$ *be a solution of the boundary-initial value problem* $(\mathcal{P}^*)$. *Then, for all* $t\in[0,\infty)$, *we have*

$$\begin{aligned}&2\int_B\varrho u_i(t)\dot{u}_i(t)dv-\frac{1}{\theta_0}\int_B K_{ij}\int_0^t g_i(z)dz\int_0^t g_j(z)dzdv\\ &\quad=2\int_0^t\int_B\{\varrho\dot{u}_i(s)\dot{u}_i(s)-[C_{ijkl}e_{ij}(s)e_{kl}(s)++\frac{c}{\theta_0}\theta(s)^2]\}dvds\\ &\quad+2\int_B\varrho u_i(0)\dot{u}_i(0)dv-2\int_0^t\int_B\theta(s)[M_{ij}e_{ij}(0)-\frac{c}{\theta_0}\theta(0)]dvds.\end{aligned}\tag{24}$$

PROOF. We start with the following identity

$$\frac{\partial}{\partial s}[\varrho u_i(s)\dot{u}_i(s)] = \varrho\dot{u}_i(s)\dot{u}_i(s) + \varrho u_i(s)\ddot{u}_i(s), \tag{25}$$

so that, by an integration over $[0,t]$, we get

$$\varrho u_i(t)\dot{u}_i(t) = \varrho u_i(0)\dot{u}_i(0) + \int_0^t [\varrho\dot{u}_i(s)\dot{u}_i(s) + \varrho u_i(s)\ddot{u}_i(s)]ds. \tag{26}$$

In view of the relations (7), (8), (13) and (17), we get

$$\varrho u_i(s)\ddot{u}_i(s) = [S_{ji}(s)u_i(s)]_{,j} - S_{ij}(s)e_{ij}(s), \tag{27}$$

and therefore, by means of the relation (15), we obtain

$$\varrho u_i(s)\ddot{u}_i(s) = [S_{ji}(s)u_i(s)]_{,j} - C_{ijkl}e_{ij}(s)e_{kl}(s) - M_{ij}e_{ij}(s)\theta(s). \tag{28}$$

now, we integrate the equation (18) over $[0,t]$, to obtain

$$M_{ij}e_{ij}(t) = -\frac{1}{\theta_0}\int_0^t q_{i,i}(z)dz + \frac{c}{\theta_0}\theta(t) + \eta_0, \tag{29}$$

where

$$\eta_0 = M_{ij}e_{ij}(0) - \frac{c}{\theta_0}\theta(0). \tag{30}$$

By combining the relations (28) and (29), we get

$$\begin{aligned}\varrho u_i(s)\ddot{u}_i(s) &= [S_{ji}(s)u_i(s) + \frac{1}{\theta_0}\theta(s)\int_0^s q_j(z)dz]_{,j} - [C_{ijkl}e_{ij}(s)e_{kl}(s) + \frac{c}{\theta_0}\theta(s)^2] \\ &\quad -\eta_0\theta(s) + \frac{1}{\theta_0}K_{ij}g_i(s)\int_0^s g_j(z)dz,\end{aligned} \tag{31}$$

where an use was made by the relations (9) and (16).

Finally, we substitute the relation (31) into (26) and integrate the result over B and then we take into account the divergence theorem and the boundary conditions (20). Thus, we are led to the identity (24) and the proof is complete. ∎

LEMMA 3 *Let* $\pi = [u_i, e_{ij}, S_{ij}, \theta, g_i, q_i]$ *be a solution of the boundary-initial value problem* $(\mathcal{P}^*)$. *Then, for all* $t \in [0,\infty)$, *we have*

$$\begin{aligned}&2\int_B \varrho u_i(t)\dot{u}_i(t)dv - \frac{1}{\theta_0}\int_B K_{ij}\int_0^t g_i(z)dz\int_0^t g_j(z)dzdv \\ &= \int_B \varrho[u_i(2t)\dot{u}_i(0) + \dot{u}_i(2t)u_i(0)]dv + \int_0^t\int_B \eta_0[\theta(t+s) - \theta(t-s)]dvds.\end{aligned} \tag{32}$$

PROOF. Obviously, we have

$$\frac{\partial}{\partial s}\{\varrho[u_i(t+s)\dot{u}_i(t-s)+\dot{u}_i(t+s)u_i(t-s)]\} \tag{33}$$
$$=\varrho[u_i(t-s)\ddot{u}_i(t+s)-u_i(t+s)\ddot{u}_i(t-s)],$$

so that, by an integration over $[0,t]$ with respect to s, we get

$$2\varrho u_i(t)\dot{u}_i(t)=\varrho[u_i(2t)\dot{u}_i(0)+\dot{u}_i(2t)u_i(0)]$$
$$+\int_0^t \varrho[u_i(t+s)\ddot{u}_i(t-s)-u_i(t-s)\ddot{u}_i(t+s)]ds. \tag{34}$$

On the other hand, by using the relations (7), (8), (13) and (17), we obtain

$$\varrho[u_i(t+s)\ddot{u}_i(t-s)-u_i(t-s)\ddot{u}_i(t+s)]=[S_{ji}(t-s)u_i(t+s)-S_{ji}(t+s)u_i(t-s)]_{,j}$$
$$+[S_{ij}(t+s)e_{ij}(t-s)-S_{ij}(t-s)e_{ij}(t+s)]. \tag{35}$$

Further, we use the relations (7) and (15) to deduce

$$S_{ij}(t+s)e_{ij}(t-s)-S_{ij}(t-s)e_{ij}(t+s)$$
$$=\theta(t+s)M_{ij}e_{ij}(t-s)-\theta(t-s)M_{ij}e_{ij}(t+s), \tag{36}$$

and, by means of the relations (16) and (29), we obtain

$$S_{ij}(t+s)e_{ij}(t-s)-S_{ij}(t-s)e_{ij}(t+s)=\eta_0[\theta(t+s)-\theta(t-s)]$$
$$+\{\frac{1}{\theta_0}[\theta(t-s)\int_0^{t+s}q_i(z)dz--\theta(t+s)\int_0^{t-s}q_i(z)dz]\}_{,i} \tag{37}$$
$$+\frac{1}{\theta_0}[g_i(t-s)K_{ij}\int_0^{t+s}g_j(z)dz-g_i(t+s)K_{ij}\int_0^{t-s}g_j(z)dz].$$

Now, we substitute the relation (37) into (35) and the result into the relation (34). Then we integrate the result over B and use the divergence theorem and the boundary conditions (20). Thus, we are led to the identity (32) and the proof is complete. ∎

COROLLARY 1 *Let* $\pi=[u_i,e_{ij},S_{ij},\theta,g_i,q_i]$ *be a solution of the boundary-initial value problem* $(\mathcal{P}^*)$. *Then, for all* $t\in[0,\infty)$, *we have*

$$2\int_0^t\int_B\{\varrho\dot{u}_i(s)\dot{u}_i(s)-[C_{ijkl}e_{ij}(s)e_{kl}(s)+\frac{c}{\theta_0}\theta(s)^2]\}dvds$$
$$=-2\int_B\varrho u_i(0)\dot{u}_i(0)dv+\int_B\varrho[u_i(2t)\dot{u}_i(0)+\dot{u}_i(2t)u_i(0)]dv \tag{38}$$
$$+\int_0^t\int_B\eta_0[2\theta(s)+\theta(t+s)-\theta(t-s)]dvds.$$

PROOF. A combination of the relations (24) and (32) implies the identity (38) and the proof is complete. ∎

4. – Asymptotic partition

In this section we derive the relations which exhibit asymptotic partition of the energy provided only that the thermoelastic process is constrained to lie in a set $\mathcal{M}$ i.e., in the set of all thermoelastic processes $\pi = [u_i,\ e_{ij},\ S_{ij},\ \theta,\ g_i,\ q_i]$ defined on $B \times [0,\infty)$ which satisfy

$$\int_0^t \int_B \frac{1}{\theta_0} K_{ij} g_i(s) g_j(s) dv ds \le M^2, \qquad \forall t \in [0,\infty). \tag{39}$$

For later convenience we shall asume in this section that $meas\ \Sigma_3 \neq 0$.

Let $\pi = [u_i,\ e_{ij},\ S_{ij},\ \theta,\ g_i,\ q_i]$ be a solution for the boundary-initial value problem $(\mathcal{P}^*)$ and let associate with it the following Cesàro means

$$\mathcal{K}_C(t) \doteq \frac{1}{2t} \int_0^t \int_B \varrho \dot{u}_i(s) \dot{u}_i(s) dv ds, \tag{40}$$

$$\mathcal{S}_C(t) \doteq \frac{1}{2t} \int_0^t \int_B C_{ijkl} e_{ij}(s) e_{kl}(s) dv ds, \tag{41}$$

$$\mathcal{T}_C(t) \doteq \frac{1}{2t} \int_0^t \int_B \frac{c}{\theta_0} \theta(s)^2 dv ds, \tag{42}$$

$$\mathcal{D}_C(t) \doteq \frac{1}{t} \int_0^t \int_0^s \int_B \frac{1}{\theta_0} K_{ij} g_i(z) g_j(z) dv dz ds. \tag{43}$$

We observe that if $meas\ \Sigma_1 = 0$, then there exists a family of rigid motions and null temperature which satisfy the equations (13) to (18) and the boundary conditions (20). For this reason, we decompose the initial data u_i^0 and $\dot{u}_i^0$ as follows

$$u_i^0 = u_i^* + U_i^0\ , \qquad \dot{u}_i^0 = \dot{u}_i^* + \dot{U}_i^0, \tag{44}$$

where u_i^* and $\dot{u}_i^*$ are rigid displacements determined in such a way that

$$\int_B \varrho U_i^0 dv = 0\ , \qquad \int_B \varrho \varepsilon_{ijk} x_j U_k^0 dv = 0,$$

$$\int_B \varrho \dot{U}_i^0 dv = 0\ , \qquad \int_B \varrho \varepsilon_{ijk} x_j \dot{U}_k^0 dv = 0, \tag{45}$$

where ε_{ijk} represents the alternating symbol.

Let us introduce the following notations:

$$\hat{\mathbf{C}}^1(B) \doteq \Big\{ \mathbf{v} = (v_1, v_2, v_3),\ v_i \in C^1(\bar{B}) : \ v_i = 0 \text{ on } \Sigma_1 \\ \text{and if } meas\Sigma_1 = 0, \text{ then } \int_B \varrho v_i dv = 0\ ,\ \int_B \varrho \varepsilon_{ijk} x_j v_k dv = 0 \Big\};$$

$\hat{C}^1(B) \doteq \{\gamma \in C^1(B) : \ \gamma = 0 \ \text{ on } \ \Sigma_3 \}$;
$\hat{\mathbf{W}}_1(B) \doteq$ the completion of $\hat{\mathbf{C}}^1(B)$ by means of $\|\cdot\|_{\mathbf{W}_1(B)}$;
$\hat{W}_1(B) \doteq$ the completion of $\hat{C}^1(B)$ by means of $\|\cdot\|_{W_1(B)}$.

In the above relations $C^1(\bar{B})$ represents the set of scalar functions which are continuous and continuously differentiable on $\bar{B}$. Moreover, $W_m(B)$ represents the familiar Sobolev space [11] and $\mathbf{W}_m(B) \doteq [W_m(B)]^3$.

We note that because C_{ijkl} is a positive definite tensor it follows that the following inequality [12] holds

$$\frac{1}{4}\int_B C_{ijkl}(v_{i,j}+v_{j,i})(v_{k,l}+v_{l,k})dv \geq m_1 \int_B v_i v_i dv,\ \mathrm{m}_1 = \mathrm{const.} > 0,\ \forall \mathbf{v} \in \hat{\mathbf{W}}_1(\mathrm{B}). \tag{46}$$

Moreover, the boundary value condition (20) coupled with the fact that *meas* $\Sigma_3 \neq 0$ and the positive definiteness of the conductivity tensor, imply that the following Poincaré inequality holds

$$\int_B K_{ij}\gamma_{,i}\gamma_{,j}dv \geq m_2 \int_B \gamma^2 dv, \quad \mathrm{m}_2 = \mathrm{const.} > 0,\ \forall \gamma \in \hat{\mathrm{W}}_1(\mathrm{B}). \tag{47}$$

If *meas* $\Sigma_1 = 0$, then we shall find it is a convenient practice to decompose $\{u_i, \theta\}$ as follows

$$u_i = u_i^* + t\dot{u}_i^* + v_i \quad , \quad \theta = \gamma, \tag{48}$$

where $\{\mathbf{v}, \gamma\} \in \hat{\mathbf{W}}_1(B) \times \hat{W}_1(B)$ represents the solution of the boundary-initial value problem $(\mathcal{P}^*)$ in which the initial conditions (19) are substituted by

$$v_i(\mathbf{x},0) = U_i^0(\mathbf{x}), \quad \dot{v}_i(\mathbf{x},0) = \dot{U}_i^0(\mathbf{x}), \quad \gamma(\mathbf{x},0) = \theta^0(\mathbf{x}), \quad \mathbf{x} \in \bar{B}. \tag{49}$$

We further introduce the total energy associated with the solution´ $\pi = [u_i,\ e_{ij},\ S_{ij},\ \theta,\ g_i,\ q_i]$ by

$$\mathcal{E}(t) \doteq \frac{1}{2}\int_B [\rho\dot{u}_i(t)\dot{u}_i(t) + C_{ijkl}e_{ij}(t)e_{kl}(t) + \frac{c}{\theta_0}\theta(t)^2]dv. \tag{50}$$

We have now assembled all the preliminary material needed to derive the asymptotic partition in terms of the Cesàro means defined by the relations (40) to (43).

THEOREM 1 *Let* $\pi = [u_i, e_{ij}, S_{ij}, \theta, g_i, q_i]$ *be a solution of the boundary-initial value problem* $(\mathcal{P}^*)$ *residing in the class* $\mathcal{M}$ *defined by the relation (39). Then, for all choices of the initial data* $\mathbf{u}^0 \in \mathbf{W}_1(B)$, $\dot{\mathbf{u}}^0 \in \mathbf{W}_0(B)$ *and* $\theta^0 \in W_0(B)$, *we have*

$$\lim_{t\to\infty} \mathcal{T}_{\mathrm{C}}(\mathrm{t}) = 0. \tag{51}$$

Further, we have:

(i) *if* *meas* $\Sigma_1 \neq 0$, *then*

$$\lim_{t\to\infty} \mathcal{K}_{\mathrm{C}}(\mathrm{t}) = \lim_{\mathrm{t}\to\infty} \mathcal{S}_{\mathrm{C}}(\mathrm{t}), \tag{52}$$

$$\lim_{t\to\infty} \mathcal{S}_{\mathrm{C}}(\mathrm{t}) = \frac{1}{2}\,\mathcal{E}(0) + \frac{1}{2}\lim_{\mathrm{t}\to\infty} \mathcal{D}_{\mathrm{C}}(\mathrm{t}); \tag{53}$$

(ii) *if* *meas* $\Sigma_1 = 0$, *then*

$$\lim_{t\to\infty} \mathcal{K}_{\mathrm{C}}(\mathrm{t}) = \lim_{t\to\infty} \mathcal{S}_{\mathrm{C}}(\mathrm{t}) + \frac{1}{2}\int_{\mathrm{B}} \varrho\dot{\mathrm{u}}_{\mathrm{i}}^*\dot{\mathrm{u}}_{\mathrm{i}}^*\mathrm{dv}, \tag{54}$$

$$\lim_{t\to\infty} \mathcal{S}_{\mathrm{C}}(\mathrm{t}) = \frac{1}{2}\,\mathcal{E}(0) + \frac{1}{2}\int_{\mathrm{B}} \varrho\dot{\mathrm{u}}_{\mathrm{i}}^*\dot{\mathrm{u}}_{\mathrm{i}}^*\mathrm{dv} + \frac{1}{2}\lim_{\mathrm{t}\to\infty} \mathcal{D}_{\mathrm{C}}(\mathrm{t}). \tag{55}$$

PROOF. We first note that the Lemma 1 and the relation (50) give

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_B \frac{1}{\theta_0} K_{ij} g_i(s) g_j(s) dv ds, \quad \mathrm{t} \geq 0. \tag{56}$$

By taking into account the relations (40) to (43), from the relation (56) we deduce that

$$\mathcal{K}_C(t) + \mathcal{S}_C(t) + \mathcal{T}_C(t) = \mathcal{E}(0) + \mathcal{D}_C(t), \quad \text{for all } t > 0. \tag{57}$$

On the basis of the relations (39), (42) and (47) it results

$$\begin{aligned} \mathcal{T}_C(t) &\leq \tfrac{1}{2t}[\tfrac{1}{\theta_0} \max_{\bar{B}} \mathrm{c}(\mathbf{x})] \textstyle\int_0^t \int_B \theta(\mathrm{s})^2 \mathrm{dvds} \\ &\leq \tfrac{1}{2tm_2}[\max_{\bar{B}} \mathrm{c}(\mathbf{x})] \textstyle\int_0^t \int_B \tfrac{1}{\theta_0} \mathrm{K_{ij} g_i(s) g_j(s) dvds} \\ &\leq \tfrac{M^2}{2tm_2}[\max_{\bar{B}} \mathrm{c}(\mathbf{x})], \qquad t > 0, \end{aligned} \tag{58}$$

and hence by making t to tend to infinity, we get the relation (51). Thus, the relation (57) implies that

$$\lim_{t\to\infty} \mathcal{K}_\mathrm{C}(\mathrm{t}) + \lim_{t\to\infty} \mathcal{S}_\mathrm{C}(\mathrm{t}) = \mathcal{E}(0) + \lim_{t\to\infty} \mathcal{D}_\mathrm{C}(\mathrm{t}). \tag{59}$$

On the other hand, from the relations (38), (40), (41) and (42), we get

$$\begin{aligned} \mathcal{K}_C(t) - \mathcal{S}_C(t) - \mathcal{T}_C(t) = & -\frac{1}{2t} \int_B \varrho u_i(0) \dot{u}_i(0) dv \\ & + \frac{1}{4t} \int_0^t \int_B \eta_0 [2\theta(s) + \; + \theta(\mathrm{t}+\mathrm{s}) - \theta(\mathrm{t}-\mathrm{s})] \mathrm{dvds} \\ & + \frac{1}{4t} \int_B \varrho [u_i(2t) \dot{u}_i(0) + \dot{u}_i(2t) u_i(0)] dv, \quad t > 0. \end{aligned} \tag{60}$$

Further, the relations (39), (47) and (56), give

$$\int_B \varrho \dot{u}_i(s) \dot{u}_i(s) dv \leq 2\mathcal{E}(0) + 2M^2, \tag{61}$$

$$\int_B \theta(s)^2 dv \leq \frac{\theta_0}{\max_{\bar{B}} \mathrm{c}(\mathbf{x})} \int_B \frac{c}{\theta_0} \theta(s)^2 dv \leq \frac{2\theta_0}{\max_{\bar{B}} \mathrm{c}(\mathbf{x})} [\mathcal{E}(0) + M^2]. \tag{62}$$

Thus, by using the Schwarz's inequality and the relations (51), (61) and (62) into the relation (60), we obtain

$$\lim_{t\to\infty} \mathcal{K}_\mathrm{C}(\mathrm{t}) - \lim_{t\to\infty} \mathcal{S}_\mathrm{C}(\mathrm{t}) = \lim_{t\to\infty} \{\frac{1}{4\mathrm{t}} \int_\mathrm{B} \varrho \dot{\mathrm{u}}_\mathrm{i}(0) \mathrm{u}_\mathrm{i}(2\mathrm{t}) \mathrm{dv}\}. \tag{63}$$

Let us first consider the point (i). Since *meas* $\Sigma_1 \neq 0$ and $\mathbf{u} \in \hat{\mathbf{W}}_1(B)$, from (46), (50) and (56), we deduce that

$$\int_B u_i(s) u_i(s) dv \leq \frac{2}{m_1} \mathcal{E}(s) \leq \frac{2}{m_1} [\mathcal{E}(0) + M^2], \tag{64}$$

so that, by means of the Schwarz's inequality, we get

$$\lim_{t\to\infty}\{\frac{1}{4t}\int_B \varrho\dot{u}_i(0)u_i(2t)dv\} = 0. \tag{65}$$

Therefore, the relations (63) and (65) lead to the relation (52). The relation (53) results now from the relations (52) and (59).

Let us further consider the point (ii). Since $meas\ \Sigma_1 = 0$, then the decompositions (48) and the relation (45) give

$$\begin{aligned}&\frac{1}{4t}\int_B \varrho\dot{u}_i(0)u_i(2t)dv\\ &= \frac{1}{4t}\int_B \varrho\dot{u}_i^* u_i^* dv + \frac{1}{4t}\int_B \varrho(\dot{u}_i^* + \dot{U}_i^0)v_i(2t)dv + \frac{1}{2}\int_B \varrho\dot{u}_i^*\dot{u}_i^* dv.\end{aligned} \tag{66}$$

On the other hand, the relations (46), (50), (56) and (39) imply

$$\int_B v_i(s)v_i(s)dv \le \frac{2}{m_1}\mathcal{E}(s) \le \frac{2}{m_1}[\mathcal{E}(0) + M^2], \tag{67}$$

so that, the relation (66) leads to

$$\lim_{t\to\infty}\{\frac{1}{4t}\int_B \varrho\dot{u}_i(0)u_i(2t)dv\} = \frac{1}{2}\int_B \varrho\dot{u}_i^*\dot{u}_i^* dv. \tag{68}$$

Therefore, if we substitute the relation (68) into (63), then we obtain the relation (54). The relation (55) follows then by coupling the relations (54) and (59). Thus, the proof of theorem 1 is complete. ∎

5. – Concluding remarks

We first note that the procedure presented in the above Theorem 1 can be extended to cover the case when $meas\ \Sigma_3 = 0$, but a detailed analysis of various situations which can appear must be considered.

The relations describing the asymptotic partition of energy for the thermoelastic processes forward in time have been established in [13] without any kind of constraint restrictions upon the class of processes. The constraint restriction used to establish the theorem 1 is to be expected in view of the results on the uniqueness and continuous dependence results obtained by Ames and Payne [1] and Ciarletta [6]. On the other hand, in view of the identity (56), the constraint restriction (39) can be substituted by the following one

$$\mathcal{E}(t) \le M^2, \quad \forall t \ge 0,$$

where $\mathcal{E}(t)$ is defined by the relation (50).

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Internal parameters and superconductive phase in metals

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1. – Introduction

Superconductivity, i.e. the nondissipative current transport in metals, is a low temperature phenomenon, due to quantum effects which become apparent at the macroscopic scale. After the pioneering work of London brothers [1], its description was based on the celebrated Ginzburg-Landau theory [2], where special attention is paied to the transition from the normal to the superconducting phase. In modern nonequilibrium thermodynamics such a fascinating phenomenon motivated a wide literature, [3,4,5]. Let us quote the series of papers by Fabrizio and co-workers [6,7,8], where a nonlocal theory of superconductivity is developed by modifying the classical London's model. In the last decade several authors approached the problem in the framework of internal variable thermodynamics, by introducing a complex internal variable which models the phase effect of quantum mechanics [9,10].

In the present paper instead we introduce in the constitutive equations a real internal variable together its gradient in order to account for weakly nonlocal effects, both in space and time, which characterize a superconductive state. A similar point of view has been applied by Kosiński and Cimmelli in the description of the properties of the superfluid helium II, [11]. We develop a phenomenological model of a rigid electromagnetic solid which is able to describe the main features of the superconducting phase, i.e. the nondissipative current transport and the expulsion of the magnetic induction from the conductor (Meissner effect). According with London's approach [1], we split the total current density $\mathbf{J}$ into a sum of a normal current $\mathbf{J}_n$ and a supercurrent $\mathbf{J}_s$, namely:

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s. \tag{1}$$

Our main hypothesis consists in assuming the vector $\mathbf{J}_s$ proportional to the gradient of the internal variable. By design $\mathbf{J}_s$ is compatible with the system of Maxwell equations and, moreover, it does not dissipate energy inside the conductor.

In Section 2 we sketch the main properties of the superconducting systems in order to point out the experimental starting points of the theory.

In Section 3 the complete set of evolution equations for a rigid electromagnetic solid

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with an internal variable is derived. The local form of second law of thermodynamics is obtained as well.
In Section 4 we postulate a Fourier's type constitutive relation between the supercurrent vector $\mathbf{J}_s$ and the gradient of the internal variable and show that the basic properties of $\mathbf{J}_s$ are traduced by a suitable initial and boundary value problem whose mathematical structure is analyzed.
In Section 5 the compatibility of the form of $\mathbf{J}_s$ with both the system of Maxwell equations and the Clausius-Duhem inequality is investigated. Such a compatibility yields a set of thermodynamic restrictions on the response functions together with some additional constraints on the material functions characterizing the model. We close the paper by a final discussion and a comparison with a different theory proposed recently in the literature.

2. – Aspects of the superconducting phase

The superconducting phase was first observed in 1911 at the Kamerlingh Onnes Laboratory of the Univesity of Leiden. Onnes itself discovered that in some metals if the temperature is lowered until a given critical value θ_c, the electrical resistance suddenly drops to zero and an electrical current can flow in the absence of dissipation. That property allows some important applications, for instance the generation of magnetic fields without the need to remove the Joule heating due to the current creating the fields [13]. In 1933 Meissner and Ochsenfeld measured the magnetic induction $\mathbf{B}$ outside a superconductor as it is cooled in an applied field. They found that the strength of $\mathbf{B}$ immediately outside the superconductor increased while its normal component on the boundary appeared to be zero, indicating the vanishing of $\mathbf{B}$ inside the specimen and the existence of a perfectly diamagnetic state of the superconductor which caused the internal field to be expelled. The final state of the superconductor was found to be independent of whether it was cooled through θ_c and then placed in a field or viceversa, since all the flux was expelled from the superconductor in either case. The phenomenon is referred to as Meissner effect.
If the magnetic field $\mathbf{H}$ increases at a constant temperature below θ_c the metal remains superconductor until a given critical field $\mathbf{H}_c$ at which the normal behaviour is restored. The transition from the superconductive to the normal phase at constant temperature and magnetic field $\mathbf{H}_c$ represents a first order phase transition since the experiments show the existence of a nonvanishing latent heat [14].
The London brothers were the first to suppose the existence of superconductive electrons flowing without resistance. In the presence of an electric field $\mathbf{E}$ these electrons obey the equation of motion:

$$m\dot{\mathbf{v}}_s = e\mathbf{E}, \tag{2}$$

where m, e and $\mathbf{v}_s$ are the mass, the electric charge and the mean velocity of the electrons. Under the hypothesis:

$$\mathbf{J}_s = n_s e\mathbf{v}_s, \tag{3}$$

where n_s means the superelectron density, we get:

$$\frac{m}{n_s e^2}\dot{\mathbf{J}}_s = \mathbf{E}. \tag{4}$$

Finally, combination of (4) with the Maxwell equations leads to:

$$\nabla \times \left(\frac{m}{n_s e^2}\mathbf{J}_s\right) = -\mathbf{B}. \tag{5}$$

Relations (4) – (5) are referred to as London's equations, to be understood as additional conditions to the Maxwell system. It was observed that the complete Maxwell-London system is overdetermined, since it exhibits more equations than unknowns [6,7]. The constant $\lambda = \sqrt{m/n_s e^2 \mu}$, where μ is the magnetic permeability, characterizes the properties of the medium and represents the penetration depth of the magnetic field in the interior of the superconductor.

3. – A rigid superconductor with internal variable

Let us consider an isotropic body $\mathcal{B}$ occupying a compact and simply connected region $\mathcal{C}$ of an Euclidean point space E^3. A vector $\mathbf{x}$ of the associated vector space $\mathbf{E}^3$ will denote the position of the points of $\mathcal{C}$. We assume that, upon the action of an electrical field $\mathbf{E}$ and a magnetic field $\mathbf{H}$, a part of the total current circulating inside $\mathcal{B}$ may flow without resistance. Its flux will be represented by the vector $\mathbf{J}_s$ appearing in (1), [1].
A thermodynamic process of $\mathcal{B}$ is an almost regular curve in a state space Σ spanned by the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$, the absolute temperature θ, an internal state variable α and the gradients $\mathbf{g} = \nabla\theta$ and $\mathbf{a} = \nabla\alpha$, [15,16,17]. The internal variable is controlled by the nonlocal kinetic equation:

$$\dot{\alpha} = f(\theta, \alpha, \mathbf{a}, \mathbf{H}), \tag{6}$$

accounting for the property of the material of shifting to the superconductive state for suitable values of the temperature and of the magnetic field. The local balance of energy reads, [7,18]:

$$\rho\dot{e} + \nabla\cdot\mathbf{q} = \rho r + \mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{J}\cdot\mathbf{E}, \tag{7}$$

where ρ is the mass density, r the radiative heat supply, $\mathbf{q}$ the heat flux vector. Moreover, the fields:

$$\mathbf{D} = \epsilon\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}, \tag{8}$$

with the scalar functions ϵ and μ representing the electric permittivity and the magnetic permeability of the medium, are the electric dispacement and the magnetic induction. Equation (6) – (8) are supplemented by the system of Maxwell equations:

$$\nabla\cdot\mathbf{D} = \rho_e, \tag{9}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{10}$$

$$\dot{\mathbf{D}} = \nabla \times \mathbf{H} - \mathbf{J}, \tag{11}$$

$$\dot{\mathbf{B}} = \nabla \times \mathbf{E}, \tag{12}$$

where ρ_e is the density of the electric charge.
Finally, the absence of free superconductive charges inside $\mathcal{B}$ forces the vector $\mathbf{J}_s$ to satisfy the boundary value problem:

$$\nabla \cdot \mathbf{J}_s = 0 \quad \text{in} \quad \mathcal{C}, \tag{13}$$

$$\mathbf{J}_s \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{C}, \tag{14}$$

with $\mathbf{n}$ unitary normal vector defined on $\partial\mathcal{C}$. Since the system constituted by (1) and (6)-(12) is not closed, we need additional constitutive equations for the set of functions $\{e, r, \mu, \epsilon, \mathbf{q}, \mathbf{J}_s, \mathbf{J}_n\}$, which will be choosen in the form:

$$\Phi = \Phi^*(\theta, \alpha, \mathbf{g}, \mathbf{a}, \mathbf{E}, \mathbf{H}). \tag{15}$$

The dissipation principle [18,19], forces (15) to be compatible with the local form of second law of thermodynamics (Clausius-Duhem inequality), which in the present case reads [7]:

$$-\rho(\dot{\psi} + s\dot{\theta}) - \frac{1}{\theta}\mathbf{q} \cdot \mathbf{g} + \mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{J} \cdot \mathbf{E} \geq 0, \tag{16}$$

where s means the specific entropy and

$$\psi = e - \theta s \tag{17}$$

is the Helmholtz free energy. In order to investigate such a compatibility, a constitutive equation for ψ (or s) must be assigned as well.

4. – A constitutive equation for $\mathbf{J}_s$

In the present section we specialize the constitutive equation (14) for the total current density $\mathbf{J}$, i.e. for the vectors $\mathbf{J}_s$ and $\mathbf{J}_n$, and investigate its compatibility with the equations (6) and (13). Let us assume:

$$\mathbf{J}_n = \sigma(\theta, \alpha, \mathbf{g}, \mathbf{a}, \mathbf{H})\mathbf{E}, \tag{18}$$

$$\mathbf{J}_s = j(\theta, \alpha, \mathbf{g}, \mathbf{E}, \mathbf{H})\mathbf{a}. \tag{19}$$

Equation (18) represents the well known Ohm's law, with σ the electrical resistance of the normal state [5]. Equation (19) is new and relates the superconductive current to the gradient of the internal variable through a linear law. Due to (19), the boundary value problem (13)-(14) specializes to:

$$\nabla j \cdot \mathbf{a} + j \nabla \cdot \mathbf{a} = 0 \quad \text{in} \quad \mathcal{C}, \tag{20}$$

$$\mathbf{a} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{C}. \tag{21}$$

Equation (21) may be used as boundary condition for (2), together with an initial condition:

$$\alpha(t_0) = \alpha_0. \tag{22}$$

Finally, since $\mathbf{J}_s$ is an observable quantity, a boundary condition for j may be deduced by the measurements of $\mathbf{J}_s$ on the boundary of the superconductor. For instance, for a wide class of superconductors, it results, [4]:

$$\mathbf{J}_s \mid_{\partial C} = \frac{1}{\lambda} \mathbf{n} \times \mathbf{H} \mid_{\partial C} . \tag{23}$$

For such systems we get the boundary condition:

$$j \mid_{\partial C} = \frac{1}{\lambda a^2} (\mathbf{n} \times \mathbf{H}) \cdot \mathbf{a} \mid_{\partial C} . \tag{24}$$

5. – Thermodynamic and electromagnetic compatibility

The compatibility of (15) with (16) may be investigated by exploiting the classical procedures of thermodynamics of irreversible processes, [12,20,21]. It can be proved that (15) is compatible with second law of thermodynamics if the following restrictions hold true:

$$s = -\frac{\partial \mathcal{F}}{\partial \theta}, \tag{25}$$

$$\mathbf{q} = \frac{\rho\theta}{\tau} \frac{\partial \mathcal{F}}{\partial \mathbf{a}}, \tag{26}$$

$$\mathbf{B} = -\rho \frac{\partial \mathcal{F}}{\partial \mathbf{H}}, \tag{27}$$

$$\mathbf{D} = -\rho \frac{\partial \mathcal{F}}{\partial \mathbf{E}}, \tag{28}$$

$$\frac{\partial \mathcal{F}}{\partial \mathbf{g}} = \mathbf{0}, \tag{29}$$

$$\mathbf{q} \cdot \frac{\partial f}{\partial \mathbf{a}} = 0, \tag{30}$$

$$\mathbf{q} \cdot \frac{\partial f}{\partial \mathbf{H}} = 0, \tag{31}$$

$$\rho \frac{\partial \mathcal{F}}{\partial \alpha} \dot{\alpha} + \left(\frac{\tau}{\theta \chi} \mathbf{q} + j \mathbf{E} \right) \cdot \mathbf{a} + \sigma E^2 \geq 0, \tag{32}$$

with:

$$\mathcal{F} =: \psi + \frac{\mu}{\rho} H^2 + \frac{\epsilon}{\rho} E^2, \tag{33}$$

and:

$$\frac{1}{\tau} =: \frac{\partial f}{\partial \theta} \quad , \quad \frac{1}{\chi} =: \frac{\partial f}{\partial \alpha}. \tag{34}$$

The left hand side of (32) represents the local entropy production $\eta_{(s)}$ in a thermodynamic process. It is in the classical form of a bilinear product of generalized forces (affinities) times fluxes or rates, introduces by Onsager in nonequilibrium thermodynamics, [12,15]. Thence we may regard the quantity $\rho\frac{\partial\mathcal{F}}{\partial\alpha}$ as the affinity conjugated to the rate of the internal variable $\dot{\alpha}$, the quantity $\frac{\tau}{\theta\chi}\mathbf{q}+j\mathbf{E}$ as the generalized force conjugated to the flux of internal variable $\mathbf{a}$ and the vector $\mathbf{E}$ as the force conjugated to the flux of normal current $\mathbf{J}_n = \sigma\mathbf{E}$. With respect to the classical case $\eta_{(s)}$ contains two additional terms due to the internal variable. Under the further constitutive assumptions:

$$\frac{\partial\mathcal{F}}{\partial\alpha} = 0, \tag{35}$$

$$\mathbf{q} = -\frac{j\theta\chi}{\tau}\mathbf{E}, \tag{36}$$

$\eta_{(s)}$ reduces to σE^2, i.e. to the classical entropy production for standard electromagnetic systems. In such a case neither the internal variable nor its gradient dissipate energy and hence the supercurrent flows without entropy production. Let us observe that since below the critical temperature function σ identically vanishes (see Sec.2), in such an interval the system is completely nondissipative.
Furthermore, as a consequence of (27), if:

$$\frac{\partial\mathcal{F}}{\partial\mathbf{H}} = 0, \tag{37}$$

then a perfectly diamagnetic state is present inside the superconductor (Meissner effect). Hence we are allowed to consider equations (35)-(37) as characterizing the superconductive phase.
Finally, we investigate the compatibility of the constitutive equation (19) with the Maxwell equation (11). We regard that compatibility as a constraint on f, which must be assigned in such a way that the initial and boundary value problems for α and j yield a vector $\mathbf{J}_s$ which is compatible with (11). Since f is scalar valued, it depends on $\mathbf{a}$ and $\mathbf{H}$ through the invariant quantities a^2, H^2 and $I =: \mathbf{a}\cdot\mathbf{H}$. Thence (19) reads:

$$\mathbf{J}_s = j\chi\dot{\mathbf{a}} - \frac{j\chi}{\tau}\mathbf{g} - \frac{j\chi}{\nu}\nabla a^2 - \frac{j\chi}{\gamma}\nabla H^2 - \frac{j\chi}{\delta}\nabla I, \tag{38}$$

with:

$$\frac{1}{\nu} =: \frac{\partial f}{\partial a^2} \quad , \quad \frac{1}{\gamma} =: \frac{\partial f}{\partial H^2} \quad , \quad \frac{1}{\delta} =: \frac{\partial f}{\partial I}. \tag{39}$$

On the other hand, substitution of (38) in (11) yields:

$$j\chi\dot{\mathbf{a}} - \frac{j\chi}{\tau}\mathbf{g} - \frac{j\chi}{\nu}\nabla a^2 - \frac{j\chi}{\gamma}\nabla H^2 - \frac{j\chi}{\delta}\nabla I = \sigma\mathbf{E} + \dot{\mathbf{D}} - \nabla\times\mathbf{H}. \tag{40}$$

Equation (40), together with the thermodynamic restrictions (30) and (31), represent five additional scalar restrictions on the five material functions $\tau, \chi, \gamma, \delta$ and ν, to be understood as compatibility conditions, allowing $\mathbf{J}_s$ to satisfy (11).

6. – Final remarks

In that paper the superconductive state of metals has been related to an intrinsic state variable, controlled by a nonlocal kinetic equation which involves the magnetic field and the absolute temperature as well. In fact, the nondissipative current flow is regarded as an intrinsic property which manifests itself only in a suitable interval of temperature and magnetic field. The model is completely compatible with both the Maxwell equations and second law of thermodynamics. It also allows the existence of a nondissipative perfectly diamagnetic state, which is characteristic of the superconduction. The main difference between the present model and other theories with internal variables consists in the mathematical nature of that variable, which here is assumed to take only real values. Indeed different authors introduce a complex internal variable in order to account for quantum effects at a macroscopic level and to obtain, as a particular case, the Ginzburg-Landau theory, [9,10]. This fact renders difficult to compare these theories and the present one. However a comparison can be performed with a macroscopic model proposed recently by Fabrizio and co-workers throught a slight modification of the classical London's theory, [6,7,8]. In that approach an additional evolution equation for $\mathbf{J}_s$ is postulated in the form:

$$\nabla \times (\Lambda_1 \mathbf{J}_s) + \Lambda_2 \dot{\mathbf{J}}_s = -\mu \mathbf{H}, \tag{41}$$

where Λ_1 and Λ_2 are suitable material functions. On the other hand, (19) is compatible with (41) if and only if the material parameters Λ_1, Λ_2 and μ are choosen in such a way that the following evolution equation holds true:

$$\nabla(\Lambda_1 j) \times \mathbf{a} + \Lambda_2 \frac{d(j\mathbf{a})}{dt} = -\frac{\partial \mathcal{F}}{\partial \mathbf{H}}. \tag{42}$$

If the three scalar differential constraints (42) are fulfilled, then both models coincide.
Finally, let us notice that, as it is typical for low temperature systems, the model should allow the propagation of thermomechanical waves together with the electromagnetic ones. The present subject is currently under investigation.

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Phase relaxation problems with memory and their optimal control

Pierluigi Colli*

1. – Introduction

I would like to introduce this note with some personal recollections of Giorgio Gentili. The first time I met Giorgio was on occasion of a meeting held just in Cortona in 1996. We started a collaboration which eventually led us to our joint paper [6]. This paper was finalized two years later and I myself was the main responsible for this delay to the point that Giorgio was joking on me comparing the paper to the Penelope's web! Giorgio was kidding, of course; however, this episode showed me how he was determined on his plans. I remember him as a serious and very active researcher, full of enthusiasm for his work, keen on Thermodynamics, but also practical and constructive in building up his fine pieces of research and life.

This paper is naturally dedicated to him as well as phase relaxation systems were just among the topics treated in our joint article [6] with Claudio Giorgi. By phase relaxation model we mean a model for solid–liquid phase transition which describes the evolution of a body in terms of two state variables, the (absolute) temperature ϑ and the phase parameter χ, via the basic energy balance equation and a time relaxation dynamics for the phase variable χ. About the physical meaning of χ, we point out that it usually represents the local proportion of one of the two phases. Hence, we can agree for instance that $\chi = 1$ in the liquid phase, in which we expect the temperature ϑ to be greater than a reference temperature ϑ_c for the phase transition, and $\chi = 0$ in the solid phase where ϑ should lie below the threshold ϑ_c. We can also postulate the existence of some intermediate range, characterized by values $0 < \chi < 1$ and temperatures ϑ staying around ϑ_c: for such a situation we speak of "mushy region" and in these cases we see some transition regions - where water and ice, say, are well mixed - instead of narrow interfaces separating the two phases. Let us quote at once some significant references for phase relaxation systems: in particular, we aim to refer to [11, 16, 17] and, for an approach which takes into account memory effects as well, to [7, 9, 3].

In a very simple setting, the partial differential equations describing the phase relaxation model with memory are

$$\partial_t(\vartheta + \chi) - \Delta(\vartheta + k * \vartheta) = f_0 \quad \text{in} \quad Q := \Omega \times (0, T), \tag{1}$$

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$$\mu \partial_t \chi + \partial I_{[0,1]}(\chi) \ni \vartheta - \vartheta_c \quad \text{in } Q, \tag{2}$$

where Ω denotes a bounded domain in $\mathbb{R}^3$ and T stands for some final time. Moreover, ∂_t indicates the time derivative and the Laplace operator Δ acts on the space variable $x \in \Omega$. The memory kernel k is a prescribed function of time, and the convolution in (1) is defined by

$$(k * \vartheta)(x,t) := \int_0^t k(t-s)\,\vartheta(x,s)\,ds, \quad t \in [0,T].$$

In the light of the heat conduction theory for materials with memory (cf., e.g., [4, 5, 13, 14] and references therein), one derives equation (1) from the energy balance

$$\partial_t e + \nabla \cdot \mathbf{q} = \tilde{f}_0 \quad \text{in } Q \tag{3}$$

where $\tilde{f}_0$ is a known heat supply, when using the linear Coleman–Gurtin law for the heat flux

$$\mathbf{q}(x,t) = -k_0 \nabla\vartheta(x,t) - \int_{-\infty}^t k(t-s)\,\nabla\vartheta(x,s)\,ds. \tag{4}$$

As usual, the internal energy e is given by

$$e = \Psi + \vartheta s$$

where $s = -\partial\Psi/\partial\vartheta$ represents the specific entropy and Ψ is the free energy. According to [11], for the free energy we assume the following dependence on ϑ and χ,

$$\Psi(\vartheta,\chi) = -\vartheta\log\vartheta + \vartheta I_{[0,1]}(\chi) - \frac{L}{\vartheta_c}(\vartheta - \vartheta_c)\chi, \tag{5}$$

thus allowing for a nonsmooth contribution of the indicator function $I_{[0,1]}(\chi)$ ($= 0$ whenever $0 \le \chi \le 1$, $= +\infty$ otherwise). Note that the free energy in (5) turns out the sum of three contributions: a purely caloric term $-\vartheta\log\vartheta$, the indicator function term with a positive factor ϑ (which is irrelevant for the values of the resulting function but plays a role in the computation of e), and a coupling term where also the latent heat L appears. Now, since $e = \vartheta + L\chi$, (1) eventually follows from a normalization of parameters L and k_0 to 1, and from the information on the past history of ϑ up to $t = 0$, so that the integral in (4) can be split into a convolution and a known contribution on the right hand side. Let us observe that in (1) the choice $k = 0$ is allowed: in this case, memory disappears and the Coleman–Gurtin law reduces to the classical Fourier law. Likewise, we wish to point out that laws like (4) and conditions on kernel k are discussed in the paper [12] in connection with the restrictions imposed by the Second Principle of Thermodynamics, and it turns out that for suitable k the resulting heat flux obeys an integral version of the Clausius–Duhem inequality.

In order to show the meaning of (2), following a procedure that is inspired by the general approach in [10], we introduce a pseudopotential of dissipation which may depend on several variables and here it depends at least on the derivative χ_t,

$$\Phi(\chi_t, \ldots) = \frac{\mu}{2}|\chi_t|^2,$$

and infer (2) from the law

$$\frac{\partial \Phi}{\partial \chi_t} + \frac{\partial \Psi}{\partial \chi} \ni 0. \tag{6}$$

Indeed, taking $L/\vartheta_c = 1$ for simplicity and making use of the subdifferential of the indicator function

$$\xi \in \partial I_{[0,1]}(\chi) \quad \Longleftrightarrow \quad \xi \begin{cases} \leq 0 & \text{if } \chi = 0 \\ = 0 & \text{if } 0 < \chi < 1 \\ \geq 0 & \text{if } \chi = 1 \end{cases},$$

we reduce to (2), which in particular says that $\mu \chi_t = \vartheta - \vartheta_c$ whenever $0 < \chi < 1$, otherwise $\chi_t = 0$. Thus, the dynamics of χ is clearly determined from the evolution of ϑ and it may happen that $\chi \in (0,1)$ in correspondence with values of ϑ that differ from the critical temperature ϑ_c. Let us also remark that the limiting case $\mu = 0$ of (2) leads to the equilibrium condition of the Stefan problem

$$\partial I_{[0,1]}(\chi) \ni \vartheta - \vartheta_c \quad \text{or equivalently} \quad \chi \in H(\vartheta - \vartheta_c),$$

where H denotes the Heaviside graph. In this case, a comparison in (6) with the help of (5) shows that we are actually looking for the minima $\chi_{min} \in [0,1]$ of the free energy, and $\chi_{min} = 0$ (resp. 1) if $\vartheta < \vartheta_c$ (resp. $> \vartheta_c$) while all values $\chi_{min} \in [0,1]$ are equally minima of the free energy at the equilibrium temperature $\vartheta = \vartheta_c$.

Consider now an initial and boundary value problem for the system (1–2) by introducing the initial conditions

$$(\vartheta + \chi)(0) = e_0 \quad \text{and} \quad \chi(0) = \chi_0 \quad \text{in } \Omega \tag{7}$$

and the boundary condition

$$\partial_\nu(\vartheta + k * \vartheta) + \alpha(\vartheta - u) = 0 \quad \text{on } \Sigma := \partial\Omega \times (0,T). \tag{8}$$

Here, the symbol ∂_ν denotes the outward normal derivative on $\partial\Omega$ and α is a positive coefficient. In condition (7), for space-time functions v the notation $v(0)$ is used in place of $v(\cdot, 0)$, and e_0 and χ_0 are prescribed initial data (with $0 \leq \chi_0 \leq 1$ in Ω). For a given $u \in L^2(\Sigma)$, existence and uniqueness of a pair (ϑ, χ) solving a suitable weak formulation of (1–2), (7–8) are well known [16, 7, 3]. Our interest now focuses on the investigation of the optimal control problem for (1–2), (7–8) (the reader who is interested in optimal control problems for PDE's can refer, e.g., to the books [2, 15]).

Note that (8) states that the heat flux on the boundary is proportional to the difference of internal temperature and some external value which will play as a control in our presentation here. Indeed, the aim of this note is that of reviewing some recent results obtained in collaboration with V. Barbu, M.L. Bernardi, G. Gilardi in [1] regarding the minimization problem for the functional

$$J(\vartheta, u) := \int_Q g(\vartheta) + \int_\Sigma h(u) \tag{9}$$

over all controls $u \in L^2(\Sigma)$ and the corresponding values of the state variables ϑ and χ, where g and h are given convex functions on $\mathbb{R}$ satisfying some growth conditions at infinity. We discuss existence of minimizers, asymptotic convergence of minimizers as the parameter μ tends to zero, and some necessary conditions for a given u^* to be a minimizer. Moreover, in the final part of the paper we address a few open questions concerning possible extensions or improvements of mentioned results to the more general phase relaxation systems examined in [6].

2. – The optimal control problem

We first introduce a variational formulation of problem (1–2), (7–8). Let $V = H^1(\Omega)$ and $H = L^2(\Omega) \cong H'$, so that $V \subset H \subset V'$ with dense and compact embeddings. Assume that $\mu > 0$ and

$$f_0 \in L^2(0,T;V'), \quad e_0, \chi_0 \in H, \quad 0 \le \chi_0 \le 1 \text{ a.e. in } \Omega, \tag{10}$$

$$k \in W^{1,1}(0,T), \quad \alpha \in L^\infty(\partial\Omega) \text{ non negative.} \tag{11}$$

Testing formally (1) by an arbitrary $v \in V$, with the help of (8) we obtain

$$(\partial_t(\vartheta+\chi)(t), v)_\Omega + (\nabla(\vartheta + k*\vartheta)(t), \nabla v)_\Omega + \int_\Gamma \alpha\,\vartheta(t)\,v = \langle f_0(t), v\rangle + \int_\Gamma \alpha\,u(t)\,v$$

for a.a. $t \in (0,T)$, where $(\cdot\,,\cdot)_\Omega$ stands for the inner product both in $H = L^2(\Omega)$ and in $(L^2(\Omega))^3$, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V' and V. Collecting then (2), (7) and defining the abstract operators $A, B : V \to V'$ and $B_0 : L^2(\Sigma) \to V'$ by

$$\langle Aw, v\rangle := (\nabla w, \nabla v)_\Omega, \quad \langle Bw, v\rangle := \int_\Gamma \alpha w v,$$
$$\langle B_0 z, v\rangle = \int_\Gamma \alpha z v \qquad \forall\, w,v \in V, \quad \forall\, z \in L^2(\Sigma),$$

to any fixed $u \in L^2(\Sigma)$ we associate a unique pair

$$\vartheta \in L^2(0,T;V) \cap H^1(0,T;V'), \quad \chi \in H^1(0,T;H) \tag{12}$$

such that

$$\partial_t(\vartheta+\chi) + A(\vartheta + k*\vartheta) + B\vartheta = f_0 + B_0 u \quad \text{in } V', \text{ a.e. in } (0,T), \tag{13}$$

$$\mu\partial_t\chi + \partial I_{[0,1]}(\chi) \ni \vartheta - \vartheta_c \quad \text{a.e. in } Q, \tag{14}$$

$$(\vartheta+\chi)(0) = e_0 \quad \text{and} \quad \chi(0) = \chi_0. \tag{15}$$

This mapping is well defined as it is shown in [16, 7, 3]) for analogous problems with different boundary conditions.

Therefore, we let u vary and consider the cost functional

$$J(\vartheta, u) = \int_Q g(\vartheta) + \int_\Sigma h(u)$$

with $g, h : \mathbb{R} \to (-\infty, +\infty]$ proper, convex, lower semicontinuous and satisfying

g has *at most* quadratic growth, $\quad h$ has *at least* quadratic growth, so that $g(\vartheta)$ is in $L^1(Q)$ and $h(r) \geq \omega|r|^2 - c$ for some positive constants ω and c, and for all $r \in \mathbb{R}$.

At this point, we can formulate the

OPTIMAL CONTROL PROBLEM *Minimize $J(\vartheta, u)$ among all possible triplets (ϑ, χ, u) satisfying (13–15).*

Here are our results, complemented with a brief sketch of proofs (see [1] for full details).

THEOREM 1 **[existence]** *There exists at least one solution to the optimal control problem, i.e., a triplet $(\vartheta^*, \chi^*, u^*)$ such that*

$$J(\vartheta^*, u^*) \leq J(\vartheta, u) \qquad \forall\, (\vartheta, \chi, u),$$

where $(\vartheta^*, \chi^*, u^*)$ and (ϑ, χ, u) solve (13–15).

PROOF. Argue by minimizing sequences. ∎

THEOREM 2 $[\mu \searrow 0]$ *For any $\mu > 0$ let $(\vartheta^*_\mu, \chi^*_\mu, u^*_\mu)$ be any solution to the optimal control problem. Then there exists a weakly convergent subsequence of $\{u^*_\mu\}$ in $L^2(\Sigma)$ as $\mu \searrow 0$. Moreover, if $u^*_\mu \rightharpoonup \overline{u}$ in $L^2(\Sigma)$ as $\mu \searrow 0$, then $\overline{u}$ is an optimal control for the limiting Stefan problem, characterized by the condition (which replaces (14))*

$$\chi \in H(\vartheta - \vartheta_c) \quad \text{a.e. in } Q.$$

PROOF. One checks that the weak limits $\overline{\vartheta}$ of $\{\vartheta^*_\mu\}$ and $\overline{\chi}$ of $\{\chi^*_\mu\}$ yield the solution to the Stefan problem with $\overline{u}$ as datum. Then, the minimum property follows from a strong convergence argument for the functions ϑ_μ, specified by the triplet $(\vartheta_\mu, \chi_\mu, \overline{u})$ that solves (13–15), and for the related integrals $\int_\Omega g(\vartheta_\mu)$. ∎

THEOREM 3 **[optimality conditions]** *In the case of $\mu > 0$ fixed, let $(\vartheta^*, \chi^*, u^*)$ be a solution to the optimal control problem. Then, there exists a pair (p, q) with*

$$p \in C^0([0,T]; V) \cap H^1(0,T;H), \quad q \in L^\infty(0,T;H), \quad \partial_t q \in \mathcal{M}(Q) \tag{16}$$

such that

$$\partial_t p - A(p + k \star p) + \frac{1}{\mu}(q - p) - Bp \in \partial g(\vartheta^*) \quad \text{a.e. in } Q, \tag{17}$$

$$\alpha p \in \partial h(u^*) \quad \text{a.e. on } \Sigma, \tag{18}$$

$$p(T) = 0, \tag{19}$$

where $\star$ denotes the "backward" convolution

$$(k \star p)(t) := \int_t^T k(s - t)\, p(s)\, ds, \quad t \in [0, T],$$

$\mathcal{M}(Q)$ stands for the space of Radon measures on Q, and ∂g, ∂h represent the subdifferentials of the convex functions g, h in $\mathbb{R} \times \mathbb{R}$.

PROOF. First, some comments are due on the adjoint problem (17–19). The inclusion in (17) means that $\partial_t p + \Delta(p + k \star p) + \mu^{-1}(q - p)$ is in $L^2(Q)$ and belongs to $\partial g(\vartheta^*)$ pointwise in the interior of the domain, besides the boundary condition $\partial_\nu(p + k \star p) + \alpha p = 0$ on Σ. Note that (18) adds another condition for the trace of function p on the boundary. In some sense, we should admit that the optimality conditions for the adjoint pair (p, q) are not so satisfactory, since the only information we have on the solution component q is the regularity property stated in (16).

We outline now the proof of Theorem 3. Approximate J by

$$J_\varepsilon(\vartheta, u) = \int_Q g_\varepsilon(\vartheta) + \int_\Sigma h(u) + \frac{1}{2}\int_\Sigma |u - u^*|^2$$

and $\partial I_{[0,1]}$ by a suitable $F_\varepsilon \in C^{1,1}(\mathbb{R})$, where $\varepsilon > 0$ is the approximation parameter, g_ε denotes the Yosida regularization of g, and u^* is fixed as in the statement. Then one solves the $\varepsilon-$ optimal control problem and gets one solution $(\vartheta_\varepsilon^*, \chi_\varepsilon^*, u_\varepsilon^*)$, that is, a triplet $(\vartheta_\varepsilon^*, \chi_\varepsilon^*, u_\varepsilon^*)$ which fulfils $J_\varepsilon(\vartheta_\varepsilon^*, u_\varepsilon^*) \le J_\varepsilon(\vartheta, u)$ for all triplets (ϑ, χ, u) satisfying the $\varepsilon-$system corresponding to (13–15). Then the main step consists in proving existence and uniqueness of a pair $(p_\varepsilon, q_\varepsilon)$ satisfying

$$\begin{gathered} p_\varepsilon \in C^0([0,T]; V) \cap H^1(0,T;H) \quad \text{and} \quad q_\varepsilon \in H^1(0,T;H), \\ \partial_t p_\varepsilon - A(p_\varepsilon + k \star p_\varepsilon) + \frac{1}{\mu}(q_\varepsilon - p_\varepsilon) - B p_\varepsilon = g_\varepsilon'(\vartheta_\varepsilon^*) \quad \text{in } V', \text{ a.e. in } (0,T), \\ \alpha p_\varepsilon + u^* - u_\varepsilon^* \in \partial h(u_\varepsilon^*) \quad \text{a.e. on } \Sigma, \\ p_\varepsilon(T) = q_\varepsilon(T) = 0, \end{gathered}$$

and

$$\partial_t q_\varepsilon + \frac{1}{\mu} F_\varepsilon'(\chi_\varepsilon^*)\,(p_\varepsilon - q_\varepsilon) = 0 \quad \text{a.e. in } Q. \tag{20}$$

Hence, one takes the limit as $\varepsilon \searrow 0$ getting $u_\varepsilon^* \to u^*$ in $L^2(\Sigma)$ and checking that the limits p and q of some subsequences of $\{p_\varepsilon\}$ and $\{q_\varepsilon\}$ in a suitable topology satisfy (16–19). As a matter of fact, we do not pass to the limit in (20), since we can just prove that $\mu^{-1} F_\varepsilon'(\chi_\varepsilon^*)\,(p_\varepsilon - q_\varepsilon)$ tends to some ζ weakly star in $\mathcal{M}(Q)$, but we cannot characterize ζ in terms of p and q. ∎

3. – Extensions and remarks

The aim of this section is to take up the general phase relaxation systems considered in [6] and comment on possible generalizations of the results presented in Section 2.

Thermodynamically consistent models scrutinized in [6] are essentially concerned with the following form of the free energy

$$\Psi(\vartheta, \chi) = -\vartheta F_0(\vartheta) + \vartheta G(\chi) - (\vartheta - \vartheta_c) F_1(\chi) \tag{21}$$

which yields an extension of (5) since $F_0(\vartheta)$ replaces $\log \vartheta$, $G(\chi)$ takes the place of the indicator function of the interval $[0,1]$, and $F_1(\chi)$ can be possibly nonlinear.

This setting corresponds to an internal energy $e = \vartheta^2 F_0'(\vartheta) + \vartheta_c F_1'(\chi)$ which is no longer linear with respect to the state variables. Here, F_0 and F_1 are assumed to be smooth functions with some monotonicity properties: in particular, it is required that the specific heat $c_v(= \partial e/\partial\vartheta) = 2\vartheta F_0'(\vartheta) + \vartheta^2 F_0''(\vartheta)$ is well defined and lies between two positive constants C_* and C^*.

The pseudopotential Φ is supposed to depend on ϑ and χ as well, and is given by the following expression

$$\Phi(\chi_t, \vartheta, \chi) = \frac{1}{2} N(\vartheta) F_2'(\chi)|\chi_t|^2, \tag{22}$$

where N is continuous with $N(\vartheta) > 0$ for all ϑ, and F_2 is a smooth monotone and strictly increasing function on the domain where χ lives. In the framework of (21) and (22), it turns out that the "rate type" constitutive equation (6) summarizes a number of well-known phase transition models in absence of interfacial energy (see [6] and references therein).

Let us now briefly elucidate some peculiar choices of $G(\chi)$ and $F_1(\chi)$ in terms of stationary values of the phase parameter χ (that is, values for which $\chi_t = 0$).

- If $G(\chi) = I_{[0,1]}(\chi)$ and $F_1'(\chi) \geq 0$ for all $\chi \in [0,1]$, then mushy region is allowed and all $\chi \in [0,1]$ are minima of the free energy (and obey (6)) at the equilibrium critical temperature $\vartheta = \vartheta_c$; no superheating or undercooling effects appear at temperatures $\vartheta \neq \vartheta_c$, where either $\chi = 0$ or $\chi = 1$ is a minimum.

- If $G(\chi)$ is a double well potential, for instance of the form $G(\chi) = \chi^2(1-\chi)^2/4$, then no mushy region is observed and $\Psi(\,\cdot\,, \chi)$ exhibits two equally preferred strict minima, for $\chi = 0$ and $\chi = 1$, at equilibrium ($\vartheta = \vartheta_c$). Moreover, if we take $F_1(\chi) = \chi^2(3 - 2\chi)/3$ (cf. [6, equation (2.15)]), then superheating and undercooling phenomena may occur provided $|\vartheta - \vartheta_c|/\vartheta_c$ is sufficiently small. In particular, there is a range of values of ϑ where the two minima, at $\chi = 0$ and $\chi = 1$, still coexist.

Note that, if the latter setting is considered, the convexity property of function G definitely fails. On the other hand, the investigation of the related optimal control problem can take advantage of the absence of subdifferential operators. In particular, the set of optimality conditions might be simpler to derive and more complete.

In general, one can assume that G is the sum of a convex, possibly nonsmooth, potential $G_0 : \mathbb{R} \to (-\infty, +\infty]$, proper and lower semicontinuous, and of a smooth non convex contribution $G_1 : \mathbb{R} \to \mathbb{R}$. Then, the general system replacing (13–15) would be the following

$$\partial_t(\vartheta^2 F_0'(\vartheta) + \vartheta_c F_1(\chi)) + A(\vartheta + k * \vartheta) + B\vartheta = f_0 + B_0 u \quad \text{in } V', \text{ a.e. in } (0,T), \tag{23}$$

$$N(\vartheta)\partial_t(F_2(\chi)) + \vartheta\partial G_0(\chi) \ni (\vartheta - \vartheta_c)F_1'(\vartheta) - \vartheta G_1'(\chi) \quad \text{a.e. in } Q, \tag{24}$$

$$(\vartheta^2 F_0'(\vartheta) + \vartheta_c F_1(\chi))(0) = e_0 \quad \text{and} \quad F_2(\chi)(0) = F_2(\chi_0). \tag{25}$$

Existence and uniqueness of solution to (23–24) are discussed in [6] under the following assumptions on the nonlinearities

$$F_0 \in C^2(0,+\infty), \text{ the function } \xi \mapsto \xi^2 F_0'(\xi) \text{ is extended to the entire } \mathbb{R}$$
$$\text{in order that the extension is a.e. differentiable}$$
$$\text{and} \quad 0 < C_* \leq \xi\,(2F_0'(\xi) + \xi F_0''(\xi)) \leq C^* \quad \text{for a.a. } \xi \in \mathbb{R}\,,$$

$$N \in C^{0,1}(\mathbb{R}) \text{ is strictly positive with } 1/N \in L^\infty(\mathbb{R}),$$

$$G_0 \equiv I_{[0,1]}, \quad F_1,\, F_2,\, G_1 \in C^2(\mathbb{R}) \cap C^{0,1}(\mathbb{R}),$$
$$F_2'(\xi) > 0 \quad \forall\, \xi \in \mathbb{R}, \quad \text{and} \quad 1/F_2' \in L^\infty(\mathbb{R})$$

for the sample problem characterized by $f_0 = 0$, $k = 0$, and $\alpha = 0$. In this framework, Gentili, Giorgi and I obtained in [6, Theorems 3.1 and 3.4] existence of a strong solution (ϑ, χ) (that is, a solution more regular than in (12) and such that all equations and boundary conditions are satisfied almost everywhere) fulfilling the positivity property $\vartheta \geq 0$ a.e. in Q (see also [8] for related results).

Of course, the case considered in [6] is not interesting for the optimal control problem since (8) reduces to a Neumann homogeneous boundary condition and thus the solution (ϑ, χ) to (23–25) would be always the same independently of the control.

Therefore, a first open problem is the following

QUESTION 1 **[well–posedness]** *For a given $u \in L^2(\Sigma)$ and under assumptions (10–11), prove existence, uniqueness, and possibly continuous dependence on the data, of a (weak) solution (ϑ, χ) to problem (23–25).*

Once this necessary and preliminary step is settled, one could address the optimal control problem for the system (23–25).

QUESTION 2 **[extensions]** *Prove results analogous to Theorems 1 and 3 for the triplets $(\vartheta^*, \chi^*, u^*)$ minimizing the cost functional J and solving (23–25). In addition, try to extend the analysis when G_0 is not necessarily the indicator function of the interval $[0,1]$, but an arbitrary proper convex lower semicontinuous function on $\mathbb{R}$ (this would of course affect also Question 1).*

As it seems rather clear, the generalization of Theorem 2 is of minor interest in this framework, where the attention is focused on the rate–type phase transition models.

QUESTION 3 **[improvement]** *In some special and significant cases for G_0, for instance when $\partial G_0(\chi) = G_0'(\chi) = \chi^3$ and G is the corresponding double well potential, improve the conclusion of Theorem 3. In particular, prove some further necessary conditions for the function q in order to completely identify the pair (p, q) from optimality conditions.*

This contribution ends with the above questions which appear rather intriguing to me. I do hope to be able to tackle them in the near future. To resolve this question would be a way, I think, to keep Giorgio's memory alive.

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Some inverse problems related to the heat equation with memory in non smooth spatial domains *

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1. – Introduction

In this note we present some inverse problems related to the heat equation with memory. The memory thermal effects are taken into account by modifying the usual heat equation with an additional convolution term.

Let Ω be an open bounded set in $\mathbf{R}^3$ and $T > 0$. We can easily deduce the evolution equation for the temperature u by the continuity equation (for $(t,x) \in [0,T] \times \Omega$)

$$D_t u(t,x) + \operatorname{div} J(t,x) - f(t,x) = 0, \tag{1}$$

in which the vector J denotes the density of heat flow per unit surface area per unit time and f is the heat source per unit volume per unit time in Ω. The well known Fourier's law is given by

$$J(t,x) = -D_1 \nabla u(t,x) \tag{2}$$

and its best modification for materials with memory, supported by experiments, leads to replace (2) by

$$J(t,x) = -D_1 \nabla u(t,x) - D_2 \int_0^t h(t-s,x) \nabla u(s,x)\, ds, \tag{3}$$

where D_1 and D_2 are given positive functions and h is the convolution kernel, which accounts for the thermal memory. To obtain the equation for the evolution of the temperature we replace (3) into the continuity equation (1) and we get

$$D_t u(t,x) = \operatorname{div} [D_1 \nabla u(t,x) + D_2 \int_0^t h(t-s,x) \nabla u(s,x)\, ds] + f(t,x). \tag{4}$$

The fundamental point, when we deal with memory effects is that the kernel h cannot be considered a known function since there are no ways to measure h

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directly. What we can do is to try to reconstruct h by additional measurements on the temperature u in a suitable subset of the body Ω. Such additional information on u can be represented in integral form as follows

$$\int_\Omega \phi(x)u(t,x)\,dx = g(t), \qquad \forall t \in [0,T], \tag{5}$$

where ϕ and g are given functions representing the type of device used to measure the temperature and the results of the measurements, respectively. The inverse problem we consider in its more general form is the following.

Determine the temperature $u : [0,T] \times \Omega \longrightarrow \mathbf{R}$ *and the convolution kernel* $h : [0,T] \times \Omega \longrightarrow \mathbf{R}$ *satisfying* (4) *and* (5) *under suitable initial boundary-conditions.*

The general problem just introduced is very difficult to solve in the general form, but some approximated models of physical interest can be considered. In recent years we have collected our results in the papers [4], [5], [6], [7].

The first attempt to solve integrodifferential inverse problems of parabolic type, using the theory of analytic semigroups as fundamental tool, has been done by A. Lorenzi and E. Sinestrari in [13].

Then several authors have worked in this area, studying the non linear version of the equation or systems with kernels independent of space variables see for example [2], [8], [10], [11], [12]. Recently in [1] has been solved an inverse problem for a Phase-Field model with memory.

Using the Laplace transform method important results can be found for example in [15], [16] and their bibliography.

The novelty of the papers [4], [5], [6], [7] is that the kernel h is supposed to depend on spatial variables.

We now show the approximated physical models without giving the precise functional setting, at the moment we are interested in pointing out the physical differences. We will be more precise in the next sections.

Problem 1. The inverse problem we have solved, in the papers [5], [6], can be formulated as follows: *Given the data* ϕ, g, u_0 *and* f *determine the couple* (u,h) *satisfying the system*

$$\begin{cases} u_t(t,x,y) = Bu(t,x,y) + \int_0^t h(t-s,x)Bu(s,x,y)ds \\ \qquad +f(t,x,y), \quad (t,x,y) \in [0,T] \times \Omega, \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in \Omega, \\ u(t,x,y) = 0, \quad (t,x,y) \in [0,T] \times \partial\Omega, \\ \int_\omega \phi(y)u(t,x,y)dy = g(t,x), \quad (t,x) \in [0,T] \times [0,L] \end{cases} \tag{6}$$

in the framework of Sobolev fractional order spaces.

Here Ω is the bounded space cylindrical $[0,L] \times \omega$ and $x \in [0,L]$ and $y \in \omega$, B is a uniformly elliptic differential linear operator, in particular B is split in the sum $B_1 + B_2$ and B_1 and B_2 commute in the sense of resolvent. In this first approximation we suppose that the term $\int_0^t \nabla h(t-s,x)\nabla u(s,x)\,ds$ can be neglected. The sum of applications has been introduced to formulate the problem in an equivalent fixed point form that is more simple to treat.

The papers [5] and [6] are based on the generation results [9].

Problem 2. The inverse problem studied in [7] can be formulated as follows: *Given the data ϕ_1, ϕ_2, g_1, g_2, ℓ, u_0 and f determine the couple (u,h) satisfying the system*

$$(7)\quad \begin{cases} u_t(t,x,y) = Au(t,x,y) + \int_0^t h(t-s,x)Bu(s,x,y)ds \\ + \int_0^t D_x h(t-s,x)Cu(s,x,y)\,ds + f(t,x,y), \quad (t,(x,y)) \in [0,T]\times\Omega, \\ u(0,x,y) = u_0(x,y), \quad (x,y)\in\Omega, \\ \partial_{\nu_A} u(t,x,y) = \ell(t,x,y), \quad (t,(x,y))\in[0,T]\times\partial\Omega, \\ \int_\Omega \phi_1(x,y)u(t,x,y)dxdy = g_1(t), \quad t\in[0,T] \\ \int_\omega \phi_2(x,y)u(t,x)dy = g_2(t,x), \quad (t,x)\in[0,T]\times[0,L] \end{cases}$$

in the framework of the space of continuous functions.

Here Ω is as above, A and B are second order operators, while C is a first order linear differential operator. The novelty with respect to the first problem is that A is split in the sum $A_1 + A_2$ with A_1 and A_2 that *do not* commute in the sense of resolvent. This problem is a generalization of the one above also for the introduction of the term $\int_0^t D_x h(t-s,x)Cu(s,x)\,ds$.

Problem 3. In the paper [4] we have considered a more general model since it contains two memory kernels but we neglect the term $\int_0^t D_x h(t-s,x)Cu(s,x)\,ds$. The inverse problem we have solved can be formulated as follows:

Given the data ϕ_1, ϕ_2, G_1, G_2, R, F, and u_0 determine the triplet (u,k_1,k_2) satisfying the system

$$(8)\quad \begin{cases} D_t u(t,x,y) = \mathcal{A}u(t,x,y) + D_t\int_0^t k_1(t-s,x)u(s,x,y)ds \\ + \int_0^t k_2(t-s,x)\mathcal{A}u(s,x,y)ds + F(t,x,y), \\ (t,(x,y))\in[0,T]\times O. \\ u(0,x,y) = u_0(x,y), \quad (x,y)\in O, \\ u(t,x,y) = R(t,x,y), \quad (t,x,y)\in[0,T]\times\partial O, \\ \int_{O_x}\phi_1(x,y)u(x,y)dy = G_1(t,x), \quad x\in[a,b], \\ \int_{O_x}\phi_2(x,y)u(x,y)dy = G_2(t,x), \quad x\in[a,b], \end{cases}$$

in the framework of the space of continuous functions.

Here O is an admissible curvilinear polygon in $\mathbf{R}^2$, $[a,b]$ is its projection onto the $x-$axis and we have set

$$(9)\quad O_x := \{y\in\mathbf{R} : (x,y)\in O\}, \quad \forall x\in[a,b],$$

finally, $\mathcal{A}$ is a suitable uniformly elliptic second-order differential operator.

The novelty of this paper is that, related to this model we prove theorems of existence and uniqueness of a solution global in time and, even though we limit ourselves to plane spatial domains we consider curvilinear polygon with arbitrary corners. The paper [4] is based on the generation results [3].

The strategy to solve the above mentioned problems consists in reformulating each one in a suitable abstract framework, related to a Banach space X. In this new form we can apply the analytic semigroup theory and optimal regularity results in Hölder or in Sobolev spaces to transform the integrodifferential problem in an equivalent fixed point system of Volterra second kind integral equations.

The equivalent system in fixed point form is a crucial step in our approach since the compatibility conditions for the equivalence are also the conditions which assure that the inverse problem is well-posed.

Then we apply fixed point arguments to prove existence and uniqueness for the Volterra system and finally thanks to suitable generation estimates, we apply the abstract theorems to the concrete case.

In Section 2 we introduce the functional setting,
in Section 3 we study problem P_1,
in Section 4 we study problem P_2 and
in Section 5 we study problem P_3.

2. – Preliminary material

In this section we define the Banach spaces that will be used in the sequel to formulate our results.

We denote by $C([0,T];X)$ the Banach space consisting of all X–valued continuous functions defined on $[0,T]$. As usual we equip $C([0,T];X)$ with the sup–norm $\|u\|_{0,T,X} := \|u\|_{C([0,T];X)}$. Moreover, with any $\beta \in (0,1)$ we define the Hölder spaces $C^\beta([0,T];X)$ as

$$(10) \qquad \{u \in C([0,T];X) : |u|_{\beta,T,X} = \sup_{0 \le s < t \le T} (t-s)^{-\beta}\|u(t)-u(s)\| < \infty\}$$

and we endow them with the norm $\|u\|_{\beta,T,X} = \|u\|_{0,T,X} + |u|_{\beta,T,X}$.

The Sobolev spaces of fractional order $W^{\sigma,p}((0,T);X)$ consist of all functions $f \in L^p((0,T);X)$ for which

$$(11) \; |f|_{W^{\sigma,p}((0,T);X)} := \Big(\int_0^T dt_1 \int_0^T |t_2 - t_1|^{-1-\sigma p}\|f(t_2) - f(t_1)\|^p dt_2\Big)^{1/p} < +\infty,$$

$W^{\sigma,p}((0,t);X)$ turn out to be a Banach spaces when equipped with the norms

$$(12) \; \|f\|_{W^{\sigma,p}((0,T);X)} = (\|t^{-\sigma}f\|^p_{L^p((0,T);X)} + |f|^p_{W^{\sigma,p}((0,T);X)})^{1/p}, \quad if\ \sigma \in (0,1/p),$$

$$(13) \quad \|f\|_{W^{\sigma,p}((0,T);X)} = (\|f\|^p_{L^p((0,T);X)} + |f|^p_{W^{\sigma,p}((0,T);X)})^{1/p}, \quad if\ \sigma \in (1/p,1).$$

We now recall some results from the analytic semigroup theory and interpolation spaces. For more details see [14]. Let $B : D(B) \subset X \to X$ be a linear closed operator (possibly with $\overline{D(B)} \neq X$) satisfying the following assumptions:

i) *there exists* $\theta \in (\pi/2, \pi)$ *such that the resolvent set of* B *contains* 0 *and the open sector* $\Sigma_\theta = \{\mu \in \mathbf{C}\backslash\{0\} : |\arg\mu| < \theta\}$;

ii) *there exists* $M > 0$ *such that* $\|(\lambda I - B)^{-1}\|_{\mathcal{L}(X)} \le M|\lambda|^{-1}$ *for any* $\lambda \in \Sigma_\theta$.

Here $\mathcal{L}(X)$ denotes the Banach space of all bounded linear operators from X into itself equipped with the sup–norm.

By virtue of assumptions **i), ii)** we can define the analytic semigroup $\{e^{tB}\}_{t \ge 0}$ of bounded linear operators in $\mathcal{L}(X)$ generated by B.

After endowing $D(B)$ with the graph–norm we can define the following family of interpolation spaces $D_B(\beta, +\infty)$ (for $\beta \in (0,1)$), which are intermediate between $D(B)$ and X:

$$\mathcal{D}_B(\beta,+\infty)=\Big\{x\in X:\ |x|_{\mathcal{D}_B(\beta,+\infty)}:=\sup_{t>0} t^{(1-\beta)}\|Be^{tB}x\|<\infty\Big\}. \tag{14}$$

Moreover, we set

$$\mathcal{D}_B(1+\beta,+\infty)=\{x\in D(B):Bx\in\mathcal{D}_B(\beta,+\infty)\}. \tag{15}$$

Consequently, $\mathcal{D}_B(n+\beta,+\infty)$ $(n\in\mathbf{N},\beta\in(0,1))$ turns out to be a Banach spaces when equipped with the norm

$$\|x\|_{\mathcal{D}_B(n+\beta,+\infty)}=\sum_{j=0}^{n}\|B^jx\|+|B^nx|_{\mathcal{D}_B(\beta,+\infty)}. \tag{16}$$

With any linear operator B satisfying assumptions of type **i)** , **ii)** we can associate the vector spaces $\mathcal{D}_B(\eta,p)$ $(\eta\in(0,1),\ p\in(1,+\infty))$, intermediate between X and $\mathcal{D}(B)$, defined by

$$\mathcal{D}_B(\eta,p)=\left\{x\in X:|x|_{\eta,p}:=\left(\int_0^{+\infty}t^{(1-\eta)p-1}\|(B-\omega I)\mathrm{e}^{t(B-\omega I)}x\|^p\,dt\right)^{1/p}<+\infty\right\}. \tag{17}$$

Analogously, we set for any $p\in(1,+\infty)$ and $\eta\in(0,1)$

$$\mathcal{D}_B(1,p)=\mathcal{D}(B),\quad \mathcal{D}_B(1+\eta,p)=\{x\in\mathcal{D}(B):Bx\in\mathcal{D}_B(\eta,p)\}. \tag{18}$$

We recall that $\mathcal{D}_B(n+\eta,p)$ $(n=0,1)$ are Banach spaces when equipped with the norms

$$\|x\|_{n+\eta,p}=\sum_{j=0}^{n}\|B^jx\|+|B^nx|_{\eta,p},\quad \eta\in(0,1),\ p\in(1,+\infty),\ n=0,1. \tag{19}$$

In the following sections we will formulate the three problems more precisely and we state the relative statements of the theorems associated.

3. – The first problem

In this section we consider bounded sets of cylindrical type $\Omega=\Omega_1\times\Omega_2$ where Ω_1 and Ω_2 denote, respectively, an open interval in $\mathbf{R}$ and open bounded set in $\mathbf{R}^n$.

The problem P_1: *Find two functions* $u:[0,T]\times\Omega\to\mathbf{R}$ *and* $h:[0,T]\times\Omega_1\to\mathbf{R}$ *satisfying the equations*:

$$\begin{aligned}D_tu(t,x,y)&=\mathcal{B}_1(x,D_x)u(t,x,y)+\mathcal{B}_2(y,D_y)u(t,x,y)\\&+\int_0^t h(t-s,x)[\mathcal{B}_1(x,D_x)u(s,x,y)+\mathcal{B}_2(y,D_y)u(s,x,y)]\,ds\\&+f(t,x,y),\quad (t,x,y)\in[0,T]\times\Omega_1\times\Omega_2,\end{aligned} \tag{20}$$

$$u(0,x,y)=u_0(x,y),\quad (x,y)\in\Omega_1\times\Omega_2, \tag{21}$$

$$u(t,x,y)=0,\quad t\in[0,T],\ (x,y)\in(\partial\Omega_1\times\Omega_2)\cup(\Omega_1\times\partial\Omega_2), \tag{22}$$

$$\int_{\Omega_2} \phi(y)u(t,x,y)dy = g(t,x) \quad (t,x) \in [0,T] \times \Omega_1, \tag{23}$$

where $f : [0,T] \times \Omega \to \mathbf{R}$, $u_0 : \Omega \to \mathbf{R}$, $g : [0,T] \times \Omega_1 \to \mathbf{R}$ and $\phi : \Omega_2 \to \mathbf{R}$ are prescribed functions. Moreover, the linear differential operators $\mathcal{B}_1(x, D_x)$ and $\mathcal{B}_2(y, D_y)$ are defined by

$$\mathcal{B}_1(x, D_x) = \sum_{h=0}^{2} a_{2-h}^{(1)}(x) D_x^h, \tag{24}$$

$$\mathcal{B}_2(y, D_y) = \sum_{h,k=1}^{n} a_{h,k}^{(2)}(y) D_{y_h} D_{y_k} + \sum_{h=1}^{n} a_h^{(2)}(y) D_{y_h} + a^{(2)}(y), \tag{25}$$

and we assume $\mathcal{B}_1(x, D_x)$ and $\mathcal{B}_2(y, D_y)$ to be uniformly elliptic in $\overline{\Omega}_1$ and $\overline{\Omega}_2$, respectively. In [2] with the specific identification problem (20)–(23) we have associated the following abstract version related to a Banach space X: *find a function $u : [0,T] \to X$ and an operator $H : [0,T] \to \mathcal{L}(X)$ satisfying the equations*:

$$u'(t) = (B_1 + B_2)u(t) + \int_0^t H(t-s)(B_1 + B_2)u(s)\, ds + f(t), \quad t \in [0,T], \tag{26}$$

$$u(0) = u_0, \tag{27}$$

$$\Phi(u(t)) = G(t), \quad t \in [0,T]. \tag{28}$$

For any fixed pair of Banach spaces X_1 and X_2 we denote by $\mathcal{L}(X_1; X_2)$ the space of all bounded linear operators endowed with the uniform norm. We also set $\mathcal{L}(X_1; X_1) = \mathcal{L}(X_1)$. The main assumptions are the following. We now endow with the graph norms the domains $\mathcal{D}(B_j) \subset X$ of the linear closed operators B_j ($j = 1, 2$). Further, we assume that $B_j : \mathcal{D}(B_j) \subset X$ ($j = 1, 2$) are sectorial operators with resolvent sets $\rho(B_j)$, respectively, related to six constants $\omega_j \in \mathbf{R}$, $\theta_j \in (\pi/2, \pi)$, $M_j > 0$ ($j = 1, 2$) such that

H1 $\rho(B_j) \supset S_{\theta_j,\omega_j} = \{\lambda_j \in \mathbf{C};\ \lambda_j \neq \omega_j,\ |\arg(\lambda_j - \omega_j)| < \theta_j\}$;

H2 $\|(\lambda_j I - B_j)^{-1}\|_{\mathcal{L}(X)} \leq M_j |\lambda_j - \omega_j|^{-1}, \quad \forall \lambda_j \in S_{\theta_j,\omega_j}$;

H3 *there esists a pair* $(\mu_1, \mu_2) \in \rho(B_1) \times \rho(B_2)$ *such that*
$\mathcal{D}((\mu_1 I - B_1)(\mu_2 I - B_2)) = \mathcal{D}((\mu_2 I - B_2)(\mu_1 I - B_1))$
and $(\mu_1 I - B_1)(\mu_2 I - B_2) = (\mu_2 I - B_2)(\mu_1 I - B_1)$;

H4 $\mathcal{D}((\mu_2 I - B_2)(\lambda_1 I - B_1)) \subset \mathcal{D}((\mu_2 I - B_2)(\mu_1 I - B_1)), \quad \forall \lambda_1 \in \rho(B_1)$;

H5 $\mathcal{D}((\mu_1 I - B_1)(\lambda_2 I - B_2)) \subset \mathcal{D}((\mu_1 I - B_1)(\mu_2 I - B_2)), \quad \forall \lambda_2 \in \rho(B_2)$;

H6 $\rho(B_1 + B_2)$ *contains a real number* $\mu > \max(0, \omega_1) + \omega_2$;

H7 $\Phi \in \mathcal{L}(X; \mathcal{L}(X)) \cap \mathcal{L}(\mathcal{D}(B_1); \mathcal{L}(\mathcal{D}(B_1))$;

H8 $B_1 \Phi[u] = \Phi[B_1 u], \quad \forall u \in \mathcal{D}(B_1)$;

H9 $H\Phi[u] = \Phi[Hu], \quad \forall u \in X,\ \forall H \in \mathcal{K}$;

H10 $H \in \mathcal{L}(\mathcal{D}(B_1))$ *and* $B_2 H = H B_2, \quad \forall H \in \mathcal{K}$;

H11 $\Phi(B_1 u_0 + B_2 u_0 - \mu u_0)$ *is invertible in* $\mathcal{K}$ *for some*

$$\mu \in (\max(0, \omega_1) + \omega_2, +\infty)$$

and its inverse operator $\Lambda(u_0) := [\Phi(B_1 u_0 + B_2 u_0 - \mu u_0)]^{-1}$ *belongs to* $\mathcal{K}$.
We recall that $\mathcal{K}$ denotes a closed subalgebra in $\mathcal{L}(X) \cap \mathcal{L}(\mathcal{D}(B_1) \cap \mathcal{D}(B_2))$.

We now list our assumptions on the data

K1 $f \in W^{\sigma,p}((0,T_0);\mathcal{D}(B_1)\cap\mathcal{D}(B_2))\cap W^{1+\sigma,p}((0,T_0);\mathcal{D}(B_2))$;

K2 $u_0 \in \mathcal{D}(B_1)\cap\mathcal{D}(B_2)\cap\mathcal{D}((B_1+B_2)B_2)$, u_0 *satisfies* $H11$;

K3 $G \in W^{2+\sigma,p}((0,T_0);\mathcal{L}(X))\cap W^{1+\sigma,p}((0,T_0);\mathcal{D}(B_1))$;

K4 $w_0 \in \mathcal{D}_{B_1}(\sigma+1/p_{++},p)\cap\mathcal{D}_{B_2}(\sigma+1/p_{++},p)$;

K5 $w_1 \in \mathcal{D}_{B_1}(\sigma-1/p,p)\cap\mathcal{D}_{B_2}(\sigma-1/p,p)$, *if* $\sigma\in(1/p,1)$;

where T_0 is a fixed positive number and

$$w_0 := (B_2-\mu I)[(B_1+B_2-\mu I)u_0+f(0)], \tag{29}$$

$$\begin{aligned} w_1 &:= (B_1+B_2-\mu I)(B_2-\mu I)[(B_1+B_2-\mu I)u_0+f(0)]+(B_2-\mu I)f''(0)\\ &-\{\Phi[(B_2-\mu I)((B_1+B_2-\mu I)u_0+f(0))]-G''(0)+B_1G''(0)+\Phi[f''(0)]\}\Lambda(u_0)\\ &\times(B_2-\mu I)(B_1+B_2-\mu I)u_0. \end{aligned} \tag{30}$$

Then we list the consistency conditions related to our problem

K6 $\Phi[u_0]=G(0)$;

K7 $(B_1+B_2-\mu I)G(0)+\Phi[B_2u_0+f(0)]=G'(0)$.

We now denote by $\mathcal{G}$ the space of our *admissible* data satisfying H1–H11 and K1–K7. We also defined the following subset $\mathcal{G}(m)$ $(m\in\mathbf{R}_+)$ by

$$\mathcal{G}(m)=\{(f,G,u_0)\in\mathcal{G}:\|[\Phi(B_1u_0+B_2u_0-\mu u_0)]^{-1}\|_{\mathcal{K}}\le m\}. \tag{31}$$

We define then the following Banach spaces, where $s\in[1,+\infty)$:

$$\mathcal{U}^{s,p}(X)=W^{s,p}((0,T);X)\cap W^{s-1,p}((0,T);\mathcal{D}((B_1)\cap\mathcal{D}((B_2)). \tag{32}$$

Finally, we give the main theorems in [6]:

THEOREM **1** *Let* $(f,G,u_0)\in\mathcal{G}(m)$ *for some* $T_0>0$, $p\in(1,+\infty)$, $m>0$, *and let* $\sigma\in(0,1)\backslash\{1/p\}$. *Then there exists* $T^*\in(0,T_0]$ *such that for any* $T\in(0,T^*]$ *problem (26)–(28) admits a unique solution* $(u,H)\in\mathcal{U}^{2+\sigma,p}(X)\times W^{\sigma,p}((0,T);\mathcal{K})$.

THEOREM **2** *For any* $m>0$ *and* $T\in(0,T^*]$ *the map* $(f,G,u_0)\to(u,H)$, *where* (u,H) *is the unique solution to problem (26)–(28), is bounded (i.e. maps bounded sets into bounded sets) and is Lipschitz continuous from* $\mathcal{G}(m)$ *into*

$$\mathcal{U}^{2+\sigma,p}(X)\times W^{\sigma,p}((0,T);\mathcal{K}).$$

The abstract theorems stated above can be applied to the concrete problem P_1 choosing as reference spaces $X=L^p(\Omega_2;W_0^{1,p}(\Omega_1))$ where $W_0^{1,p}$ are the well known Sobolev spaces and $p>1$. The chosen spaces seem to be unusual, one could think that the most natural spaces are $X=L^p(\Omega_1\times\Omega_2)$, but this turns out to be not correct because the evolution equation does not have meaning here. The above choice is necessary, because the kernel h depends on the spatial variable x.

4. – The second problem

The paper [7] is concerned with the identification of an unknown coefficient h (the *relaxation coefficient,* depending on time and one space variable) appearing in the following integrodifferential equation related to the *convex cylindrical* domain $\Omega = (0,\ell) \times \omega \subset \mathbf{R}^n$ $(n \geq 2)$, ω being an open bounded convex set in $\mathbf{R}^{n-1}$.

The problem P_2: *Find two functions* $u : [0,T]\times\Omega \to \mathbf{R}$ *and* $h : [0,T]\times\Omega_1 \to \mathbf{R}$ *satisfying the equations*:

$$D_t u(t,x,y) = \mathcal{A}u(t,x,y) + \int_0^t h(t-s,x)\mathcal{B}u(s,x,y)\,ds$$

$$+\int_0^t D_x h(t-s,x)\mathcal{C}u(s,x,y)\,ds + f(t,x,y), \quad (t,x,y) \in [0,T]\times\Omega, \tag{33}$$

$$u(0,x,y) = u_0(x,y), \quad (x,y)\in\Omega, \tag{34}$$

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}}(t,x,y) = \frac{\partial u_1}{\partial \nu_{\mathcal{A}}}(t,x,y), \quad (t,x,y)\in[0,T]\times\partial\Omega, \tag{35}$$

$$\Phi[u(t,\cdot)](x) = \varphi(t,x), \qquad (t,x)\in[0,T]\times(0,\ell), \tag{36}$$

$$\Psi[u(t,\cdot,\cdot)] = \psi(t), \quad t\in[0,T], \tag{37}$$

where $u_0 : \overline{\Omega} \to \mathbf{R}$ and $u_1 : [0,T]\times\overline{\Omega} \to \mathbf{R}$ are given (smooth) functions and $\nu_{\mathcal{A}}$ denotes the *conormal* vector associated with $\mathcal{A}$ and Ω.

To determine the relaxation coefficient h we assume additional information available by Φ and Ψ. The linear operator Φ acts on the variable y only, while Ψ is a linear functional acting on all the space variables. The operators are:

$$\mathcal{A} = D_x[a_{1,1}(x)D_x] + \sum_{j=1}^{n-1} D_x[a_{1,1+j}(x,y)D_{y_j}] + \sum_{i=1}^{n-1} D_{y_i}[a_{1+i,1}(x,y)D_x]$$

$$+\sum_{i,j=1}^{n-1} D_{y_i}[a_{1+i,1+j}(x,y)D_{y_j}] + a_1(x,y)D_x + \sum_{j=1}^{n-1} a_{1+j}(x,y)D_{y_j} + a_0(x,y), \tag{38}$$

$$\mathcal{B} = D_x[b_{1,1}(x,y)D_x] + \sum_{j=1}^{n-1} D_x[b_{1,1+j}(x,y)D_{y_j}] + \sum_{i=1}^{n-1} D_{y_i}[b_{1+i,1}(x,y)D_x]$$

$$+\sum_{i,j=1}^{n-1} D_{y_i}[b_{1+i,1+j}(x,y)D_{y_j}] + b_1(x,y)D_x + \sum_{j=1}^{n-1} b_{1+j}(x,y)D_{y_j} + b_0(x,y), \tag{39}$$

$$\mathcal{C} = c_1(x,y)D_x + \sum_{j=1}^{n-1} c_{1+j}(x,y)D_{y_j} + c_0(x,y). \tag{40}$$

We note that $\mathcal{C}$ is a linear (formal) first–order differential operator, while $\mathcal{A}$ and $\mathcal{B}$ are two linear (formal) second–order operators with principal parts in *divergence form.* We emphasize that the coefficient $a_{1,1}$ in $\mathcal{A}$ depends on x, only, instead of on the pair (x,y) as in the general case. In particular, $\mathcal{A}$ turns out to be the sum of two linear differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$, the first acting on the variable x only, while the latter is differential in y with coefficients depending on both x and y, so that $\mathcal{A}_1$ and $\mathcal{A}_2$ do *not* commute. Operator $\mathcal{A}$ is supposed to be uniformly elliptic.

Examples of *admissible* Φ and Ψ are in Section 1 Problem 2.

In this second problem we have considered, besides the sum of non commuting operators the term $\int_0^t D_x h(t-s,x)\mathcal{C}u(s,x,y)\,ds$ which introduces serious difficulties in formulating directly the inverse problem in an abstract setting. In fact we are forced to introduce an intermediate step, consisting in an equivalent problem in non abstract form for the unknowns $D_t u$, $D_x h(0,x)$ and $h(t,0)$. To identify the convolution kernel h we have to identify $D_x h(0,x)$ and $h(t,0)$, for this reason we must give two additional measurements on the temperature.

We then give an abstract formulation and we formulate a second equivalent problem in abstract form. Starting from it we apply fixed point arguments to prove our results.

Since the consistency conditions and the regularity conditions are very complicated we do not report them in the following but we refer the reader to the original paper.

The set of admissible data $\mathcal{X}$ are in [7]:

i) assumptions (1.5),(1.6), (2.1)–(2.9), (2.11)–(2.15) be fulfilled,

ii) the consistency conditions (1.16)–(1.21) be fulfilled and satisfy the inequalities (2.20), (2.21).

The result we obtain in this case is the following

THEOREM **3** *Suppose the data belong to $\mathcal{X}$. Then the identification problem P_2 admits a unique solution*

$$(u,h) \in [C^{2+\beta}([0,T];L^2(\Omega)) \cap C^{1+\beta}([0,T];H^2(\Omega))] \times C^{\beta}((0,T^*);H^1((0,\ell))),$$

for some $T^* \in (0,T]$ and $\beta \in (1/2,1)$. Moreover, (u,h) continuously depends on the data with respect to suitable norms.

5. – The third problem

The last problem introduced in Section 1 can be formulated in abstract form in the following way. *Determine two functions $u : [0,T] \to X$ and $k : [0,T] \to Z$ satisfying the system:*

$$\left\{ \begin{array}{ll} u'(t) = Au(t) + k * (Bu + r)(t) + f(t), & t \in [0,T], \\ u(0) = u_0, & \\ \Phi[u(t)] = g(t), & t \in [0,T]. \end{array} \right. \tag{41}$$

The abstract version of the problem is studied under the following hypothesis:

(H1) X, Y, Z are Banach spaces, $(k,y) \to ky$ is a bilinear and continuous mapping from $Z \times Y$ to X;

(H2) $A : D(A) \subseteq X \to X$ is a closed linear operator such that the resolvent set $\rho(A)$ of A contains the sector $\Sigma := \{\mu \in \mathbf{C} \setminus \{0\} : |\mu| \geq \lambda_0, \ |arg\mu| \leq \frac{\pi}{2}\}$ and there exist $m > 0$ such that $\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq m|\lambda - \lambda_0|^{-1}$ for any $\lambda \in \Sigma$;

(H3) $B \in \mathcal{L}(D(A);Y)$;

(H4) $\Phi \in \mathcal{L}(X, Z)$;
(H5) $u_0 \in D(A)$ *and the linear continuous mapping of* Z *into itself*

$$k \to \Phi[k(Bu_0 + r(0))]$$

is one to one and onto. We shall indicate with M *the inverse mapping;*
(H6) there exist $\gamma \in (0,1)$, A_1 *closed linear operator in* Z *with domain* $D(A_1)$ *and* $\Psi \in \mathcal{L}(D_A(\gamma,\infty); Z)$ *such that, for every* $u \in D(A)$, *we have*

$$\Phi(Au) = A_1\Phi[u] + \Psi u; \tag{42}$$

(H7) concerning A_1 *we also assume the following: let* $h \in C^1([0,T]; Z) \cap C([0,T]; D(A_1))$ *be such that*

$$\begin{cases} h'(t) = A_1 h(t) \quad \forall t \in [0,T], \\ h(0) = 0. \end{cases}$$

Then $h(t) = 0, \quad \forall t \in [0,T]$;
(H8) $r \in C^{1+\beta}([0,T]; Y)$, *for some* $\beta \in (0,1)$. If $h : [0,T] \to Z$ and $u : [0,T] \to Y$,

we set, for $t \in [0,T]$,

$$h * u(t) := \int_0^t h(t-s)u(s)ds \tag{43}$$

whenever the integral in (43) is defined in the sense of Bochner. According to $H2$, it is possible to define the semigroup $\{e^{tA}\}_{t\geq 0}$, of bounded linear operators in $\mathcal{L}(X)$, so that $t \to e^{tA}$ is an analytic function from $(0,\infty)$ to $\mathcal{L}(X)$. For more details see ([14]). The main abstract result of this paper is the following

THEOREM 4 *Assume that* $(H1) - (H8)$ *hold. Suppose, moreover, that*
(H9) $v_0 := Au_0 + f(0) \in D(A)$;
(H10) $f \in C^{1+\beta}([0,T]; X)$;
(H11) $g \in C^{2+\beta}([0,T]; Z) \cap C^{1+\beta}([0,T]; D(A_1))$;
(H12) $\Phi[u_0] = g(0)$, $\Phi[v_0] = g'(0)$;
(H13) $Av_0 + f'(0) + k_0(Bu_0 + r(0)) \in D_A(\beta,\infty)$, *where* k_0 *is defined as* $k_0 := M(g''(0) - \Phi[Av_0 + f'(0)])$, *and* M *is defined in H5. Then problem (41) has a unique solution* $(u,k) \in [C^{2+\beta}([0,T]; X) \cap C^{1+\beta}([0,T]; D(A))] \times C^{\beta}([0,T]; Z)$.

This theorem is proved in Section 5 of [4]. Thanks to the generation results proved in [3] we can choose as reference space $X = L^\infty(O)$ which is an algebra and our concrete,problem P_3 in Section 1 is well posed in this functional setting.

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On the minimal free energy and the Saint-Venant principle in linear viscoelasticity *

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1. – In memory of Giorgio

The heritage that Giorgio Gentili left is certainly important not only from the scientific point of view. This contribution is dedicated to his memory and to his family. We know that He wants us to remember him in joy, but even though this is the case, it is not without commotion that we write these notes, to which Giorgio gave his significative contribution.

2. – Introduction

Work on deriving clearly quantified expressions of Saint-Venant's principles for elastic and other materials has been ongoing for some decades [24]. Rigorous proofs of of such a 'principles' for particular classes of linear viscoelastic solids have been provided in the past by many authors [see e. g. [26] and references cited therein]. In spite of the general shape of the analyzed bodies, main issues, such as proving a Saint-Venant principle for general dissipative relaxation functions, have not been solved yet. A very intuitive form of the Saint-Venant principle for a linear elastic cylinder maybe seen in several papers, such as [25]. In these cases, such a cylinder is assumed to be free from constraints and loaded on one basis only by a self equilibrated traction field; the spatial decay properties of the stored energy are then investigated. In particular, the state of points on the cross sections of the cylinder are considered. The rate of spatial decay of the energy is determined along the direction of the axis. In the case of a semi-infinite solid this argument shows that the energy stored in the solid delimited by the loaded basis and a given cross section

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approaches its value at the natural state, as the distance of the given cross section increases from the loaded basis. It follows that the corresponding state of points on the same cross section approaches the natural state. In the context just described, the state, called elastic state, is given by the triple $\{\mathbf{u}(\mathbf{x},t), \mathbf{E}(\mathbf{x},t), \mathbf{T}(\mathbf{x},t)\}$ (see [2]). As far as linear viscoelastic materials are concerned, two replacements have to be done in order to establish a Saint Venant principle: (i) the stored energy has to be replaced by some free energy, and at the same time (ii) a notion of *linear viscoelastic state* has to be provided. About (i), it is very well known that there are different (but related) possibilities of defining the free energy for a linear viscoelastic material [4]pir3, and the issue of its non-uniqueness arises [23], [15]], [16], [20], [21], [22],[14], [7], [32],[1]. About (ii), a notion of state for linear viscoleastic materials has been provided in [4] by particularizing the concept of state proposed by Noll [3].

The two issues (i) and (ii) have then been faced at once in [4, 8]: in particular, for the considered set of free energies, which are functions of the state in the sense of Noll, the existence of both the maximal and the minimal element is ensured, the latter being the maximum recoverable work from a given state. An explicit expression for the isothermal minimum free energy of a linear viscoelastic material has been given in [1] for the case of a scalar constitutive equations, and the same problem has been solved for general tensorial stresses, strains and relaxation functions in [27]. There, a characterization in the frequency domain for the state in the sense of Noll has also been provided, and the resultant expression for the minimal free energy has shown to be a quadratic form in the variable characterizing the state in the abovementioned sense. References [26, 27] form the basis of the present work, the aim of which is to utilize the explicit expression for the minimum free energy and its properties in obtaining an energy decay estimate.

The desired estimate can be established on the minimal free energy provided that a time-integral inequality, involving the difference between this free energy and the inner product between the values of stress and strain, is verified. It remains not clear whether or not this is the case for all relaxation functions satisfying the minimal thermodynamic restrictions; it is certainly the case for a special class of functions, and this will be discussed in the sequel.

In the case of a semi-infinite cylynder, the state of points on a given cross section of the cylinder "far enough" from the loaded basis turns out to approach the natural state. This is infact the case, because the associated minimal free energy provides a norm in the state space [27], and the above mentioned result shows that the minimal free energy approaches its value at the natural state. The estimate for the spatial decay of the minimal free energy, integrated over a portion of the given cylinder, is obtained in this paper in terms of the constant derived in [26]. The result allowing for the stated decay property entails a new estimate on the magnitude of the stress at a given state in terms of the corresponding value of the minimal free energy density.

In a future paper [11],the remaining authors will face and solve other two problems which had been already discussed and almost solved with Gentili. The first one may regard the quasi-static case in which the self-balanced tractions do act with just one frequency. For this case the frequency-dependence of the spatial decay of the minimal free energy will be obtained. It will also be shown that any other free energy does decay more rapidly than the minimal one, allowing for establishing the largest-influence zone for the given frequency of the applied tractions. The inertial case will also be discussed in a future paper together with the resulting domain of influence and spatial decay theorems are discussed. The energy measure involving the minimum free energy, rather than the Morro-Vianello function used in [26], obeys a differential inequality that is stronger than that given in [26].

3. – Notation and basic assumptions for a linear viscoelastic solid

Let Sym be the space of symmetric second order tensors acting on $\mathcal{R}^3$ viz. $Sym := \{\mathbf{M} \in Lin(\mathcal{R}^3) : \mathbf{M} = \mathbf{M}^\top\}$, where the superscript "$\top$" denotes the transpose. Operating on Sym is the space of the fourth order tensors $Lin(Sym)$.

It is well known that Sym is isomorphic to $\mathcal{R}^6$. In particular, for every $\mathbf{L}, \mathbf{M} \in Sym$, if $\mathbf{C}_i$, $i = 1, ..., 6$ is an orthonormal basis of Sym with respect to the usual inner product in $Lin(\mathcal{R}^3)$, namely $tr(\mathbf{L}\mathbf{M}^\top)$, it is clear that the representation

$$\mathbf{L} = \sum_{i=1}^{6} L_i \mathbf{C}_i \,, \qquad \mathbf{M} = \sum_{i=1}^{6} M_i \mathbf{C}_i \tag{1}$$

is such that $tr(\mathbf{L}\mathbf{M}^\top) = \sum_{i=1}^{6} L_i M_i$. Therefore, henceforth we treat each tensor of Sym as a vector in $\mathcal{R}^6$ and denote by $\mathbf{L} \cdot \mathbf{M}$ the inner product between elements of Sym, viz.

$$\mathbf{L} \cdot \mathbf{M} = tr(\mathbf{L}\mathbf{M}^\top) = tr(\mathbf{L}\mathbf{M}) = \sum_{i=1}^{6} L_i M_i$$

and $|\mathbf{M}|^2 = \mathbf{M} \cdot \mathbf{M}$. Consequently any fourth order tensor $\mathbb{K} \in Lin(Sym)$ will be identified with an element of $Lin(\mathcal{R}^6)$ by the representation

$$\mathbb{K} = \sum_{i,i=1}^{6} K_{ij} \mathbf{C}_i \otimes \mathbf{C}_j \tag{2}$$

and $\mathbb{K}^\top$ means the transpose of $\mathbb{K}$ as an element of $Lin(\mathcal{R}^6)$. According to (2), the norm $|\,\mathbb{K}|$ of $\mathbb{K} \in Lin(Sym)$ may be given by

$$|\,\mathbb{K}|^2 = tr\left(\,\mathbb{K}\,\mathbb{K}^\top\right) = \left(\sum_{i,j=1}^{6} K_{ij} K_{ji}\right) .$$

In the sequel we deal with complex valued tensors. Denoting by Ω the complex plane and by $Sym(\Omega)$ and $Lin(Sym(\Omega))$ respectively the tensors represented by the forms (1) and (2) with $L_i, M_i, K_{ij} \in \Omega$, then the norms $|\mathbf{M}|$ and $|\,\mathbb{K}|$ of $\mathbf{M} \in Sym(\Omega)$ and $\mathbb{K} \in Lin(Sym(\Omega))$ will be given respectively by

$$|\mathbf{M}|^2 = \left(\mathbf{M} \cdot \overline{\mathbf{M}}\right) , \qquad |\,\mathbb{K}|^2 = tr\left(\,\mathbb{K}\,\mathbb{K}^*\right) = \left(\sum_{i,j=1}^{6} K_{ij} \overline{K}_{ji}\right) , \tag{3}$$

where the overhead bar indicates complex conjugate and $\mathbb{K}^* = \overline{\mathbb{K}}^{\top}$ is the hermitian conjugate.

The above representations allows results of [6] to be easily extended to tensors belonging to $Lin(Sym(\Omega))$.

The symbols $\mathcal{R}^+$ and $\mathcal{R}^{++}$ denote the non-negative reals and the strictly positive reals, respectively, while $\mathcal{R}^-$ and $\mathcal{R}^{--}$ denote the non-positive and strictly negative reals.

For any function $f : \mathcal{R} \to \mathcal{V}$, where $\mathcal{V}$ is a finite-dimensional vector space, in particular in the present context Sym or $LinSym$, let f_F, denote its *Fourier transform* viz. $f_F(\omega) = \int_{-\infty}^{\infty} f(s)e^{-i\omega s}ds$. Also, we define

$$
\begin{aligned}
f_+(\omega) &= \int_0^\infty f(s)e^{-i\omega s}ds, & f_-(\omega) &= \int_{-\infty}^0 f(s)e^{-i\omega s}ds \\
f_s(\omega) &= \int_0^\infty f(s)\sin\omega s\, ds, & f_c(\omega) &= \int_0^\infty f(s)\cos\omega s\, ds
\end{aligned}
\tag{4}
$$

The relations defining f_F and (4) are to be understood as applying to each component of the tensor quantities involved. Some constraint must be placed on these components to ensure that the Fourier transforms exist. It is assumed that all components of tensors in the time domain belong to $L^2(\mathcal{R})$ (or $L^2(\mathcal{R}^\pm)$ in the case of $f_\pm$) so that in the frequency domain, they belong to $L^2(\mathcal{R})$ (or $L^2(\mathcal{R}^\pm)$) [12]. Further restrictions on the allowed function spaces will be imposed below.

When $f : \mathcal{R}^+ \to \mathcal{V}$ we can always extend the domain of f to $\mathcal{R}$, by considering its *causal* extension *viz.*

$$
f(s) = \begin{cases} f(s) & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases}
\tag{5}
$$

in which case

$$
f_F(\omega) = f_+(\omega) = f_c(\omega) - if_s(\omega)
\tag{6}
$$

We shall need to consider the Fourier transform of functions that do not go to zero at large times and thus do not belong to L^2 for the appropriate domain. In particular, let $f(s)$ in (5) be given by a constant a for all s. The standard procedure is adopted of introducing an exponential decay factor, calculating the Fourier transform and then letting the time decay constant tend to infinity. Thus, we obtain

$$
\begin{aligned}
f_+(\omega) &= \frac{a}{i\omega^-} \\
\omega^- &= \lim_{\alpha\to 0}(\omega - i\alpha)
\end{aligned}
\tag{7}
$$

The corresponding result for a constant function defined on R^- is obtained by taking the complex conjugates of this relationship. Also, if f is a function defined on $\mathcal{R}^-$

and if $\lim_{s\to-\infty} f(s) = b$ where the components of the function $g : R^+ \to \mathcal{V}$ defined by $g(s) = f(s) - b$ belong to $L^2(\mathcal{R}^+)$, then

$$f_F(\omega) = g_F(\omega) - \frac{b}{i\omega^+} \tag{8}$$

Again, taking complex conjugates gives the result for functions defined on $\mathcal{R}^+$. This procedure amounts to defining the Fourier transform of such functions as the limit of the transforms of a sequence of functions in L^2. The limit is to be taken after integrations over ω are carried out if the ω^{-1} results in a singularity in the integrand. Generally, in the present application, the ω^{-1} produces no such singularity and the limiting process is redundant.

The comples ω plane, denoted by Ω, will play an important role in our discussions. We define the following sets:

$$\Omega^+ = \{\zeta \in \Omega : \Im m\zeta \geq 0\}\ , \quad \Omega^{(+)} = \{\zeta \in \Omega : \Im m\zeta > 0\}\,. \tag{9}$$

Analogous meanings are assigned to Ω^- and $\Omega^{(-)}$.

The quantities $f_\pm$ defined by (4) are analytic in $\Omega^{(\mp)}$ respectively. This analyticity is extended by assumption to $\Omega^\mp$. The function f_+ may be defined by (4) and analytic on a portion of Ω^+ if for example f decays exponentially at large times. Over the remaining portion of Ω^+, on which the integral definition is meaningless, f_+ is defined by analytic continuation.

4. – Relaxation functions, histories and states

A linear viscoelastic material is described by the classical Boltzmann-Volterra constitutive equation between the current value of the stress tensor $\mathbf{T}(t) \in Sym$, the current strain $\mathbf{E}(t) \in Sym$ and the *past strain history* up to the time t, i.e. $\mathbf{E}^t : \mathcal{R}^{++} \to Sym$, of the form:

$$\begin{aligned} \mathbf{T}(t) &= \mathbb{G}_0\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t(s)ds \\ &= \mathbb{G}_\infty\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t_r(s)ds \end{aligned} \tag{10}$$

where

$$\mathbf{E}^t(s) := \mathbf{E}(t-s),\quad \mathbf{E}^t_r(s) := \mathbf{E}^t(s) - \mathbf{E}(t),\quad s \in \mathcal{R}^{++}. \tag{11}$$

We refer to $\mathbf{E}^t_r$ as the relative strain history. The fourth order tensor $\dot{\mathbb{G}} : \mathcal{R}^{++} \to LinSym$ is assumed to be integrable, so its primitive, the *relaxation function* $\mathbb{G} : \mathcal{R}^{++} \to LinSym$ is absolutely continuous and it is defined as

$$\mathbb{G}(t) := \mathbb{G}_0 + \int_0^t \dot{\mathbb{G}}(s)\,ds \tag{12}$$

where $\mathbb{G}_0 = \mathbb{G}(0)$ is the *instantaneous elastic modulus.* Moreover there exists the limit

$$\mathbb{G}_\infty := \lim_{t\to\infty} \mathbb{G}(t) \in LinSym \tag{13}$$

where $\mathbb{G}_\infty$ is the *equilibrium elastic modulus.* We require the property that [27]

$$0 < \left| \int_0^\infty s\dot{\mathbb{G}}(s)\, ds \right| < \infty. \tag{14}$$

The rightmost inequality follows from the assumed analyticity (and therefore differentiability) of $\dot{\mathbb{G}}_F$.

The Fourier transform of $\dot{\mathbb{G}}(t)$, namely $\dot{\mathbb{G}}_F(\omega) = \dot{\mathbb{G}}_c(\omega) - i\dot{\mathbb{G}}_s(\omega)$, for real ω, belongs to $L^2(\mathcal{R})$, according to our earlier assumptions. It is clear that $\dot{\mathbb{G}}_c(\omega)$ is even as a function of ω and $\dot{\mathbb{G}}_s(\omega)$ is odd. The quantity $\dot{\mathbb{G}}_s(\omega)$ therefore vanishes at the origin. In fact, a consequence of our assumption of analyticity of Fourier transformed quantities on the real axis of Ω, it vanishes al least linearly at the origin. The leftmost inequality in (14) implies that it vanishes no more strongly than linearly.

Thermodynamic properties of the linear viscoelastic materials imply that [13, 14]

$$\mathbb{G}_0 = \mathbb{G}_0^\top, \quad \mathbb{G}_\infty = \mathbb{G}_\infty^\top, \quad \dot{\mathbb{G}}_s(\omega)\mathbf{E}\cdot\mathbf{E} < 0 \quad \forall \mathbf{E} \in Sym \ \ \forall \omega \in \mathcal{R}^{++}. \tag{15}$$

An important consequences of $(15)_3$ is [14]

$$\dot{\mathbb{G}}(0)\mathbf{E}\cdot\mathbf{E} \leq 0 \ \ \forall \mathbf{E} \in ym \setminus \{\mathbf{0}\}. \tag{16}$$

Also [14]

$$\mathbb{G}_\infty - \mathbb{G}_0 = \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega \frac{\dot{\mathbb{G}}_s(\omega)}{\omega} \tag{17}$$

giving, by virtue of $(15)_3$

$$\mathbb{G}_0\mathbf{E}\cdot\mathbf{E} > \mathbb{G}_\infty\mathbf{E}\cdot\mathbf{E} \ \ \forall \mathbf{E} \in Sym \setminus \{\mathbf{0}\}. \tag{18}$$

We assume a stronger relation than $(18)_1$, namely

$$\dot{\mathbb{G}}(0)\mathbf{E}\cdot\mathbf{E} < 0, \quad \forall \mathbf{E} \in Sym \setminus \{\mathbf{0}\} \tag{19}$$

We also consider linear viscoelastic solids, so that enforce the following inequality to be satisfied:

$$\mathbb{G}_\infty\mathbf{E}\cdot\mathbf{E} > 0, \quad \forall \mathbf{E} \in Sym \setminus \{\mathbf{0}\}. \tag{20}$$

We will allow the extra generality of inhomogeneity in some later sections, so that $\mathbb{G}$ may depend on $\mathbf{x}$. This dependence is omitted except where explicitly required.

Let us extend the integral in (10) to $\mathcal{R}$ by identifying $\dot{\mathbb{G}}$ with its odd extension while taking $\mathbf{E}^t$ to be zero on $\mathcal{R}^-$. We now apply Plancherel's theorem, noting that $\dot{\mathbb{G}}_F(\omega) = -2\dot{\mathbb{G}}_s(\omega)$, to obtain[7].

$$\begin{aligned} \mathbf{T}(t) &= \mathbb{G}_0\mathbf{E}(t) + \frac{i}{\pi}\int_{-\infty}^{\infty} \dot{\mathbb{G}}_s(\omega)\mathbf{E}_+^t(\omega)ds \\ &= \mathbb{G}_\infty\mathbf{E}(t) + \frac{i}{\pi}\int_{-\infty}^{\infty} \dot{\mathbb{G}}_s(\omega)\mathbf{E}_{r+}^t(\omega)ds \\ \mathbf{E}_{r+}^t &= \mathbf{E}_+^t(\omega) - \frac{\mathbf{E}(t)}{i\omega^-} \end{aligned} \tag{21}$$

where $\mathbf{E}^t_{\tau+}$ is the Fourier transform of $\mathbf{E}^t_\tau$, defined in (10), as can be seen from (7).

For simplicity, we let $\mathbb{G}(t)$ be symmetric for all values of t. As well as belonging to $L^2(\mathcal{R}^+)$, we assume that $\mathbf{E}^t \in L^1(\mathcal{R}^+) \cap C^1(\mathcal{R}^+)$ and that its derivative also belong to $L^1(\mathcal{R}^+)$ [12].

If we define the vector space

$$\Gamma := \left\{ \mathbf{E}^t : \mathcal{R}^{++} \to Sym\,; \left| \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,ds \right| < \infty \;\; \forall \tau \geq 0 \right\} \tag{22}$$

the Boltzmann-Volterra equation (10) defines the linear functional $\tilde{\mathbf{T}} : Sym \times \Gamma \to Sym$ such that

$$\tilde{\mathbf{T}}(\mathbf{E}(t), \mathbf{E}^t) = \mathbb{G}_0 \mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t(s)ds \tag{23}$$

The concept of the state of a linear viscoelastic solid has been discussed by various authors [15, 4, 17, 3]. We briefly recall some basic propositions.

Remark 1. *According to the definition in [15] and [14], a process P of finite duration d, is given by $\dot{\mathbf{E}}_P : [0, d) \to Sym$. Given the couple $(\mathbf{E}(t), \mathbf{E}^t) \in Sym \times \Gamma$, related to the strain $\mathbf{E}(\tau)$, $\tau \leq t$, we associate with P the quantity*

$$\mathbf{E}_P : (0, d) \to Sym, \qquad \mathbf{E}_P(\tau) = \mathbf{E}(t) + \int_0^\tau \dot{\mathbf{E}}_P(s')\,ds' \quad \tau \in (0, d] \tag{24}$$

The strain $\mathbf{E}_f(\tau') = (\mathbf{E}_P * \mathbf{E})(\tau')$, $\tau' \leq t + d$, yielded by $\mathbf{E}^t$ and $\dot{\mathbf{E}}_P$, is given by

$$\mathbf{E}_f(t+d-s) = (\mathbf{E}_P * \mathbf{E})(t+d-s) := \begin{cases} \mathbf{E}_P(d-s) & 0 \leq s < d \\ \mathbf{E}(t+d-s) & s \geq d \end{cases} \tag{25}$$

Thus, $\mathbf{E}_f$ is related to the couple $(\mathbf{E}_P(d), (\mathbf{E}_P * \mathbf{E})^{t+d})$.

We denote by Π the set of all processes of finite duration, and by Π_∞ the set of processes of infinite duration *i.e.* the ones related to $\dot{\mathbf{E}}_P : \mathcal{R}^+ \to Sym$.

In the sequel we use the symbol" $*$ " to denote both the combination of two histories and the combination (continuation) of two processes [4].

Definition 1. *Two histories $\mathbf{E}^t_1$ and $\mathbf{E}^t_2$ are said to be equivalent if for every $\mathbf{E}_P : (0, \tau] \to Sym$ and for every $\tau > 0$, they satisfy [16]*

$$\tilde{\mathbf{T}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E}_1)^{t+\tau}) = \tilde{\mathbf{T}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E}_2)^{t+\tau}) \tag{26}$$

As a consequence, it is easy to show that $\mathbf{E}^t$ is equivalent to the zero history $\mathbf{0}^\dagger$ if

$$\mathbf{I}^t(\tau) = \int_\tau^\infty \dot{\mathbb{G}}(s)\mathbf{E}^{t+\tau}(s)\,ds = \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,ds = 0 \qquad \forall\tau > 0 \tag{27}$$

Equation (27) represents an equivalence relation. Two histories $\mathbf{E}_1^t$ and $\mathbf{E}_2^t$ are said to be equivalent if their difference $\mathbf{E}^t = \mathbf{E}_1^t - \mathbf{E}_2^t$ satisfies (27).

According to the definition of the state σ given by Noll [3], two couples $(\mathbf{E}_1(t), \mathbf{E}_1^t)$ and $(\mathbf{E}_2(t), \mathbf{E}_2^t)$ such that $\mathbf{E}_1(t) = \mathbf{E}_2(t)$ and $\mathbf{E}_1^t - \mathbf{E}_2^t$ satisfies (27), are represented by the same state $\sigma(t)$. In this sense, $\sigma(t)$ may be thought as the "minimum" set of variables allowing a univocal relation between $\dot{\mathbf{E}}_P : [0,\tau) \to Sym$ and the stress $\mathbf{T}(t+\tau) = \tilde{\mathbf{T}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E})^{t+\tau})$ for every $\tau > 0$.

In other words [4, 17], denoting with Γ_0 the set of all the past histories of Γ satisfying (27), and by Γ/Γ_0 the usual quotient space, the state σ of a linear viscoelastic material is an element of [1]

$$\Sigma := Sym \times (\Gamma/\Gamma_0) \tag{28}$$

Henceforth, we also view P as an endomorphism on Σ so that $P\sigma \in \Sigma$ will denote the state yielded by P starting from $\sigma \in \Sigma$.

The work done on the material by the strain history $\mathbf{E}(\tau)$, $\tau \le t$ is

$$\begin{aligned} \widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) &:= \int_{-\infty}^t \mathbf{T}(\tau)\cdot\dot{\mathbf{E}}(\tau)d\tau \\ &= \frac{1}{2}\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t) + \int_{-\infty}^t\int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^\tau(s)\cdot\dot{\mathbf{E}}(\tau)ds d\tau\,. \end{aligned} \tag{29}$$

It will be clear from the representation of $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$ in the frequency domain, given below, that it is a non-negative quantity. We will restrict our considerations to histories such that $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) < \infty$.

The properties of the work have been extensively studied in [4].

A representation of the work $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$, defined by (29), in the frequency domain has been obtained in [7] (see also [18]) in terms of integrals over $\mathcal{R}^+$. Using symmetry arguments, this representation can be expressed as an integral over $\mathcal{R}$ of the form

$$\begin{aligned} \widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^\infty \mathbb{H}(\omega)\mathbf{E}_{r+}^t(\omega)\cdot\overline{\mathbf{E}}_{r+}^t(\omega)d\omega \\ &= S(t) + \frac{1}{2\pi}\int_{-\infty}^\infty \mathbb{H}(\omega)\mathbf{E}_{+}^t(\omega)\cdot\overline{\mathbf{E}}_{+}^t(\omega)d\omega \\ \phi(t) &= \frac{1}{2}\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t) \\ S(t) &= \mathbf{T}(t)\cdot\mathbf{E}(t) - \frac{1}{2}\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t) \end{aligned} \tag{30}$$

---

[1] It is worth noting that, by virtue of (22) and Definition 2.1, the space of the states Σ depends on the memory kernel $\mathbb{G}$ characterising the material by means of (10). This property distinguishes (28) from the usual fading memory spaces

where, for each given $\omega \in \mathcal{R}$, the fourth order tensor $\mathbb{H}(\omega) \in Lin(Sym)$ is defined as :

$$\mathbb{H}(\omega) := -\omega \dot{\mathbb{G}}_s(\omega) \tag{31}$$

The equivalence of the two forms of (30) follows from (17) and (21). They reduce to relations of Golden [1] in the scalar case.

5. – Explicit expression for the minimum free energy

From a result in [27], based on a theorem of Gohberg and Kreĭn [6], we have that $\mathbb{H}(\omega)$ can be factorized as follows:

$$\mathbb{H}(\omega) = \mathbb{H}_+(\omega)\, \mathbb{H}_-(\omega) \tag{32}$$

with

$$\mathbb{H}_+(\omega) = \mathbb{H}^*_-(\omega) \tag{33}$$

where the matrix functions $\mathbb{H}_{(\pm)}$ admit analytic continuations, are holomorphic in the interior and continuous up to the boundary of the corresponding complex half planes $\Omega^{\pm}$, and are such that

$$\det\, \mathbb{H}_{(\pm)}(\zeta) \neq 0 \quad , \quad \zeta \in \Omega^{\pm} \tag{34}$$

Similarly $\mathbb{H}$ has a right factorization with corresponding properties [27].

The notation for $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ follow the convention used in [1], *i.e.* the sign indicates the half plane where any singularities of the tensor and any zeros in the determinant of the corresponding matrix occur.

Consider now the second order symmetric tensor $\mathbf{P}^t(\omega) = \mathbb{H}_-(\omega)\mathbf{E}^t_{r+}(\omega)$, whose components are continuous by virtue of the properties of $\mathbb{H}_-(\omega)$ and $\mathbf{E}^t_{r+}(\omega)$. The Plemelj formulae [19] give that

$$\mathbf{P}^t(\omega) := \mathbb{H}_-(\omega)\mathbf{E}^t_{r+}(\omega) = \mathbf{p}^t_-(\omega) - \mathbf{p}^t_+(\omega) \tag{35}$$

where

$$\mathbf{p}^t(z) := \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbf{P}^t(\omega)}{\omega - z} d\omega \ , \quad \mathbf{p}^t_{\pm}(\omega) := \lim_{\alpha \to 0^{\mp}} \mathbf{p}^t(\omega + i\alpha) \tag{36}$$

Moreover, $\mathbf{p}^t(z) = \mathbf{p}^t_+(z)$ is analytic in $z \in \Omega^{(-)}$ and $\mathbf{p}^t(z) = \mathbf{p}^t_-(z)$ is analytic in $z \in \Omega^{(+)}$. Both are analytic on the real axis by virtue of the assumption in section 3 on the analyticity of Fourier-transformed quantities on the real axis and an argument given in [1]. The minimum free energy has the form

$$\psi_m(t) = \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}^t_-(\omega)|^2\, d\omega =: \tilde{\tilde{\psi}}_m(\mathbf{E}(t), \mathbf{p}^t_-). \tag{37}$$

where $\tilde{\tilde{\psi}}_m$ is the symbol denoting the minimal free energy as a functional of the state variables $\mathbf{E}(t), \mathbf{p}^t_-$. The main developments in [27] are in terms of the history $\mathbf{E}^t$, although the result (37) in terms of the relative history $\mathbf{E}^t_r$ is also given.

It was demonstrated in [27] that ψ_m satisfies both the criteria required of a free energy in the sense of Graffi [20, 21] (also [22, 7]) and in the sense of Coleman and Owen [9, 4].

6. – State in the frequency domain

An argument used in [10], following the first characterization produced in [27] of the state of linear viscoleastic solids in the frequency domain, ensures that for every viscoelastic material with a symmetric relaxation function, a given couple $(\mathbf{E}, \mathbf{E}^t)$ is equivalent to the zero couple $(\mathbf{0}, \mathbf{0}^\dagger)$ if and only if the $\mathbf{p}^t_-$ related to $\mathbf{E}^t_r$, by (35)-(36) is such that

$$\mathbf{p}^t_-(\omega) = 0 \quad , \quad \forall\omega \in \mathcal{R} \tag{38}$$

and $\mathbf{E}(t) = 0$. Because the concept of state (at least in its original version) is stress-based, the response functional (10) has to have a representation formula in terms of $\mathbf{p}^t_-(\omega)$. This can be obtained by noting that by (27) and (23) the stress can be epressed as follows:

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \mathbf{I}^t(0). \tag{39}$$

By particularizing Eqn. (7.7) obtained in [27] (Sect.7) the stress takes the form:

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{q}^t_-(\omega)d\omega, \tag{40}$$

where $\mathbf{q}^t_-(\omega)$ and $\mathbf{p}^t_-(\omega)$ are related as follows (see e.g. [27] Sect. 8, [10], Sect. 6):

$$\mathbf{q}^t_-(\omega) = \mathbf{p}^t_-(\omega) - \frac{\mathbb{H}_-(\omega)}{i\omega}\mathbf{E}(t). \tag{41}$$

An expression for the stress in terms of $\mathbf{E}(t)$ and $\mathbf{p}^t_-$ can also be obtained, just by substituting (41) into (40) to get:

$$\mathbf{T}(t) = \mathbb{G}_\infty\mathbf{E}(t) + \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}^t_-(\omega)d\omega =: \tilde{\tilde{\mathbf{T}}}(\mathbf{E}(t), \mathbf{p}^t_-) \tag{42}$$

and this will be considered in the following Proposition, providing an estimate for the magnitude of the stress in terms of the minimal free energy $\psi_m(t) = \tilde{\tilde{\psi}}_m(\mathbf{E}(t), \mathbf{p}^t_-)$ defined in (37) .

PROPOSITION 1. *Let* $\sigma(t) = (\mathbf{E}(t), \mathbf{p}^t_-)$ *a given state. Then the following estimate holds true:*

$$|\tilde{\tilde{\mathbf{T}}}(\mathbf{E}(t), \mathbf{p}^t_-)|^2 \leq c_0\tilde{\tilde{\psi}}_m(\mathbf{E}(t), \mathbf{p}^t_-), \tag{43}$$

where

$$c_0 := 2\max\{|\mathbb{G}_\infty|, |\mathbb{G}_\infty - \mathbb{G}_0|\} \tag{44}$$

The proof of this Proposition is contained in [11].

In order to set the stage for a Saint-Venant principle a notion of *triple* related to the concept of state discussed in the previous sections may also be introduced.

7. – A Saint-Venant principle for the non inertial case

We consider an open regular bounded cylindrical region $\mathcal{B}_0$, which is occupied by an anisotropic and inhomogeneous linear viscoelastic solid in its reference configuration. The relaxation tensor at every point $\mathbf{x}$ of such a solid is $\mathbb{G}(\mathbf{x}, \cdot)$. It is assumed that $\mathbb{G}$ satisfies the thermodynamic restrictions outlined in section 5; and also that $\mathbb{G}_0(\mathbf{x})$ and $\mathbb{G}_\infty(\mathbf{x})$ are continuous on $\bar{\mathcal{B}}_0$, the closure of $\mathcal{B}_0$. The boundary of $\mathcal{B}_0$ is denoted by $\partial\mathcal{B}_0$ and it is partitioned as follows:

1. lateral surface of the cylinder $\partial\mathcal{B}_{lat}$,
2. "initial" cross section of the cylinder $\mathcal{S}_0$,
3. "final" cross section of the cylinder $\mathcal{S}_l$,

such that

$$\partial\mathcal{B} = \mathcal{S}_0 \cup \partial\mathcal{B}_{lat} \cup \mathcal{S}_l \tag{45}$$

and

$$\partial\mathcal{B} = \mathcal{S}_0 \cap \partial\mathcal{B}_{lat} \cap \mathcal{S}_l = \oslash. \tag{46}$$

We note that the set $\partial\mathcal{B} = \mathcal{S}_0 \cap \partial\mathcal{B}_{lat} =: \partial\mathcal{S}_0$ is the planar curve delimiting the boundary of the initial cross section, whereas $\partial\mathcal{B}_{lat} \cap \mathcal{S}_l =: \partial\mathcal{S}_l$ is the planar curve delimiting the boundary of the final cross section.

Let $\mathbf{u}$ be a displacement vector field defined on $\bar{\mathcal{B}}\times\mathcal{R}$ and let $\mathbf{E}$ be the corresponding strain tensor, i.e. such that $\mathbf{E} = sym\nabla\mathbf{u} \in Sym$. We may denote the current displacement at $\mathbf{x}$ by $\mathbf{u}(\mathbf{x}, t)$ and the history of displacement at the same point with $\mathbf{u}^t(\mathbf{x}, \cdot)$ and we may consider the *triple* $((\mathbf{u}(\mathbf{x}, t), \mathbf{u}^t(\mathbf{x}, \cdot)), (\mathbf{E}(\mathbf{x}, t), \mathbf{E}_r^t(\mathbf{x}, \cdot)), \mathbf{T}(t)))$, where $\mathbf{E}_r^t$ is the relative history defined as in (11). Let

$$\left(\left(\mathbf{u}_1(\mathbf{x}, t), \mathbf{u}_1^t(\mathbf{x}, \cdot)\right), \left(\mathbf{E}_1(\mathbf{x}, t), \mathbf{E}_{r1}^t(\mathbf{x}, \cdot)\right), \mathbf{T}_1(t))\right)$$

and

$$\left(\left(\mathbf{u}_2(\mathbf{x}, t), \mathbf{u}_2^t(\mathbf{x}, \cdot)\right), \left(\mathbf{E}_2(\mathbf{x}, t), \mathbf{E}_{r2}^t(\mathbf{x}, \cdot)\right), \mathbf{T}_2(t))\right)$$

be two triples and let $\mathbf{p}_1^t$ and $\mathbf{p}_2^t$ the corresponding state variables via formula (42). We say that two triples are equivalent if they determine the same state at $\mathbf{x}$, that is to say $\tilde{\mathbf{u}} := \mathbf{u}_1 - \mathbf{u}_2$ is such that the corresponding current strain

$$\tilde{\mathbf{E}}(\mathbf{x}, t) := sym\nabla(\mathbf{u}_1(\mathbf{x}, t) - \mathbf{u}_2(\mathbf{x}, t)), \tag{47}$$

is zero and the state variable

$$\tilde{\mathbf{p}}^t(\mathbf{x}, \omega) := \mathbf{p}_1^t(\mathbf{x}, \omega) - \mathbf{p}_2^t(\mathbf{x}, \omega) \tag{48}$$

is zero for almost all the $\omega \in \mathcal{R}$.

The body is assumed not to have prescribed displacements in any part of its boundary, the body forces are assumed to be zero at every point in $\mathcal{B}_0$ at every time, the lateral surface of the cylinder $\partial\mathcal{B}_{lat}$ is assumed to be traction free ans well as $\mathcal{S}_l$. The only non-zero tractions are assumed to be acting on points belonging to the intial cross section $\mathcal{S}_0$, i.e.

$$\mathbf{s}(\mathbf{x},\tau) \neq \mathbf{0} \text{ for some } \tau \in [0,T],\ \mathbf{x} \in \mathcal{S}_0. \tag{49}$$

where T is some given positive number and $[0,T]$ is a bounded and closed time interval during which the abovementioned tractions are applied. It is necessary for the equilibrium of the solid for those tractions to be self equilibrated, i.e.:

$$\begin{aligned} \mathbf{R}(t) &= \int_{\mathcal{S}_0} \mathbf{s}(\mathbf{x},t)\, dS = \underset{\sim}{0} \\ \mathbf{M}(t) &= \int_{\mathcal{S}_0} \mathbf{x} \times \mathbf{s}(\mathbf{x},t)\, dS = \underset{\sim}{0}; \end{aligned} \tag{50}$$

here $\mathbf{R}(t)$ and $\mathbf{M}(t)$ denote the total load and moment acting on $\mathcal{S}_0$ respectively, $-\mathbf{n}$ is the outward normal on $\mathcal{S}_0$, and $\mathbf{n}$ denotes the unit vector parallel to the axis of the cylinder and orienting then axis from $\mathcal{S}_0$ to $\mathcal{S}_l$.

Let r be a fixed value of the rectilinear abscissa mapping the axis of the body oriented as above. Consider now an open subregion $\mathcal{B}_r$ of $\mathcal{B}_0$ such that

$$\mathbf{x} \in \mathcal{B}_r \text{ if } \mathbf{x}\cdot\mathbf{n} > r. \tag{51}$$

The initial cross section $\mathcal{S}_r$ of $\mathcal{B}_r$ is the set of points in $\mathcal{B}_0$ such that

$$\mathbf{x} \in \mathcal{B}_0 \text{ if } \mathbf{x}\cdot\mathbf{n} = r, \tag{52}$$

the final cross section remains $\mathcal{S}_l$, whereas its lateral boundary $\partial\mathcal{B}_r$ is a subset of $\partial\mathcal{B}_0$ whose points $\mathbf{x}$ satisfy $\mathbf{x}\cdot\mathbf{n} \geq r$. Necessary conditions for the equilibrium of $\mathcal{B}_r$ are

$$\begin{aligned} \int_{\mathcal{S}_r} \mathbf{s}(\mathbf{x},t)\, dS &= \mathbf{R}(t) \\ \int_{\mathcal{S}_r} \mathbf{x} \times \mathbf{s}(\mathbf{x},t)\, dS &= \mathbf{M}(t) \end{aligned} \tag{53}$$

Let $\left(\left(\mathbf{u}(\mathbf{x},t), \mathbf{u}^t(\mathbf{x},\cdot)\right), \left(\mathbf{E}(\mathbf{x},t), \mathbf{E}^t_r(\mathbf{x},\cdot)\right), \mathbf{T}(t)\right)$ be any triple determining the given state $\sigma(t) = (\mathbf{E}(t), \mathbf{p}^t_-)$ of each point of the solid at the beginning of loading. In other words, t corresponds to the initial time $\tau = 0$ from which the surface tractions specified in 36 may be applied on $\mathcal{S}_0$. These tractions do induce a deformation process $\mathbf{E}_P$ at each point of the body: let $\{\mathbf{u}_P(\mathbf{x},\tau), \mathbf{E}_P(\mathbf{x},\tau), \mathbf{T}(\mathbf{x},\tau)\}$ the triple associated with the induced deformation process, i.e. $\mathbf{E}_P = sym\nabla\mathbf{u}_P \in Sym$ and $\mathbf{T}(\mathbf{x},\tau) = \tilde{\mathbf{T}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E})^{t+\tau})$.

From now on the triple $\{\mathbf{u}_P(\mathbf{x},\tau), \mathbf{E}_P(\mathbf{x},\tau), \mathbf{T}(\mathbf{x},\tau)\}$ will be denoted by $\{\mathbf{u}, \mathbf{E}, \mathbf{T}\}$. The fields $\mathbf{u}, \mathbf{E}, \mathbf{T}$ will be considered to satisfy the balance of linear momentum:

$$\nabla\cdot\mathbf{T}(\mathbf{x},t) = 0, (\mathbf{x},t) \in \mathcal{B}_0 \times [0,T]. \tag{54}$$

with boundary conditions

$$\mathbf{s}(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial\mathcal{B}_0\backslash\mathcal{S}_0, t \in [0,T] \tag{55}$$

and the global balance (50), i.e.

$$(56)\qquad \begin{aligned} \int_{\mathcal{S}_0} \mathbf{s}(\mathbf{x},t)\, dS &= \int_{\mathcal{S}_r} \mathbf{s}(\mathbf{x},t)\, dS = 0 \\ \int_{\mathcal{S}_0} \mathbf{x}\times\mathbf{s}(\mathbf{x},t)\, dS &= \int_{\mathcal{S}_r} \mathbf{x}\times\mathbf{s}(\mathbf{x},t)\, dS = 0, \quad t\in[0,T]. \end{aligned}$$

where (54) holds true by neglecting the inertia forces.

Because $\psi_m(t) = \tilde{\tilde{\psi}}_m(\mathbf{E}(t), \mathbf{p}^t_-)$ is the minimal free energy per unit volume we may define the foollowing function:

$$(57)\qquad \Psi_m(r;T) := \int_0^T \int_{\mathcal{B}_r} \psi_m(\tau)\, dV d\tau$$

The result of Berdichevskii [28] is needed in order to prove the Saint-Venant principle. Berdichevskii's theorem ensures that for all vector fields $\mathbf{v}$ on a bounded domain Γ that satisfies the constraints

$$(58)\qquad \int_\Gamma \mathbf{v}\, dV = 0, \qquad \int_\Gamma \mathbf{x}\times\mathbf{v}\, dV = 0$$

the inequality

$$(59)\qquad b\int_\Omega |\mathbf{v}|^2\, dS \le \int_\Gamma \mathbf{E}\cdot\mathbb{G}_\infty\mathbf{E}\, dV$$

holds where $\mathbf{E} = sym\nabla\mathbf{v} \in Sym$, $\Omega\subset\partial\Gamma$ and b is a constant depending on Γ, Ω and the positive-definite tensor $\mathbb{G}_\infty \in Sym(Sym)$.

PROPOSITION 2. *Suppose that the relaxation tensor* $\mathbb{G}$ *satisfies the conditions* (63) *and let* $\mathbf{u}, \mathbf{E}, \mathbf{T}$ *be any triple related to a process satisfying* (54) - (56). *Then for a general history*

$$(60)\qquad \Psi_m(r;T) \le \Psi_m(0;T)e^{-r/\alpha}, \qquad 0\le r\le L-l$$

where

$$(61)\qquad \alpha = \frac{4c_0}{\beta}, \quad \beta = \min_{0\le r\le L-l} b(r), \quad l>0$$

whenever

$$(62)\qquad \int_0^T \psi_m(t)dt \le \int_0^T \mathbf{T}(t)\cdot\mathbf{E}(t)dt,'$$

where c_0 *is defined by* (44) *and* $b(r)$ *is the optimal choice of the constant in* (59) *for* $\Omega = \mathcal{S}_r$ *and* $\Gamma = \mathcal{B}_r$.

Analoguous conclusions were obtained in [26], although the just stated Proposition generalizes the decay estimate to a family of energy measures involving free energies which are also functions of state. In particular, because ψ_m establishes the $L^2 - type$ coarser possible norm in the space of the states (see e. g. [27]), the spatial decay estimate (60) entails the estimate of the *spatial decay of the state of the material points over the whole body.*

It is worth remarking that (62) certainly holds when the relaxation function obeys the following restrictive assumptions:

$$(63) \qquad \dot{\mathbb{G}}(t)\mathbf{E}\cdot\mathbf{E} < 0, \quad \ddot{\mathbb{G}}(t)\mathbf{E}\cdot\mathbf{E} \geq 0, \quad \forall\mathbf{E} \in Sym\setminus\{\mathbf{0}\},\ \forall t \in \mathcal{R}^+$$

These assumption are sufficient for the Graffi-Volterra function ψ_G to be a free energy according to the Graffi's definition. Because ψ_m is the minimal free energy also according to both the Coleman-Owen and the Graffi's definition, we obviously have $\psi_m \leq \psi_G$. When (63) are verified, inequality (62) follows as it has been proved in [26], where ψ_G replaces ψ_m. The authors in [11] are exploring the general implications of inequality (62).

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Gentili's norm on the process and state spaces in linear viscoelasticity

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1. – Introduction

In this paper we study an a priori characterization of the process and state spaces for a system with memory. In particular, we want to put in evidence the meaning and importance of determining a natural norm on the space of processes and states, which has been introduced by Gentili in [15] for a material with fading memory. We will show how the idea, considered by Gentili in [15], of introducing a topology first on the space of the processes through the continuity of the work provides the definition of a topology on the space of the states, which Gentili proves to be the dual one. This point of view seems particularly useful in the study of PDEs connected with such problems. For this reason this paper ends with an application to the integro-differential equation characteristic of viscoelasticity. For such problem, by means of this new spaces and without any hypothesis on the value of $G'(0)$ and on the regularity of $G''(s)$, but using the thermodynamic restriction

$$G'_s(\omega) = \int_0^\infty G'(s) \sin \omega s ds < 0 \ , \quad \text{for all } \omega \in \mathbb{R}^{++}$$

we are able to prove the existence, uniqueness and asymptotic decay of the solutions.

2. – Fading memory and thermodynamics

Consider a material with fading memory occupying the domain $\Omega \in \mathbb{R}^3$. This material is defined by a constitutive equation which relates the stress tensor T and the deformation gradient F by a functional of the type

$$T(x,t) = \hat{T}(F^t(x)),$$

where $F^t(x,s) = F(x,t-s), s \in \mathbb{R}^+$, is the history of F. In the linear case

$$\begin{aligned} T(x,t) &= G_0(x)E(x,t) + \int_0^\infty G'(x,s)E^t(x,s)\, ds \\ &= G_\infty(x)E(x,t) + \int_0^\infty G'(x,s)(E^t(x,s) - E(x,t))ds, \end{aligned} \tag{1}$$

where $E = \frac{\nabla u + (\nabla u)^T}{2}$ is the infinitesimal strain tensor.
Moreover, $G_0(x)$ and $-G'(x,s)$ are symmetric and positive define tensors, as well as

$$G_\infty(x) := G_0(x) + \int_0^\infty G'(x,s)ds$$

Henceforth, it is understood that the statements are relative to any fixed point $x \in \Omega$.

DEFINITION 1 A viscoelastic material is characterized by the constitutive equation $T(t) = \hat{T}(E^t)$ such that

a - the domain $\mathcal{D}$ of $\hat{T}$ is a set of histories so that

$$\mathcal{D} \supset L^\infty(\mathbb{R}^+)$$

moreover, for any $E^t \in \mathcal{D}$, the static continuation $E^t_a \in \mathcal{D}$, where

$$E^t_a(s) = \begin{cases} E^t(s-a), & \text{if} \quad s > a \\ E(t), & \text{if} \quad s \leq a \end{cases}$$

b - there exists a constitutive equation $\tilde{T}(E(t))$ of an elastic material such that

$$\lim_{a \to \infty} \hat{T}(E^t_a) = \tilde{T}(E(t))$$

c - if $E^t \in \mathcal{D}$, then the static relaxation ${}_bE^t \in \mathcal{D}$, where

$${}_bE^t(s) = \begin{cases} E^t(s), & 0 \leq s < b \\ E(t-b), & b \leq s \end{cases}$$

and

$$\lim_{b \to \infty} \hat{T}({}_bE^t) = \hat{T}(E^t)$$

A map $P : [0, d_p) \to \mathrm{Sym}$ piecewise continuous on $[0, d_p)$ and defined as:

$$P(t) = \dot{E}(t), \qquad t \in [0, d_p)$$

is called *kinetic process* of duration $d_p \in \mathbb{R}^+$. In the following we use the notation $P_{[t_1,t_2)}$ in order to denote the restriction of P to $[t_1, t_2) \subset [0, d_p)$. For ease in writing P_t stands for $P_{[0,t)}$. Given two processes P_1, P_2 the composition $P_1 * P_2$ of P_1 with P_2 is defined as .

$$P_1 * P_2(t) = \begin{cases} P_1(t)\ , & if \quad t \in [0, d_{p_1}) \\ P_2(d_{p_1} - t), & if \quad t \in [d_{p_1}, d_{p_1} + d_{p_2}) \end{cases}$$

DEFINITION 2 A fading memory material is given by the set $(\Pi, \Sigma, \hat{\rho}, \hat{T})$

a - Π is the set of kinetic processes defined as: $\Pi = \{P : [0, d_p) \to \mathrm{Sym}$ piecewise continuous and such that if $P_1, P_2 \in \Pi$, then $P_1 * P_2 \in \Pi$; and if $P \in \Pi$, $P_{[t_1,t_2)} \in \Pi$, $[t_1, t_2) \subset [0, d_p)\}$

b - Σ is the state space, whose elements are given by $\sigma = (E^t)$ and called states.

c - the map $\hat{\rho} : \Sigma \times \Pi \to \Sigma$ is called evolution function, such that if σ^i is the initial state and P is a process: $\hat{\rho}\,(\sigma^i, P) = \sigma^f$,

d - the map $\hat{T} : \Sigma \to \mathrm{Sym}$ is given by the constitutive equation:

$$T(t) = \hat{T}\left(E^t\right)$$

For any state σ and process P, the function $\hat{\rho}$ determines the one parameter family of states $\sigma(t) = \hat{\rho}(\sigma, P_t)$, $t \in [0, d_p)$.

If $\hat{\rho}(\sigma, P) = \sigma$, then the family of states is called a cycle. Moreover, we consider the space

$$\Sigma_\sigma = \{\sigma' \in \Sigma\, ; \exists\ P \in \Pi,\ \ such\, that\, \sigma' = \hat{\rho}(\sigma, P)\}$$

Second Law for isothermal processes
For every cycle $(\sigma, P) \in \Sigma \times \Pi$ *the inequality*

$$\oint_0^d T(\sigma(\tau), P(\tau)) \cdot L(\tau)\ d\tau \geq 0 \tag{2}$$

holds.

For materials with fading memory, cycles are quite rare, because usually the material gets to a state, which is different from the initial state, although "close" to it. This is the reason why it is more convenient to use the following (see[3])

Strong Form of the Second Law (for isothermal processes).
The set of the works done in passing from σ *to any state* $\sigma' \in \Sigma_\sigma$

$$\mathcal{W}(\sigma) := \{W(\sigma, P)\, ;\ P \in \Pi\} \tag{3}$$

is bounded below. There exists a state $\sigma^\dagger$*, called zero state, such that*

$$\inf \mathcal{W}\left(\sigma^\dagger\right) = 0,\ \ and\ \ W\left(\sigma^\dagger, P\right) > 0\ ,\ \ for\, all\ P \neq 0$$

REMARK 1 For materials with fading memory the zero state is given by the history $E^\dagger(s) = 0$ for all $s \in [0, \infty)$.

As show in ([11]) from the Second Law we have

$$G'_s(\omega) = \int_0^\infty G'(s) \sin \omega s ds < 0\ ,\ \ for\, all\ \omega \in \mathbb{R}^{++} \tag{4}$$

while from Strong Form of the Second Law, we have

$$G_\infty > 0 \tag{5}$$

DEFINITION 3 A function $\psi : \mathcal{S}_\psi \to \mathbf{R}^+$ is called free energy if

a - the domain $\mathcal{S}_\psi \subset \mathcal{D}$ is invariant under ρ, namely for every $\sigma_1 \in \mathcal{S}_\psi$ and $P \in \Pi$, the state $\sigma = \hat{\rho}(\sigma_1, P) \in \mathcal{S}_\psi$, $\sigma^\dagger \in \mathcal{S}_\psi$, and $\psi\left(\sigma^\dagger\right) = 0$,

b - for any pair σ_1 , $\sigma_2 \in \mathcal{S}_\psi$ and $P \in \Pi$ such that $\hat{\rho}(\sigma_1, P) = \sigma_2$ we have:

$$\psi(\sigma_2) - \psi(\sigma_1) \leq W(\sigma_1, P).$$

In linear viscoelasticity there are many free energies. The family $\mathcal{F}$ of the free energies is a convex set. $\mathcal{F}$ has a minimum and a maximum element ψ_M, ψ_m . The maximum free energy was considered in [3]

$$\begin{aligned}\psi_M\left(E^t\right) = &\ \frac{1}{2}G_\infty E\left(t\right)\cdot E\left(t\right)\\ &-\frac{1}{2}\int_0^\infty\int_0^\infty G'\left(|s-s'|\right)\left(E^t\left(s\right)-E\left(t\right)\right)\cdot\left(E^t\left(s\right)-E\left(t\right)\right)dsds'\end{aligned} \tag{6}$$

3. – Maximum Recoverable Work and Minimum Free Energy

From here on, we denote with $G(|s|)$ the extension of $G(s)$ to an even function on IR, and we suppose any process $P\in\Pi$ defined over all $[0,\infty)$, by means of the trivial extension

$$P(t)=\begin{cases} P(t)\ , & t\in[0,d_p)\\ 0, & t\in[d_p,\infty)\end{cases} \tag{7}$$

This new space will be denoted by $\overset{\circ}{\Pi}$. Now let us consider the work $W(\sigma_0,\tilde{P})$, where $\sigma_0=(\mathbf{E}^0)$ is the history in $t=0$, and $\tilde{P}\in\overset{\circ}{\Pi}$ is a process such that $P(t)=E(t),\ t\in[0,d_p)$.

We regard $(E^t)=\hat{\rho}(E^0,P_t)$ and the stress

$$T(E^t)=G_0E(t)+\int_0^t G'(s)E^t(s)\,ds+I_0(t,E^0) \tag{8}$$

where

$$\tilde{I}^0(t,E^0)=-\int_0^\infty G'(t+\tau)E^0(\tau)\,d\tau \tag{9}$$

Moreover, from the hypothesis (7) on P, there exists the limit $E(\infty)=\lim_{t\to+\infty}E(t)$, and

$$\begin{aligned}W(\sigma,P)\ &=\int_0^\infty\left\{G_0E(t)+\int_0^t G'(s)E^t(s)\,ds\right\}\cdot\dot{E}(t)\,dt-\int_0^\infty\tilde{I}^0(t,E^0)\cdot\dot{E}(t)\,dt\\ &=\int_0^\infty(G_0E(t)\cdot\dot{E}(t)dt\\ &\quad+\int_0^\infty\left\{G(s)E(t-s)\ |_0^t+\int_0^t G(s)\dot{E}^t(s)\,ds)\right\}\cdot\dot{E}(t)\,dt\\ &\quad-\int_0^\infty\tilde{I}^0(t,E^0)\cdot\dot{E}(t)\,dt\\ &=\frac{1}{2}\int_0^\infty\int_0^\infty G(|t-\tau|)\dot{E}(t)\cdot\dot{E}(\tau)d\tau dt-\int_0^\infty I^0(t,E^0)\cdot\dot{E}(t)\,dt\end{aligned} \tag{10}$$

where

$$I^0(t,E^0)=-G(t)E(0)+\tilde{I}^0(t,E^0)\,. \tag{11}$$

In order to obtain the maximum recoverable work from the state $\sigma_0=(E^0)$, we consider the maximum of $-W(\sigma_0,P)$ respect to the set of functions $E(t)\in\mathcal{M}=L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)\cap C^1(\mathbb{R}^+)$ given by

$$E(t)=E^{(m)}(t)+\varepsilon e(t)\ \ ,\quad t\in[0,\infty)$$

where ε is a real parameter and $e \in \mathcal{M}$ is such that $e(0) = 0$. If $\dot{E}^{(m)}$ is a process for which we obtain the maximum recoverable work, we have

$$\frac{d}{d\varepsilon}\left[-W(\sigma_0, P)\right]_{\varepsilon=0} = \tag{12}$$
$$-\int_0^\infty \int_0^\infty G(|t-\tau|)\dot{E}^{(m)}(t) \cdot \dot{e}(\tau)\, d\tau\, dt + \int_0^\infty \tilde{I}^0(t, E^0) \cdot \dot{e}(t)\, dt = 0$$

by the arbitrariness of $\dot{e}(t)$, necessarily we obtain

$$\int_0^\infty G(|t-\tau|)\dot{E}^{(m)}(\tau)\, d\tau = I^0(t, E^0). \tag{13}$$

The equation (13) is a Wiener-Hopf equation, whose solution makes maximum the recovered work. Because

$$\psi_m(\sigma) = -\min\left\{W(\sigma, P),\ \forall P \in \Pi\right\},$$

we have from (10), (13)

$$\psi_m(E^0) = \frac{1}{2}\int_0^\infty \int_0^\infty G(|t-\tau|)\dot{E}^{(m)}(t) \cdot \dot{E}^{(m)}(\tau) d\tau dt, \tag{14}$$

where $\dot{E}^{(m)}$ is now the solution of the equation (13).

The minimum free energy was determined by Golden in [4]. He begins by considering the two spaces $\Omega^+, \Omega^{(+)}$ of the complex plane Ω, defined by

$$\Omega^+ = \left\{\omega \in \Omega\, ;\ \mathrm{Im}\,\omega \in \mathbb{R}^+\right\}$$

$$\Omega^{(+)} = \left\{\omega \in \Omega\, ;\ \mathrm{Im}\,\omega \in \mathbb{R}^{++}\right\}$$

Similarly, Ω^- and $\Omega^{(-)}$ are the lower half-planes including and excluding the real axis, respectively.

For any $f \in L^2(\mathbb{R})$, we denote its Fourier transforms by

$$\hat{f}(\omega) = \int_{-\infty}^\infty \exp(-i\omega u) f(u) du = f_+(\omega) + f_-(\omega)$$

where

$$f_+(\omega) = \int_0^\infty \exp(-i\omega u) f(u) du, \quad f_-(\omega) = \int_{-\infty}^0 \exp(-i\omega u) f(u) du$$

The quantities $f_\pm$ are analytic in $\Omega^{(\mp)}$ respectively.

In the following we denote with $f^F(t)$ the inverse Fourier trasform of $\hat{f}(\omega)$ defined as

$$\mathcal{F}^{-1}(\hat{f})(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(\omega)\exp(i\omega t) d\omega$$

Consider the new function

$$H(\omega) = -\omega G'_s(\omega) \tag{15}$$

It is a positive, even function of the frequency. We have [14]

$$G'(0) = -\lim_{\omega\to\infty} H(\omega) \tag{16}$$

If $G(s),\ s \in \mathbb{R}^+$ is extended to the even function $G(|s|)$ on $\mathbb{R}$, then $G'(|s|)$ is an odd function with Fourier transform given by

$$\hat{G}'(\omega) = -2iG'_s(\omega) \tag{17}$$

We will be using the Fourier transforms of the strain history and continuation defined by

$$\begin{aligned} E^t_+(\omega) &= \int_0^\infty \exp(-i\omega s)\, E^t(s)ds \\ E^t_-(\omega) &= \int_{-\infty}^0 \exp(-i\omega s)\, E^t(s)ds \end{aligned} \tag{18}$$

The quantity E^t_+ is analytic on $\Omega^{(-)}$, while E^t_- is analytic on $\Omega^{(+)}$. They are assumed to be analytic on $\mathbb{R}$ and analytic at infinity, so that $E^t(0)$ is finite. Assuming that the strain history has a derivative which is continuous and belongs to $L^1(\mathbb{R}^+)$, then

$$\frac{d}{dt}E^t_+(\omega) = -i\omega E^t_+(\omega) + E(t) \tag{19}$$

It is well known that (1) can be written in the form

$$T = G_0 E - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{H(\omega)}{\omega} E^t_+(\omega) d\omega \tag{20}$$

where the oddness of H/ω has been used.

Now consider the Wiener-Hopf equation 13 (see [14])

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} G(|t-\tau|)E^0(\tau)d\tau &= \mathrm{r}(t) \\ E^0(\tau) &= E^{(m_1)}(-\tau), \qquad \tau \in \mathbb{R}^{--} \\ \mathrm{r}(t) &= 0, \qquad\qquad t \in \mathbb{R}^{-} \end{aligned} \tag{21}$$

This relation defines r on $\mathbb{R}^+$. Taking Fourier transforms and multiplying across by ω, we obtain, with the aid of (17) and (15),

$$2iH(\omega)\left(E^0_+(\omega) + E^{(m)}(\omega)\right) = \omega \mathrm{r}_+(\omega) \tag{22}$$

where $E^{(m)}(\omega)$ is the Fourier transform of $E^0(\tau)$ defined by (21) on $\mathbb{R}^{--}$ and is the quantity we wish to determine. Also, $\mathrm{r}_+(\omega)$ is analytic on ω and by assumption also on $\mathbb{R}$.

Following [5], the tensor H (which is isomorphic to a matrix in $\mathbb{R}^6 \times \mathbb{R}^6$) can be factorized as follows: $H(\omega) = H_+(\omega)H_-(\omega)$, where $H_\pm$ is analytic. We multiply (22) by $[H_+(\omega)]^{-1}$ to obtain

$$H_-(\omega)\left(E^0_+(\omega) + E^{(m)}(\omega)\right) = \frac{\omega}{2i}\left[H_+(\omega)\right]^{-1} \mathrm{r}_+(\omega) \tag{23}$$

With the aid of the Plemelj formulae [12], we write

$$\begin{aligned} Q(\omega) := &\quad H_-(\omega)E_+^0(\omega) = q_-(\omega) - q_+(\omega) \\ q_\pm(\omega) &= \lim_{z\to\omega^\mp} q \\ q(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{Q(\omega')}{\omega' - z}d\omega' \end{aligned} \tag{24}$$

where q_- is analytic on $\Omega^{(+)}$ and q_+ is analytic on ω. Substituting (24) into (23) we obtain

$$K(\omega) = q_-(\omega) + H_-(\omega)E^{(m)}(\omega) = q_+(\omega) + \frac{\omega}{2i}\left[H_+(\omega)\right]_+^{-1} \mathrm{r}(\omega) \tag{25}$$

The function $K(\omega)$ is analytic on ω^- by virtue of the first relation and analytic on ω^+ by virtue of the second. It is therefore analytic over the entire complex plane. By Liouville's theorem it must be a polynomial. However, for $|\omega| \to \infty$, $K(\omega) \to 0$ as $1/\omega$ since q_- and $E^{(m)}$ have this property. Hence, it must vanish everywhere so that

$$H_-(\omega)E^{(m)}(\omega) + q_-(\omega) = 0 \tag{26}$$

By application of the convolution theorem, Plancherel's theorem and (26), the minimum free energy may be represented in the form

$$\begin{aligned} \psi_m(E^0) &= S(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)E^{(m)}(\omega)\cdot \bar{E}^{(m)}(\omega)d\omega \\ &= S(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty} q_-(\omega)\cdot \bar{q}_-(\omega)d\omega \end{aligned} \tag{27}$$

where $S(0) = T(t)\cdot E(t) - \frac{1}{2}G_0E(t)\cdot E(t)$.

4. – Notion of equivalence states

In a material with fading memory the state is usually defined by means of the history E^t. In this section we show that it is possible to arrive to a new definition of state for this materials. Following [1]

DEFINITION 4 Two strain histories E_1^t , E_2^t are said to be equivalent if $E_1(t) = E_2(t)$ and for every process $\dot{E}^P : [0, d_P) \to Sym$, they satisfy

$$\hat{T}\left(E_1^t, \dot{E}^P\right) = \hat{T}\left(E_2^t, \dot{E}^P\right). \tag{28}$$

Gentili in [15] proved as the last identity (28) is equivalent to the following

$$W\left(E_1^t, \dot{E}^P\right) = W\left(E_2^t, \dot{E}^P\right).$$

In the linear case, Del Piero and Deseri [8] observed that two histories E_1^t , E_2^t are equivalent if and only if $E_1(t) = E_2(t)$ and

$$\int_0^\infty G'(s+\tau)E_1^t(s)ds = \int_0^\infty G'(s+\tau)E_2^t(s)ds, \ \forall\tau \geq 0$$

In [16] the state is represented by the pair

$$\sigma = (E(t), \tilde{I}^t(\tau))$$

where

$$\tilde{I}^t(\tau) = -\int_0^\infty G'(s+\tau)E^t(s)ds$$

In order to obtain the notion of minimal state, we consider the function

$$I^t(\tau) = -G(\tau)E(t) - \int_0^\infty G'(s+\tau)E^t(s)ds$$

The state $\sigma = (E(t), \tilde{I}^t(\tau))$ can be identify by means of the function $I^t(\tau, E^t)$. So that, $\lim_{\tau\to\infty} I^t(\tau, E^t) = -G_\infty E(t)$, and $\tilde{I}^t(\tau) = I^t(\tau, E^t) - \lim_{\tau\to\infty} I^t(\tau, E^t)$. The function $I^t(\tau)$ is able to rapresent the equivalent class of histories, because any hystory which belongs to this class have the same value of $I^t(\tau)$. For this reason $I^t(\tau)$ will be called minimal state.

If P_τ^0 denotes process $P_\tau^0(t) = \dot{E}^0(t) = 0$, for $t \in [0, \tau)$, then

$$\begin{aligned} I^t(\tau, E^t) &= -T(E^t, P_\tau^0) = -\left(G_0 E(t+\tau) + \int_0^\infty G'(s)E^{t+\tau}(s)ds\right) \\ &= -\left(G_0 E(t) + \int_0^{t+\tau} G'(s)E(t)ds + \int_{t+\tau}^\infty G'(s)E^{t+\tau}(s)ds\right) \end{aligned}$$

In other words the quantity $I^t(\tau)$ describes the stress associated to the static continuation in the interval $[0, \tau)$.

Gentili considered in [15] the following definition of finite work process, from which he will be able to obtain a natural topology on the process space.

A process $\dot{E}^P : [0, \infty) \to Sym$ is said to be a finite work process if

$$W(0^\dagger, \dot{E}^P) = \int_0^{d_P} T(0^\dagger, \dot{E}^P_{[0,\tau)}) \cdot \dot{E}^P(\tau)d\tau < \infty$$

From the Strong Dissipation Principle, we have for any $\dot{E}^P \neq 0$

$$W(0^\dagger, \dot{E}^P) > 0.$$

Moreover, Gentili [15] observes that the work $W(0^\dagger, \dot{E}^P)$ can be written as

$$\begin{aligned} W(0^\dagger, \dot{E}^P) &= \frac{1}{2}G_\infty E(d_P) \cdot E(d_P) + \\ &+ \frac{1}{2}\int_0^\infty \int_0^\infty \check{G}(|\tau - s|)\dot{E}^P(\tau) \cdot \dot{E}^P(s)d\tau ds + \\ &= \frac{1}{2}G_\infty E(d_P) \cdot E(d_P) + \frac{1}{2\pi}\int_0^\infty \check{G}_c(\omega)\dot{E}_+^P(\omega) \cdot \overline{\dot{E}_+^P(\omega)}d\omega \end{aligned}$$

Then, he defines the process space as the set

$$\mathcal{H}_G(\mathbb{R}^+) = \left\{\dot{E}^P(x);\ \int_0^\infty \check{G}_c(\omega)\dot{E}_+^P(\omega) \cdot \overline{\dot{E}_+^P(\omega)}d\omega < \infty\right\}$$

when we consider the norm

$$\left\|\dot{E}^P\right\|^2_{\mathcal{H}_G} = \int_0^\infty \breve{G}_c(\omega)\dot{E}^P_+(\omega)\cdot\overline{\dot{E}^P_+(\omega)}d\omega$$

then the set $\mathcal{H}_G(\mathbb{R}^+)$ becomes a Hilbert space.

The domain of definition of the states is the set of all strain histories rendering the work well defined, when the process belongs to $\mathcal{H}_G(\mathbb{R}^+)$. Gentili in [15] proves that

$$W(E^t,\dot{E}^P) = \frac{1}{2}\int_0^\infty\int_0^\infty G(|\tau-s|)\dot{E}^P(\tau)\cdot\dot{E}^P(s)d\tau ds + \int_0^\infty I^t(\tau,E^t)\cdot\dot{E}^P(\tau)d\tau$$

or equivalently

$$\begin{aligned} W(E^t,\dot{E}^P) &= \frac{1}{2}G_\infty E(\infty)\cdot E(\infty) + \frac{1}{2}\int_0^\infty\int_0^\infty \breve{G}(|\tau-s|)\dot{E}^P(\tau)\cdot\dot{E}^P(s)d\tau ds \\ &\quad + G_\infty E(0)\cdot E(\infty) - \frac{1}{2\pi}\int_{-\infty}^\infty \tilde{I}^t_+(\omega)\cdot\overline{\dot{E}^P_+(\omega)}d\omega < \infty, \end{aligned}$$

where $\tilde{I}^t_+(\omega,E^t) = \int_0^\infty \tilde{I}^t(\tau,E^t)e^{i\omega\tau}d\tau$.
Therefore the set of admissible states $(E(0),\tilde{I}^t(\cdot))$ belongs to the set $Sym\times\mathcal{H}'_G(\mathbb{R}^+)$, where $\mathcal{H}'_G(\mathbb{R}^+)$ is the dual of $\mathcal{H}_G(\mathbb{R}^+)$, namely

$$\mathcal{H}'_G(\mathbb{R}^+) = \left\{\tilde{I}^t(\cdot)\,;\ \int_0^\infty \tilde{I}^t(\tau)\cdot\dot{E}^P(s)d\tau < \infty, \forall\dot{E}^P\in\mathcal{H}_G(\mathbb{R}^+)\right\}$$

On this space, we consider the norm

$$\begin{aligned} \left\|\tilde{I}^t(\cdot)\right\|^2_{\mathcal{H}'_G} &= \int_0^\infty\int_0^\infty \breve{G}^*(|\tau-s|)\tilde{I}^t(\tau)\cdot\tilde{I}^t(s)d\tau ds \\ &= \frac{1}{2\pi}\int_0^\infty \breve{G}_c^{-1}(\omega)\tilde{I}^t_+(\omega)\cdot\overline{\tilde{I}^t_+(\omega)}d\omega \end{aligned} \tag{29}$$

where $\breve{G}^*(s) = \mathcal{F}^{-1}(\breve{G}_c^{-1})(s)$.

Finally, we can prove that there exists a new free energy Ψ_c, obtained by means of the maximum recoverable work on the process space $\mathcal{H}_G(\mathbb{R}^+)$, given by

$$\Psi_c(I^t(\tau)) = \frac{1}{2}G_\infty E(t)\cdot E(t) + \frac{1}{2}\int_0^\infty\int_0^\infty \breve{G}^*(|\tau-s|)\tilde{I}^t(\tau)\cdot\tilde{I}^t(s)d\tau ds$$

5. – Applications to linear viscoelasticity

Consider the partial differential equation connected with a viscoelastic material

$$\begin{aligned} \ddot{u}(x,t) &= \nabla\cdot(G_0(x)\nabla u(x,t) + \int_0^\infty G'(x,s)\nabla u^t(x,s)ds) \\ &= \nabla\cdot(G_0(x)\nabla u(x,t) + \int_0^t G'(x,s)\nabla u^t(x,s)ds) + f(x,t) + \nabla\cdot\tilde{I}^0(x.t), \end{aligned} \tag{30}$$

where $f(x,t)$ is a given function and $\tilde{I}^0(x,t) = \int_0^\infty G'(x,s+t)\nabla u^{t=0}(x,s)ds$. Moreover, we assume the initial conditions

$$u(x,0) = u_0(x),\ \ \dot{u}(x,0) = \dot{u}_0(x)\ \ ;\ \ u^{t=0}(x,s) = u^0(x,s),\ s\in\mathbb{R}^+ \tag{31}$$

and the boundary conditions

(32) $$u|_{\partial\Omega} = 0,$$

In order to obtain a rigorous definition of weak solution in the time interval $\mathbb{R}^+$, we have to introduce the space functions

$$\mathcal{G}(\mathbb{R}^+, L^2(\Omega)) = \Big\{ v \in L^2_{loc}(\mathbb{R}^+, L^2(\Omega)); \int_0^{\infty} (1+\omega^2) \int_{\Omega} \left|\omega^{-1}\hat{G}'_s(\omega)\right| \hat{v}(\omega)(\hat{v}(\omega))^* d\omega dx < \infty \Big\},$$

where $\hat{v}(\omega)$ is the Fourier transform of the causal function $v \in L^2_{loc}(\mathbb{R}^+, L^2(\Omega))$. Moreover, we denote by

$$\mathcal{L}(\mathbb{R}^+, H^1_0(\Omega)) = \left\{ u \in L^2_{loc}(\mathbb{R}^+, H^1_0(\Omega)), \dot{\mathrm{u}}, \nabla \mathrm{u}, \in \mathcal{G}(\mathbb{R}^+, \mathrm{L}^2(\Omega)) \right\}$$

and the dual of $\mathcal{G}(\mathbb{R}^+, L^2(\Omega))$ by $\mathcal{G}'(\mathbb{R}^+, L^2(\Omega))$. Moreover let me consider the new space $\hat{\mathcal{L}}(\mathbb{R}, H^1_0(\Omega))$ obtained as the set of the Fourier transforms of all $u \in \mathcal{L}(\mathbb{R}^+, H^1_0(\Omega))$. Finally, we denote with $\hat{\mathcal{L}}'(\mathbb{R}, H^1_0(\Omega))$ the dual of $\hat{\mathcal{L}}(\mathbb{R}, H^1_0(\Omega))$. Of course Plancherel's theorem for the Fourier transforms defines a natural isomorphism between $\mathcal{L}(\mathbb{R}^+, H^1_0(\Omega))$ and $\hat{\mathcal{L}}(\mathbb{R}, H^1_0(\Omega))$.

DEFINITION 5 A function $u \in \mathcal{L}(\mathbb{R}^+, H^1_0(\Omega))$ is called a weak solution of the initial-boundary value problem (30)-(32) with data $u_0 \in H^1(\Omega)$, $\dot{u}_0 \in L^2(\Omega)$, and $u^0(x, \cdot)$ such that $\tilde{I}^{t=0}(x,\tau) = -\int_0^\infty G'(x, s+\tau) u^0(x,s) ds \in \mathcal{H}'_G(\mathbb{R}^+, L^2(\Omega))$, if $u(x,0) = u_0(x)$ almost everywhere in Ω and

$$\text{(33)} \quad \int_0^\infty \int_\Omega \dot{u}(x,t) \cdot \dot{\phi}(x,t) - \left\{ G_0 \nabla u(x,t) + \int_0^t G'(s) \nabla u^t(x,s) ds \right\} \cdot \nabla\phi(x.t) dx$$
$$= \; -\int_\Omega \dot{u}_0(x) \cdot \phi(x,0) dx + \int_0^\infty \int_\Omega \tilde{I}^{t=0}(x,\tau) \cdot \nabla\phi(x,\tau) d\tau dx$$

for all $\phi \in \mathcal{L}'(\mathbb{R}^+, H^1_0(\Omega))$.

If we denote with $a(u,\phi)$ the sesquilinear form on $\mathcal{L}(\mathbb{R}^+, H^1_0(\Omega))$

$$a(u,\phi)$$
$$= \int_0^\infty \int_\Omega \dot{u}(x,t) \cdot \dot{\phi}(x,t) - \left\{ G_0 \nabla u(x,t) + \int_0^t G'(s) \nabla u^t(x,s) ds \right\} \cdot \nabla\phi(x.t) dx$$

for all $\phi \in \mathcal{L}'(\mathbb{R}^+, H^1_0(\Omega)$ and such that $\varphi(x,0) = 0$, then the equation (33) can be written as

$$a(u,\phi) = -\int_\Omega \dot{u}_0(x) \cdot \phi(x,0) dx + \int_0^\infty \int_\Omega \tilde{I}^{t=0}(x,\tau) \cdot \nabla\phi(x,\tau) d\tau dx$$

Now we are in a position to claim the following

THEOREM 1 *Under the hypothesis* (4)-(5) *for the relaxation function* G, *the problem* (30)-(32) *have one and only one weak solution* $u \in \mathcal{L}(\mathbb{R}^+, H^1_0(\Omega))$.

By a simple change of the unknowns, it is always possible to obtain zero initial data. Accordingly, without any significant loss in generality we let $u_0 = 0$, $\dot{u}_0 = 0$. We denote with $\hat{a}$ the following sesquilinear form on $\hat{\mathcal{L}}(\mathbb{R}, H_0^1(\Omega))$

$$
\begin{aligned}
\hat{a}\left(\hat{u},\hat{\varphi}\right) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\Omega} -i\omega\hat{u}\left(x,\omega\right)\left[i\omega\hat{\varphi}\left(x,\omega\right)\right]^{*} dxd\omega + \\
& + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\Omega}\left[G_0\left(x\right)+\hat{G}'\left(x,\omega\right)\right]\nabla\hat{u}\left(x,\omega\right)\cdot\nabla\hat{\varphi}^{*}\left(x,\omega\right)\omega dxd\omega
\end{aligned} \tag{34}
$$

for all $\hat{\varphi} \in \hat{\mathcal{L}}(\mathbb{R}, H_0^1(\Omega))$. Plancherel's theorem applied to (33) gives

$$
\hat{a}\left(\hat{u},\hat{\varphi}\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\Omega}\tilde{I}_{+}^{t=0}(x,\omega)\cdot\nabla\hat{\varphi}^{*}\left(x,\omega\right)dxd\omega \tag{35}
$$

Lemma 1 *A function $\hat{u} \in \hat{\mathcal{L}}(\mathbb{R}, H_0^1(\Omega))$ is the Fourier transform of a weak solution of the initial boundary-value problem* (30)-(32) *in the sense of Definition 5 if and only if equality* (35) *holds for all $\hat{\varphi} \in \hat{\mathcal{L}}'(\mathbb{R}, H_0^1(\Omega))$.*

Taking in (35) $\hat{\varphi}\left(x,\omega\right) = \varphi_1\left(x\right)\varphi_2\left(\omega\right)$, with $\varphi_1 \in H^1\left(\Omega\right)$ and $\varphi_2 \in L^2(\mathbb{R})$, by the arbitrariness choice of φ_2 it follows that, for almost all $\omega \in \mathbb{R}$, the following identity holds:

$$
\begin{aligned}
& \int_{\Omega}\omega^2\hat{u}\left(x,\omega\right)\hat{\varphi}_1^{*}\left(x\right)dx + \int_{\Omega}\left[G_0\left(x\right)+\hat{G}'\left(x,\omega\right)\right]\nabla\hat{u}\left(x,\omega\right)\cdot\nabla\hat{\varphi}_1^{*}\left(x\right)dx \\
= {} & -\int_{\Omega}\tilde{I}_{+}^{t=0}(x,\omega)\cdot\nabla\varphi_1^{*}\left(x\right)dx,
\end{aligned} \tag{36}
$$

for every $\varphi_1 \in H^1\left(\Omega\right)$. But identity (36) means that $\hat{u}\left(\cdot,\omega\right)$ is a generalized solution in $H^1\left(\Omega\right)$ for the elliptic problem

$$
-\omega^2\hat{u}\left(x,\omega\right) - \nabla\cdot\left\{\left[G_0\left(x\right)+\hat{G}'\left(x,\omega\right)\right]\nabla\hat{u}\left(x,\omega\right)\right\} = \nabla\cdot\tilde{I}_{+}^{t=0}(x,\omega),\ \mathrm{x}\in\Omega \tag{37}
$$

$$
\hat{u}\left(x,\omega\right) = 0, \qquad \mathrm{x}\in\partial\Omega. \tag{38}
$$

Following the proof of Theorem 3 of [17] we are able to prove

Lemma 2 *For every $\omega \in \mathbb{R}$ problem* (38) *has one and only one solution $\hat{u}\left(\cdot,\omega\right) \in H^1\left(\Omega\right)$. Besides, the following inequality*

$$
\begin{aligned}
& \left\|\hat{G}_c^{\frac{1}{2}}(\omega)(1+\omega)\nabla\hat{u}\left(\omega\right)\right\|_{L^2}^2 + \left\|\hat{G}_c^{\frac{1}{2}}(\omega)(1+\omega)\hat{u}\left(\omega\right)\right\|_{L^2}^2 \\
& \qquad \leq C\int_{-\infty}^{\infty}\int_{\Omega}(1+\omega)\left|\tilde{I}_{+}^{t=0}(x,\omega)\cdot\nabla\hat{u}\left(x,\omega\right)\right|d\omega dx
\end{aligned} \tag{39}
$$

holds, where C is a suitable constants.

Finally, Lemma 4 and the hypotheses on the data provide the inequality

$$
\left\|\hat{G}_c^{\frac{1}{2}}(\omega)(1+|\omega|)\nabla\hat{u}\left(\omega\right)\right\|_{L^2}^2 + \left\|\hat{G}_c^{\frac{1}{2}}(\omega)(1+|\omega|)\hat{u}\left(\omega\right)\right\|_{L^2}^2 \leq C\left\|\tilde{I}_{+}^{t=0}(\omega)\right\|_{\mathcal{H}'_G}^2 < \infty,
$$

from which $\hat{u} \in \hat{\mathcal{L}}(\mathbb{R}, H_0^1(\Omega))$ and the isomorphism between $\mathcal{L}(\mathbb{R}^+, H_0^1(\Omega))$ and $\hat{\mathcal{L}}(\mathbb{R}, H_0^1(\Omega))$ guarantees that $\hat{u}$ is the Fourier transform of the solution $u \in \mathcal{L}(\mathbb{R}^+, H_0^1(\Omega))$ of problem (30)–(32) and it is unique.

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Unified dynamics of particles and photons

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1. – Introduction - Scalar particles

The present approach starts from the relativistic unified Dynamics of scalar particles and photons [2]; the subsequent step is to extend the previous unification to *particles and photons with multipole structure.* Thus particle dynamics with arbitrary structure can be extended, with a few modifications, to polarized photons, without postulating the corresponding relative dynamics.

Let us consider a curved space-time V_4, where the geometry realizes a good unification for all physical fields; particle-flux can be essentially unified with light-flux, from both the *kinematical* (linear velocity, spin and deformation [1]) *and the dynamical point of view* (evolution equations). In order to clearly state the problem, here we restrict to *particles with a vector structure*; the case of scalar particles is in fact well known [2]. First of all, let us fix a *frame of reference* $\Gamma(\gamma)$; we can obtain the unification of particles and photons by introducing Cattaneo's *relative time* T as the privileged parameter along the world-line of the particle (in place of the proper time):

$$dT = -\frac{1}{c}\gamma_\alpha dx^\alpha, \tag{1}$$

T is defined by integration on the world -line (such time-parameter can be pantopic only for special frames of reference). So the unified dynamics is based on the two classical theorems, that of *impulsion*: $\mathbf{p} = m\mathbf{v}$ and that of *energy*: $\varepsilon = mc^2$:

$$\dot{\mathbf{p}}^{\perp} = \mathbf{F}, \qquad \dot{\varepsilon} = W, \tag{2}$$

where a dot denotes the T-derivation and $\dot{(\ )}^{\perp}$ stands for *constrained derivation* (see [3]); in particular, for vector fields, we have:

$$\dot{\mathbf{p}}^{\perp} \stackrel{def}{=} \dot{\mathbf{p}} + \dot{\mathbf{p}} \cdot \gamma\gamma. \tag{3}$$

Equations (2) form the *common basis* for particles-photons dynamics; of course, for a particle, the *relative mass* m is defined from the *proper mass* m_0:

$$m \stackrel{def}{=} m_0\eta, \qquad \eta \stackrel{def}{=} \frac{1}{\sqrt{1 - v^2/c^2}}; \tag{4}$$

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where η is the Lorentz factor, so that the structure is defined by the invariant m_0 only (we are considering a scalar particle) and *the proper mass enters to modify the classical energy theorem*, i.e. the power W is given by

$$W = \mathbf{F} \cdot \mathbf{v} + q. \tag{5}$$

Thus we not only have the *mechanical power*, but also the thermic power q, which vanishes only if $m_0 = const$, i.e. if there is no structure. So far we have talked about a material particle ($m_0 > 0$) and photons are not included in the considered scheme. However, by using the parameter T, the relative velocity $\mathbf{v}$ is still meaningful for a photon, while (4) has no meaning, being $v = c$. Vice versa, the primitive quantity now is the *relative energy* ε (and not m_0):

$$\varepsilon = h\nu, \tag{6}$$

where h is the Planck constant and ν the *relative frequency*, which follows the following transformation law ([4], 116):

$$\nu' = \nu \frac{1 - \mathbf{u} \cdot \mathbf{v}/c^2}{\sqrt{1 - u^2/c^2}}; \tag{7}$$

where the vector $\mathbf{u}$ is the *relative velocity* of the galileian reference frame $R'(\gamma')$, related to the frequency ν'. Of course, being $v = c$, for a photon neither the proper mass or the proper frequency ν_0 are meaningful.

In conclusion, the fundamental equations (2) also hold for a photon, provided mass is now defined by means of energy according to (6) and velocity v is equal to c:

$$m \overset{def}{=} \frac{\varepsilon}{c^2}, \qquad v = c. \tag{8}$$

2. – Vector particles and photons

In order to obtain unified dynamics, let us consider now the case of a vector particle; if the particle is just a material one, along the world line of the center of mass we have a geometrical structure defined not only by the proper-mass m_0, but also by *three director vectors*: the *generalized impulsion* $\mathbf{P}$:

$$\mathbf{P} = \mathbf{I} + \mathbf{W}, \qquad \mathbf{I} = m_0 \mathbf{V}; \tag{9}$$

where $\mathbf{V}$ denotes s the 4-velocity: $\mathbf{V} = \eta(\mathbf{v} + c\gamma)$ and $\mathbf{W}$ the impulsion increase, which is orthogonal to $\mathbf{V}$:

$$\mathbf{W} \cdot \mathbf{V} = 0; \tag{10}$$

the *static impulsion* $\mathbf{S}$, and, finally, the *angular velocity* $\boldsymbol{\omega}$ (both unified by the total spin tensor $S^{\alpha\beta}$).

Now let us start from Papapetrou's dynamical equations[1] (see [5], p. 258); the first step is to translate the dynamical equations in *relative terms*, by using the

[1] An exact, i.e. not multipole, approach, in principle can be obtained from Dixon's theory of extended bodies [6]

standard time (1). The subsequent step is *to choose additional equations,* because the general dynamical picture needs, in order to have unicity, three *constitutive constraints.* So, given the relative dyamical picture of the material case, by the procedure (8) we also obtain the dynamical equations for a spinning photon; therefore the world line l now becames *null type,* and the *ordinary impulsion* $\mathbf{I}$:

$$\mathbf{I} = \mathbf{p}+\frac{\varepsilon}{c}\boldsymbol{\gamma}, \qquad \mathbf{p} =\frac{\varepsilon}{c^2}\mathbf{v} \tag{11}$$

is now isotropic ($\mathbf{I}\cdot\mathbf{I}$ =0); we have:

$$p^2 - \frac{\varepsilon^2}{c^2} = 0 \quad \sim \quad v = c. \tag{12}$$

As for the *structure vectors*: $\mathbf{P}, \mathbf{S}$ and $\boldsymbol{\omega}$, we need no additional hypotesis, because the picture is the same.

Now let us consider our *first step* explicitely; first of all we have *the natural decomposition* of $\mathbf{P}$:

$$\mathbf{P} =\tilde{\mathbf{P}} + M\boldsymbol{\gamma}, \qquad M > 0, \qquad M^2 = \tilde{P}^2 - N\mathbf{P} > 0, \tag{13}$$

which corresponds to expression (9), by means of (11)-(12). We have:

$$\varepsilon = mc^2, \tag{14}$$

and

$$\mathbf{P} = \mathbf{p}+\frac{\varepsilon}{c}\boldsymbol{\gamma} + \mathbf{W}, \qquad \mathbf{W}\cdot(\mathbf{v}+c\boldsymbol{\gamma}) = 0. \tag{15}$$

Thus, multiplying for $\mathbf{V}$ both expressions above we deduce the following general connection between M *and* ε:

$$Mc = \tilde{\mathbf{P}}\cdot\mathbf{v}+ \left(1 - v^2/c^2\right)\varepsilon. \tag{16}$$

Then the scalar M is a well determined function of the energy $\varepsilon = mc^2$, of the velocity $\mathbf{v}$ *and of the relative impulsion* $\tilde{\mathbf{P}}$.

In the framework of Papapetrou's theory [7], the relative dynamical equations are the following:

$$\begin{cases} \dot{\tilde{\mathbf{P}}}^{\perp} = \mathbf{F} - M\,\dot{\boldsymbol{\gamma}}, & \dot{M} = \frac{W}{c} - \tilde{\mathbf{P}}\cdot\dot{\boldsymbol{\gamma}}, \qquad \dot{(\;)} = d/dT \\ \dot{\boldsymbol{\omega}}^{\perp} = \tilde{\mathbf{P}}\times\mathbf{v} - \mathbf{S}\times\dot{\boldsymbol{\gamma}} & \dot{\mathbf{S}}^{\perp} = c\tilde{\mathbf{P}} - M\mathbf{v} + \boldsymbol{\omega}\times\dot{\boldsymbol{\gamma}}, \end{cases} \tag{17}$$

where the force $\mathbf{F}$ and the power W (both relative to the reference frame) are given by:

$$F^\alpha \overset{def}{=} \frac{1}{2}(v^\sigma + c\gamma^\sigma)\mathcal{R}_{\sigma\beta}\gamma^{\alpha\beta}, \qquad W \overset{def}{=} -\frac{c}{2}\mathcal{R}_{\sigma\alpha}v^\sigma\gamma^\alpha, \tag{18}$$

with

$$\mathcal{R}_{\sigma\alpha} \overset{def}{=} S^{\mu\nu}R_{\mu\nu\sigma\alpha}, \qquad \gamma^{\alpha\beta} \overset{def}{=} g^{\alpha\beta} + \gamma^\alpha\gamma^\beta \quad . \tag{19}$$

$\gamma^{\alpha\beta}$ denotes the *spatial metric*; $\mathcal{R}_{\sigma\alpha}$ depends on the *Riemann tensor* $R_{\mu\nu\sigma\alpha}$ and on the *total spin* $S^{\mu\nu}$, which unifies both the *angular velocity* $\omega^{\mu\nu}$ and the *static impulsion* S^μ:

$$S^{\mu\nu} = \omega^{\mu\nu} + 2S^{[\mu}\gamma^{\nu]}. \tag{20}$$

We note that equations (17) constitute a first order differential system for ten unknown quantities: $M, \tilde{\mathbf{P}}, \boldsymbol{\omega}, \mathbf{S}$ (the last three are spatial, i.e. 3-dimensional, vector fields); *the velocity* $\mathbf{v}$ here *works as a parameter,* while the reference frame $\Gamma(\dot{\gamma})$ is a known field, together with the metric $g_{\alpha\beta}$. In any case, the reference frame appears explicitely only by means of its derivative $\dot{\gamma}$, which has the meaning of relative *gravitational field* (see [3]).

3. – Tulczyjew scheme: resolutive equation

Let us discard the case of more general constitutive equations (see [3]) and assume here *Tulczyjew hypothesis* [8]:

$$S^{\alpha\beta} P_\beta = 0, \tag{21}$$

which give a *well determined scheme*; more precisely, the structural conditions (21) allows, first of all, to determine $\mathbf{S}$ as a function of M, $\tilde{\mathbf{P}}$, and $\boldsymbol{\omega}$:

$$M\mathbf{S} = \tilde{\mathbf{P}} \times \boldsymbol{\omega} \tag{22}$$

so that system (17) gives he following *resolutive equation*:

$$\left(\tilde{P}^2 - M^2\right)\mathbf{v} + \left(Mc - \tilde{\mathbf{P}}\cdot\mathbf{v}\right)\tilde{\mathbf{P}} - \mathbf{F}\times\boldsymbol{\omega} + \frac{W}{c}\mathbf{S}, \tag{23}$$

where $\mathbf{S}$ is given by (22).

Equation (23) is not explicit type with respect to the vector $\mathbf{v}$, because the velocity is involved in the definition of $\mathbf{F}$ and W according to (18), and, by (16), in that of M; anyway equation (23) *is equivalent to a linear system for the three components of* $\mathbf{v}$, and, to prove that the solution of the Papapetrou-Tulczyjew equations is uniquely determined, one only needs to test that *the third order coefficient matrix is regular*, as it is shown in [7]. Such matrix is in fact singular only in a few special cases which are still uninterpreted from the mechanical point of view.

So, determied this way the principal function $\mathbf{v}$, of course in function of M, $\tilde{\mathbf{P}}, \boldsymbol{\omega}$ and of the curvature tensor, system (17) is automatically reduced of one unity, because the fourth equation can be replaced by the first integral (22); moreover by eliminating both $\mathbf{v}$ and $\mathbf{S}$, we have a well determined first order differential system for the unknown fields M, $\tilde{\mathbf{P}}$, and $\boldsymbol{\omega}$:

$$\begin{cases} \dot{\tilde{\mathbf{P}}}^{\perp} = \mathbf{F}\left(M, \tilde{\mathbf{P}}, \boldsymbol{\omega}\right) - M\,\dot{\gamma} \\ \dot{M} = \frac{1}{c} W\left(M, \tilde{\mathbf{P}}, \boldsymbol{\omega}\right) - \tilde{\mathbf{P}}\cdot\dot{\gamma} \\ \dot{\boldsymbol{\omega}}^{\perp} = \tilde{\mathbf{P}} \times \mathbf{v}\left(\mathbf{M}, \tilde{\mathbf{P}}, \boldsymbol{\omega}\right) - \frac{1}{M}\left(\tilde{\mathbf{P}}\times\boldsymbol{\omega}\right)\times\dot{\gamma}\,. \end{cases} \tag{24}$$

The static impulsion $\mathbf{S}$ is then determined a posteriori by (22); we note that $\mathbf{S}$ is orthogonal to both $\tilde{\mathbf{P}}$ and $\boldsymbol{\omega}$, but in the general case $\tilde{\mathbf{P}}\cdot\boldsymbol{\omega} \neq 0$, so we have not an orthogonal triad.

Similarly from equation (16) we have the material energy ε (with $v < c$), i.e. m, or equivalently m_0, and finally from (15) we have the difference vector $\mathbf{W}$.

Once verified uniqueness for the solution of the Papapetrou-Tulczyjew equations, let us consider the case of a *spin photon* (polarized light). First of all we have:

$$\mathbf{v} = c\mathbf{u}, \tag{25}$$

with $\mathbf{u}$ unitary, so that the resolutive equation (23) remains substantially unchanged, but for the replacement of $\mathbf{v}$ by $\mathbf{u}$:

$$\left(\tilde{P}^2 - M^2\right)\mathbf{u} + \left(M - \tilde{\mathbf{P}}\cdot\mathbf{u}\right)\tilde{\mathbf{P}} - \mathbf{F}_u \times \boldsymbol{\omega} + \frac{1}{cM} W_u \tilde{\mathbf{P}} \times \boldsymbol{\omega} = 0, \tag{26}$$

where

$$F_u^\alpha = \frac{1}{2}\left(u^\sigma + \gamma^\sigma\right)\mathcal{R}_{\sigma\beta}\gamma^{\alpha\beta}, \qquad W_u \stackrel{def}{=} -\frac{c}{2}\mathcal{R}_{\sigma\alpha}v^\sigma\gamma^\alpha, \tag{27}$$

and with $\mathcal{R}_{\sigma\beta}$ defined by (19).

After all, as in the material case, the *kinetic field* $\mathbf{u}$ is determined as a function of $\mathbf{P}$ (i.e. M and $\tilde{\mathbf{P}}$), of $\boldsymbol{\omega}$ and of the curvature tensor (which is given, as is the metric $g_{\alpha\beta}$); the vectors $\mathbf{P}$ and $\boldsymbol{\omega}$, in turn, are determined by the differential system (24), from their initial values $\mathbf{P}_0$ and $\boldsymbol{\omega}_0$. Clearly the resolutive equation has a fundamental role, both for the differential system (24), which governs $\mathbf{P}$ and $\boldsymbol{\omega}$, and for the evolution of the static impulsion $\mathbf{S}$, which is determined by means of (22). Similarly, the difference vector $\mathbf{W}$ is directly determined a posteriori by (15), after the determination of the motion, i.e. of $\mathbf{u}$, by means of (26), and that of the energy $\varepsilon = mc^2$, through equation (16). Such constraints stress that, *being* $\varepsilon = h\nu$ *in principle finite, the difference* $M - \tilde{\mathbf{P}}\cdot\mathbf{u}$ *also must be finite* for $v \longrightarrow c$; in fact we have both the expressions:

$$Mc - \tilde{\mathbf{P}}\cdot\mathbf{u} = -\mathbf{P}\cdot\mathcal{V} \equiv -\mathbf{I}\cdot\mathcal{V}, \tag{28}$$

where $\mathbf{I} \equiv \mathbf{p} + \frac{\varepsilon}{c}\gamma$ is the *ordinary impulsion* and $\mathcal{V}$ is the *relative 4-velovity:*

$$\mathcal{V} \stackrel{def}{=} \mathbf{v} + c\gamma, \tag{29}$$

which are both meaningful also for a photon. In other words equation (28) stresses first of all the kinematic meaning of the diference $Mc - \tilde{\mathbf{P}}\cdot\mathbf{u}$, and, second, it proves that such difference is correct also for a photon, at least in the general case. Anyway now a new fact occurs, in comparison with the scalar case: while for a material photon ($v << c$) equation (16) allows one to determine the energy ε, *for a photon* ($v = c$) *such link has no more sense.* The mass M is instead given by the fundamental system (24) , as in the material case. Therefore, in accordance with system (24), we must suppose (in order to have unification) that the solution satisfies the following two limit conditions:

$$\lim_{v\to c}\left(Mc - \tilde{\mathbf{P}}\cdot\mathbf{v}\right) = 0, \qquad \lim_{v\to c}\frac{Mc - \tilde{\mathbf{P}}\cdot\mathbf{v}}{1 - v^2/c^2} > 0; \tag{30}$$

which allow to *determine the energy* ε *also for a photon*:

$$\varepsilon = \lim_{v\to c}\frac{Mc - \tilde{\mathbf{P}}\cdot\mathbf{v}}{1 - v^2/c^2}. \tag{31}$$

4. – Intrinsic equations

Clearly in all the construction stated above the reference frame is entirely free, i.e. *equations are formally invariant with respect to the choice of* γ; to fix the ideas, once we have studied the conditions for to shift from the material case to photons, let us consider the case in which the vector γ *is parallel to* $\mathbf{P}$ (of course along the worldline), which is of special interest. The frame $\mathbf{P}$ is in this case similar to a rest frame, we have in fact:

$$\mathbf{P} = M\gamma \quad \sim \quad \tilde{\mathbf{P}} = 0, \quad M > 0. \tag{32}$$

Here the frame of reference is deduced from the motion, i.e. it is not given a priori: both M and the versor γ, subordinate to system (23) (*intrinsic equations*), are unknown quantities. Since we have $\mathbf{S} = 0$ the resolutive equation (26) assumes the following *reduced form*:

$$M^2\mathbf{v} + \mathbf{F} \times \boldsymbol{\omega} = 0. \tag{33}$$

Here we have a linear system on the componentd v_k which, *in the material case is regular* [7], i.e. the coefficients determinant is non-zero; more precisely such system is of the following kind:

$$v_k\left(B^i{}_k - 2M^2\delta^i_k\right) = -B^\sigma{}_k\gamma_\sigma \qquad (\gamma_\sigma = P_\sigma/M), \tag{33'}$$

where the matrix $B^i{}_k$ is a well determined function of the angular velocity and the spatial curvature tensor: $B^i{}_k \overset{def}{=} \omega_{lj}R^{ljih}\omega_{hk}$. From here, by adjoint operation, we have the following canonical decomposition:

$$B^i{}_k = \omega^i R_k - H\delta^i_k, \tag{34}$$

where R_k and H are *defined by means of the angular velocity and of the spatial curvature tensor:*

$$R_k \overset{def}{=} \frac{1}{2}\omega^i G_{ik}, \qquad H \overset{def}{=} \omega^i R_i, \qquad G_{ik} \overset{def}{=} R_{ik} - \frac{1}{2}R\gamma_{ik}. \tag{35}$$

The special structure of tensor (34) gives rise to the following *principal invariants* $B^i{}_k$:

$$I_1 = -2H, \qquad I_2 = H^2, \qquad I_3 = 0, \tag{36}$$

so that the determinant $\mathcal{D}$ of the secular matrix (33$'$) is given by:

$$\mathcal{D} = -2M^2\left(2M^2 + H\right)^2 \leq 0. \tag{37}$$

Therefore $\mathcal{D}$ *is in general negative, and it is null only if* M *assumes the following special value:*

$$2M^2 = -\omega^i R_i > 0, \tag{38}$$

which depends on both the angular velocity and the curvature tensor (by means of G_{ik}). This case discarded, equation (33) is univocally solvable with respect to $\mathbf{v}$, so that the differential system (24), i.e.:

$$\dot{M} = \frac{1}{c}W(M, \boldsymbol{\omega}), \qquad M\dot{\gamma} = F(M, \boldsymbol{\omega}), \qquad \dot{\boldsymbol{\omega}}^\perp = 0, \tag{39}$$

can be used to determine of M and γ, i.e. $\mathbf{P}$. Moreover the vector $\boldsymbol{\omega}$ is Fermi-Walker transported along l and the motion is characterized by (33), through integration of the differential system (39).

We note that, as for the reference frame, *choice (32) is not consistent with a photon*, becouse for $v = c$ equation (16) implies the condition $M = 0$, which invalidates the above mentioned choice.

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The Problem of the Rate of Thermalization, and the Relations between Classical and Quantum Mechanics

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1. – Introduction

Ther problem of the rate of thermalization consists in determining how long it takes for a system to reach thermal equilibrium. Thus enunciated, the problem appears to have no special relevance, because it is clear that every system will have some characteristic relaxation time, which should be estimated and suitably taken into account. The situation is however more delicate when relaxation times are met which are enormously large, as for example in the case of glasses. Indeed, these are fluids with the special property that the viscosity increases by even 18 orders of magnitude when the absolute temperature is reduced by only a factor of two, for example from 600 K to 300 K; thus the relaxation time to equilibrium can be extraordinarily large, and one meets with situations of a kind of metaequilibrium or metastability, for which a description by the ordinary methods of equilibrium statistical mechanics is not feasible.

The main motivation of the present paper consists in illustrating how such a situation of metaequilibrium is not confined to "strange" systems such as glasses, but is met in all common situations involving harmonic oscillators, when they present deviations from a classical behaviour by manifesting a quantum behaviour. Thus, the property of presenting enormously long relaxation times to equilibrium somehow seems to correspond to the manifestation of a quantum behaviour, and as such deserves to be investigated as a property of a general relevant interest.

It will be recalled below how the problem of the thermalization rate was actually introduced by Boltzmann just within such a foundational perspective, then pursued by Jeans and Rayleygh, and finally abandoned with a public retractation by Jeans, struck as he had been by a fundamental paper of Poincaré on the necessity of quantization. It will then be recalled how the problem of the thermalization rate reemerged, but only within a sectorial perspective, in the problem of the sound dispersion in gases, and how finally the problem was reconsidered in the original

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foundational spirit of Boltzmann, after the work of Fermi Pasta Ulam (1954) and the impact of the modern theory of dynamical systems. Finally, some perspectives will be given concerning the relations between classical and quantum mechanics in connection with the problem of the rates of relaxation.

2. – The first phase: from Boltzmann to the retractation of Jeans

Boltzmann was confronted with an essential qualiltative difficulty of classical statistical mechanics in connection with the equipartition principle. Indeed, according to classical statistical mechanics, equilibrium should be governed by the Maxwell–Boltzmann probability distribution, the main prediction of which is the equipartition of energy: every degree of freedom contributes to energy (in the harmonic approximation) with a term $1/2\ kT$ (or kT in the case of harmonic oscillators), where k is the Boltzmann constant and T the absolute temperature. So, at equilibrium the total energy of a system should be proportional to temperature, and thus the specific heat be constant, independent of temperature (and of frequency, for systems of harmonic oscillators). Instead, it was found experimentally that the specific heat of polyatomic molecules (whose internal motions could be assimilated to harmonic oscillators or rotators) was decreasing with decreasing temperature. An analogous qualitative discrepancy was also found to occur in the black–body spectrum, because the black–body can be assimilated to a system of harmonic oscillators with frequencies ranging from 0 to ∞, and the energy of the high frequency oscillators appeared to go to zero exponentially fast as frequency increases.

This is known to everybody, because it is exactly in this connection that quantum mechanics originated, on october 19, 1900, when Planck found empirically, by means of a skillfull interpolation, that the mean energy $U(\omega, T)$ of a harmonic oscillator of angular frequency ω at absolute temperature T is well fitted by Planck's law

$$U(\omega, T) = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} = kT\,\frac{x}{e^x - 1} \qquad (x = \beta\hbar\omega,\ \beta = 1/kT)\,,$$

where $\hbar$ is the (rationalized) Planck's constant. The correspondence principle is saved because, from the second expression above, Planck's formula is seen to reduce to the "classical" formula kT for $x \ll 1$, i.e. for high temperatures or low frequencies, while decreasing exponentially fast to zero for $x \gg 1$, i.e. for low temperatures or high frequencies. Two months later Planck introduced quantization, i.e. understood that his law could be obtained by the "simple" prescription that the energy of an oscillator be quantized, in the sense that only discrete values of energy, actually $E_n = n\hbar\omega$, $n = 1, 2, \cdots$, should be allowed (later on, in his "second theory" of the year 1912, he added the controversial "zero–point energy" $1/2\ \hbar\omega$, which gives the "energy levels" $E_n = (n + 1/2)\,\hbar\omega$). Planck was originally concerned with the black–body problem, but his argument was extended by Einstein in the year 1906 to "material" harmonic oscillators. In the meantime, with the celebrated paper on the photon (1905) Einstein had also shown that quantization was a "real" fact, and not just a formal one (to this we will however come back below). At the first Solvay conference (1911, see [1]) the existence of quanta was finally sanctioned by the scientific community.

This is well known. Less familiar is instead the fact that Boltzmann had previously looked for a possible escape from the difficulties of the equipartition problem in a fully classical context. Indeed, he was just suggesting that the lack of equipartition could very simply be due to the fact that the system had not reached equilibrium within the measurement time; a hint in this direction was even given by Maxwell in the last page of his third memoir on kinetic theory, where he speaks of a relaxation time of 675 years [2]. The main idea was that the relaxation rates to equilibrium should be highly nonuniform with respect to frequency and temperature: for example, in the case of polyatomic molecules equilibration should be quite rapid for the "external" degrees of freddom such as those of the center of mass, but very slow for the "internal" motions, i.e. rotations and especially vibrations. This is well witnessed by a famous letter of Boltzmann to Nature (1895) [3], where he speaks of times of the order of years. Here is the quotation: "*But how can the molecules of a gas behave as rigid bodies? Are they not composed of smaller atoms? Probably they are; but the vis viva of their internal vibrations is transformed into progressive and rotatory motion so slowly that when a gas is brought to a lower temperature the moleecules may retain for days, or even for years, the higher vis viva of their internal vibrations corresponding to the original temperature.*" Even more interesting are the sections 43, 44, 45 of his Lectures on Gas Theory, Vol II, where one finds [4]: "*The constituents of the molecule are by no means connected together as absolutely undeformable bodies, but rather this connection is so intimate that during the time of observation these constituents do not move noticeably with respect to each other, and later on their thermal equilibrium with the progressive motion is established so slowly that this process is not accessible to observation*"; and finally, the most significant statement: " *The hypothesis proposed here would be confirmed experimentally if it were to be shown that, for any gas for which* κ (the ratio C_p/C_V of specific heats) *varies with temperature, observations extended over a larger period of time give a smaller value than for those of shorter duration*".

To this last remark we will come back below. For what concerns quantitative estimates for the phenomenon predicted by Boltzmann, we could find no trace in his works. However, estimates were soon provided by Jeans [5], who was able to prove that the times needed to reach equilibrium were exponentially long with frequency and (essentially) inverse temperature. This was obtained by showing that the energy δe exchanged between external and internal degrees of freedom through collisions is exponentially small. As a prototype example consider the head–on collision of a particle with a spring of frequency ω; by extremely elementary considerations Jeans shows that, in a first approximation, the exchanged energy δe is nothing but the square of the Fourier transform (evaluatred at ω) of the function $F(t)$ expressing, as a function of time, the force acting on the free end of the spring by interaction with the impinging particle. Thus the exchanged energy δe turns out to be exponentially small with ω, namely of the form $\delta e = A\exp(-\tau\omega)$, just in virtue of a general property of the Fourier transform of an analytic function (the intermolecular potential having been assumed to be analytic). Here τ is a characteristic time of interaction, of the order l/v, where l is the range of the potential and v the velocity of the impinging particle. To have an idea of the relevance attributed by Jeans to such considerations, here is a quotation from the *incipit* of his paper: " *A steel ball* *dropped on a rigid steel plate will rebound perhaps half a dozen times before its energy is appreciably lessened; this is because of the great elasticity of steel. If the kinetic theory of gases is true, a system of molecules must rebound from one another and from rigid walls many billions of times before the total energy is appreciably lessened. The aim of the present paper is to show that, in so far as the data available enable us to judge, molecules will possess sufficient elasticity for this to occur.*"

The point of view of Boltzmann and Jeans was amply discussed at the first Solvay conference (1911), after the report of Jeans ([1], page 74) and after the reading of a letter that Rayleigh had sent in support of the nonequilibrium point of view ([1], page 51). Especially relevant was the opinion expressed by Nernst, who remarked: "*up to now it has never been observed that the measured values of the specific heat increase*" (with the time of measurement); in particular, he added, this was true for gases not obeying the equipartition principle, for which there were available experimental methods involving measurement times ranging from a millionth of a second to several minutes. An even stronger argument was given by Nernst in connection with the fusion temperature and the vapor tension. Indeed, as such quantities are well known from thermodynamics to depend on the specific heat, he pointed out that, if the specific heat were changing with time, a difference between the fusion temperature of natural minerals and that of synthetic compounds should have been observed, which was not the case. So the phenomenology appeared to require that the equilibration times should be longer than millions of years for some components of energy, while other components should equilibrate "immediately", and this, Nernst concluded, "*is very little probable*". By the way, this seems not to be so clear today, with the present popularity of the studies on glasses.

A very skeptical comment on the nonequilibrium interpretation was also made by Poincaré after the report of Jeans. Actually, just under the influence of the discussions at the Solvay conference, Poincaré himself was very soon led to perform a deep investigation which, in his opinion, constituted a proof of the necessity of quantization; not only quantization produces Planck's law, but conversely, Poincaré claims, quantization necessarily follows if Planck's law is assumed to hold at a phenomenological level [6] (see also [7]). A similar argument had previously been given by Ehrenfest [8]. This was the end of the story, because Poincaré's paper made so strong an impression on Jeans that he felt the need of making a public retractation. This occurred on the occasion of the meeting of the British Association of Physics of the year 1913, a report of which, published in Nature [9] (see also [10]), goes as follows: "*On Friday morning the most important discussion of Section A, if not of the whole meeting, took place. The subject was radiation and it was opened by Mr. J.H. Jeans in a masterly and concise manner. The discussion turned on the question of the validity of the laws which have hitherto been believed to be the ultimate laws of nature. The problem at its simplest occurs in the case of black body radiation. Mr. Jeans regarded the work of Poincaré as conclusive:when starting with the mean energy of each vibration of specified wave-length he deduces the quite definite result that the exchange of energy must take place by finite jumps.This leads directly to the quantum hypothesis which the opener assumed in its entirety.*" Moreover a few years later, in publishing the third edition of his Dynamical Theory of Gases, he introduced a very drastic change by completely eliminating the chapter 16 of the first two editions, by the title "*The transfer of energy and the propagation of sound*" [11], where the problem of the dependence of the specific heat on the time of measurement was discussed in connection with the dispersion of sound. See also [12].

3. – The second phase: from physics to chemistry; relaxation times in sound dispersion

After the retractation of Jeans, the problem of the times of relaxation to equilibrium in polyatomic molecules disappeared from the domain of fundamental physics, people being convinced that equilibrium was reached "immediately". Typically, in the case of gases such as air at ordinary conditions of pressure and temperature, by "immediately" one meant just "after a time of the order of 10^{-10} seconds"; this is indeed the mean *collision time*, i.e. the mean free time between two collisions, which gives the order of magnitude of the equilibration time for the center–of–mass energy

The problem then reappeared twelve years later, in the year 1925, but only as a sectorial one within a particular subject, namely dispersion and absorption of ultrasound (see [13][14] [15], or the very concise review in sect. 4 of [16]). Indeed, by using ultrasounds with frequencies of the order of the megahertz, which had just become available, Pierce [17] discovered an anomalous absorption which could not be explained in terms of the "classical" mechanisms of viscosity and thermal conduction, already familiar from the times of Kirchhoff and Stokes. After a long discussion of about five years, people finally became convinced that such an anomalous absorption, which was intimately related with a corresponding phenomenon of dispersion, should be explained as due to the existence of a retardation in the equilibration of the internal degrees of freedom of polyatomic molecules with the external (or translational) ones, more or less in the way conceived by Boltzmann, Rayleigh and Jeans, although such authors and their foundational perspective essentially were never mentioned. The corresponding relaxation times were found, quite unexpectedly, to be of the order of 10^{-5} or 10^{-3} seconds, namely about 5 or 7 orders of magnitude larger than "immediately" [18].

Such a phenomenon of a slow relaxation to equilibrium for the internal motions was then investigated from a macroscopic point of view, in the frame of continuum mechanics. A good survey containing a detailed historical part can be found in Kneser [14] (see also [19]. Quite relevant are the work of Mandelstam and Leontovich, a summary of which is reported by Landau and Lifshitz in connection with the "second viscosity" [20], and the standard reference book of Herzfeld and Litovitz [21] (see also [22] and [23]).

Many works were also performed from a microscopic point of view, namely with the aim of determining the relaxation rate by estimating the exchange of energy between external and internal degrees of freedom through molecular collisions. The fundamental reference here is the paper of Landau and Teller of the year 1936 [24]. They consider the problem of the exchange of energy, according to quantum mechanics, in a collision between a harmonic oscillator and an impinging particle, *interacting through a smooth potential*; they claim that the result can be estimated classically, and give a classical estimate which is essentially equivalent to the one given 33 years before by Jeans, though apparently unaware of it. The only difference is that they take into account the Maxwell–Boltzmann distribution of the velocities of the impinging particle, which entails that only the collisions with particles of extremely high energy are relevant; this by the way is the reason why the calculation can be performed classically. In such a way Landau and Teller find that the exchanged energy is exponentially small with ω/kT (actually, a suitable power of it), but refrain from giving any quantitative estimates.

Quantitative theoretical estimates for the relaxation times, to be confronted with the experimental ones for several kinds of diatomic molecules, were later given by many people. An almost complete review up to the year 1969 can be found in Rapp and Kassal [25]. The essential conclusion of all such works is that everything is apparently in order, because the theoretical estimates are found to be in a more or less good agreement with the experimental data. Things are however more complicated, as we will try to illustrate below; a hint can be found in a standard book such as [15], where one finds the sentence: "*Even if a completely satisfactory theory were available, its quantitative application would be severely limited by the lack of accurate and realistic interatomic potentials.*"

4. – The third phase: back to physics; Fermi Pasta Ulam and the theory of dynamical systems; Einstein's interpretation of Planck's law

In the meantime, the problem of the relaxation times to equilibrium had emerged again, as a problem of a general interest, with the work of Fermi–Pasta–Ulam (FPU, 1954) [26]. Such authors were making a numerical investigation of the relaxation to equilibrium in a one-dimensional model of a nonlinear crystal, namely a chain of a certain number N of equal mass points on a line, with a coupling due to nonlinear springs: taking initial data with the energy concentrated on the low–frequency modes, they found by numerical integration of the equations of motion that energy did not flow to the high–frequency modes within the times they could attain. As Ulam reports in his preface to the work of FPU, reprinted in Vol. 2 of Fermi's collected papers (N. 266): "*The results of the calculations ... were interesting and quite surprising to Fermi. He expressed to us the opinion that they really constitute a little discovery, in providing intimations that the prevalent beliefs in the universality of mixing and thermalization in nonlinear systems may not be always satisfied.*"

We don't have time to enter here a discussion of the many works written on the FPU problem (see for example [27][28][29][30]), mostly with the intent of understanding whether classical mechanics really predicts extremely long relaxation times or not. It is just in this connection that many studies in the mathematical theory of dynamical systems were performed, in the spirit of modern perturbation theory, with reference to KAM theory and to the notion of weak stability in Nekhoroshev's sense [31][32]. Thus the works of Boltzmann and Jeans were rediscovered (see [33][34][35]), and the analogy with glasses was first mentioned [36].

There still remained a fundamental problem, namely how could one give a thermodynamic description for systems being in a state of metaequilibrium, such as glasses. Indeed, on the one hand it is found that, due to te exponentially small exchanges of energy, the high–frequency oscillators give an exponentially small contribution to the specific heat, which seems to be in a qualitative agreement with the description given by Planck's law; on the other hand, the dynamical laws describing the exponentially small exchanges of energy turn out to contain a dependence on the molecular parameters, which is not the case for the laws of themodynamics. A first step towards the solution of this fundamental problem was accomplished quite recently in the work [37](see also [38][39][40]), where it was shown that the elementary mechanical laws governing molecular collisions entail a functional relation between mean exchanged energy and energy fluctuations, which has exactly the analytic from corresponding to Planck's formula, in a way pointed out by Einstein in his contribution to the Solvay conference.

Let us briefly illustrate this point. It is known since its original derivation that Planck's law can be regarded as a solution of the differential equation

$$\frac{\mathrm{d}U}{\mathrm{d}\beta} = -(\epsilon U + U^2/N)\,,$$

with $\epsilon = \hbar\omega$. On the other hand it was pointed out by Einstein (see [1]) that such an equation should be better read as split up into the two relations

$$\frac{\mathrm{d}U}{\mathrm{d}\beta} = -\sigma_E^2\,,\quad \sigma_E^2 = \epsilon U + U^2/N\,,$$

where σ_E^2 is the variance of energy; indeed the first one should be considered as a relation of a general thermodynamic character, while the second one should be looked at as having a dynamical character and might in principle be deducible from a microscopic dynamics. In his very words: these two relations "*exhaust the thermodynamic content of Planck's*" formula; and: "*a mechanics compatible with the energy fluctuation $\sigma_E^2 = \epsilon U + U^2/N$ must then necessasily lead to Planck's*" formula.

Now, in the paper [37] it was shown that the mechanics that leads to such a functional relation is nothing but the ordinary Newtonian mechanics. Indeed, considering for example the prototype model mentioned above, of a particle impinging on a spring, one easily shows (at least in a first approximation) that the equations of motion produce for the energy exchanged during a collision a certain expression, which in turn, by averaging over the collisions, leads exactly to the Einstein's functional relation between the mean exchanged energy and the corresponding variance. In such a way, one obtains for the mean exchanged energy an expression having the analytical form of Planck's law; the way in which Planck's constant should enter is not yet clear, although it might be worth mentioning that Planck's constant is known to be contained in the values of the actual molecular potentials.

5. – Perspectives: analogy with glasses and with stellar dynamics, Einstein versus Poincaré

Which conclusions can then be drawn from a complicated situation as the one described above? The first one seems to be that one of the main effects predicted by Boltzmann really exists. We refer to the prediction that "*observations extended over a larger period of time* (for the specific heat should) *give a smaller value than* *for those of shorter duration*". Indeed it is just this phenomenon that constitutes sound dispersion, that now is commonly observed in polymers, where it is described under the name of "*time–dependent specific heat*" [41][23]. By the way, we are ourselves trying to observe the analogous phenomenon in crystals, in collaboration with G. Carini and F. Ragusa. There remains however the problem that the observed relaxation times are still somehow microscopic, being of the order of 10^{-3} or 10^{-5} seconds instead of the ones that would be needed to account for the actual lack of equipartition; for example, times of the order of billions of years as mentioned by Jeans would make the job.

This is the main problem with which we are presently confronted. In this connection, we are now trying to produce analytical and numerical estimates for the energy exchanges, in correspondence with realistic interatomic potential, and we are meeting with an apparently paradoxical situation. Indeed the estimates are found to depend in an incredibly strong way on the values of the parameters used, extremely small variations of the parameters leading to sharp changes in the order of magnitude of the exhanged energy. We might perhaps describe such a situation by saying that a *principle of unpredictability of the thermalization rate* seems to hold. The minimum we can say at present is that the agreement between theory and experimental data which is allegedly found in the literature might just be due to the fact that the relevant parameters are actually fitted to the data rather than taken as given in advance.

What we hope is that the relaxation times of classical mechanics, accurately calculated using realistic interatomic potentials without free parameters, can be proven to be extremely long, entailing extremely large time–scales as those occurring in glassy systems. But in such a case one would be confronted with the further problem of explaining the much shorter times which are actually obserbved, namely, as mentioned above, of the order of 10^{-3} or 10^{-5} seconds. Our conjecture is that according to classical mechanics one meets with two time–scales. The first is a short one, which leads to the reaching a state of metaequilibrium described by a quasi–thermodynamics in qualitative agreement with Planck's law, through a dynamical mechanism as the one conceived by Einstein and illustrated above. The second time–scale, which might be extremely large as in glassy systems, would instead lead to equipartition. An analogous situation seems to occur in stellar dynamics, where one meets with a rapid, violent, relaxation to a Fermi–like distribution, i.e. the Lynden–Bell distribution (see [42] and the review [43]); this is obtained in the approximation in which the collisions are neglected, while an extremely slow relaxation to a classical equilibrium occurs later, under the action of the interstellar collisions. Moreover, for what concerns the Planck–like distribution describing the metaequilibrium in systems of harmonic oscillators, we would like to mention that, according to the scenario described above, the observations should exhibit what we like to call the Jeans effect (see [44], quoted in [45]; see also [46]), namely a *plateau* in the low frequency region of the energy spectrum, which advances, with an extremeley slow pace, towards the high frequency region. Indeed, the exponentially long relaxation times causing the exponential smallness of the thermodynamic energy are expected to occur only for frequencies above a certain threshold, below which it equipartition should hold; on the other hand such an "equipartition front" has to advance at all, if a global equipartition has eventually to occur. Actually it seems to us that such an effect might already be visible in certain experimental data available for plasmas [47].

In closing the present review, we would like to mention two quotations from Einstein and Poincaré, concerning the possibility of obtaining Planck's law without introducing quantization, i.e. a dicretization of energy. For what concerns Einstein, we have already recalled how, since his contribution to the Solvay conference, he was striving to obtain a classical understanding. We show now that his attitude did not change up to his last years. This is witnessed by the following quotations from his scientific autobiography, which was written a few years before his death [48]. Indeed, he first recalls how, by inventing the photon, he had given some concreteness to the discretization of energy, previously introduced by Planck at a purely formal level. In his very words (see [48]): "*This way of considering the problem showed in a definitive and direct way that it is necessary to attribute a certain immediate concreteness to Planck's quanta and that, under the energetic aspect, radiation possesses a sort of molecular structure*". But after a few lines he adds: "*This interpretation, that almost all contemporary physicists consider as essentially definitive, to me appears instead as a simple provisional way out*". A further very impressive quotation concerning a classical understanding of the photon, still taken from his contribution to the Solvay conference, is reported in [39].

For what concerns Poincaré, we have already recalled how in his fundamental paper [6] he claimed that quantization should be necessary if Planck's law is assumed to hold. Actually, it is well known that he had a general negative attitude towards the metastability scenario of Boltzmann and Jeans, as is witnessed by the following quotation from a paper of a less technical type [49], written just after the one mentioned above: "*Jeans tried to reconcile things, by supposing that what we observe is not a statistical equilibrium, but a kind of provisional equilibrium. It is difficult to take this point of view; his theory, being unable to foresee anything, is not contradicted by experience, but leaves without explanation all known laws ...* ". However, curiously enough, the final words of the same paper have the following tone: "*Will discontinuity reign over the physical universe and will its triumph be definitive? Or rather will it be recognised that such a discontinuity is only an appearence and that it dissimulates a series of continuous processes? The first person that saw a collision believed to be observing a discontinuous phenomenon, although we know today that the person was actually seeing the effect of very rapid changes of velocity, yet continuous ones*", with the conclusion: "*To try to express today an opinion about these problems would mean to be wasting one's ink.*"

Now, our admiration for Poincaré is unlimited, but our personal feeling, or rather hope, is that perhaps on this point Einstein was seeing farther than him.

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Solid–Solid Phase Transition in a Mechanical System

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1. – Introduction

We are concerned with phase–field models for binary alloys exposed to thermomechanical loads. Phase field models describe the morphology by means of an order parameter that indicates the present phase at time t and at any point x of the alloy. Our aim is to introduce briefly the modeling of the tin/lead alloy given in [5] and to discuss the corresponding mathematical model from the viewpoint of mathematical analysis following [2].

The variables of the tin/lead alloy are the fields

$\mathbf{u}(x,t)$	(mechanical) displacement
$\chi(x,t)$	(tin) concentration.

The field equations rely on the static momentum balance and on the conservation law of the tin content. They read

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad \text{and} \qquad \frac{\partial \chi}{\partial t} + \frac{\partial J_k}{\partial x_k} = 0$$

where the repeated index convention is in force.

Let us describe the ingredients of such equations. The stress tensor is given by Hooke's law including eigenstrains that result here from different thermal expansions of the phases

$$\sigma_{ij} = C_{ijhk}(\chi)(\varepsilon_{hk} - \varepsilon^*_{hk}(\chi)) \qquad \text{with} \qquad \varepsilon_{hk} = \frac{1}{2}\left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right).$$

Both the stiffness matrix C and the eigenstrains ε^* depend on the concentration because the phases behave differently. Hence, we assume the representations

$$C_{ijhk}(\chi) = \Theta(\chi)\, C^{\alpha}_{ijhk} + (1 - \Theta(\chi))\, C^{\beta}_{ijhk}$$
$$\text{with the shape function} \quad \Theta(\chi) = \frac{c^{\beta} - \chi}{c^{\beta} - c^{\alpha}}.$$

In the above equation C^{α}_{ijhk} and C^{β}_{ijhk} denote the stiffness matrices of the cubic α-phase and of the tetragonal β-phase, respectively. The concentrations c^{α} and c^{β}

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appearing in the shape function are the temperature dependent equilibrium concentrations of the tin/lead phase diagram.

The eigenstrains are assumed to be given by

$$\begin{aligned} &\varepsilon^*_{hk}(\chi) = \alpha_{hk}(\chi)\,(T - T_R) \\ &\text{with} \qquad \alpha_{hk}(\chi) = \Theta(\chi)\,\alpha^\alpha_{hk} + (1 - \Theta(\chi))\,\alpha^\beta_{hk}. \end{aligned}$$

The matrices of thermal expansion coefficients of the phases are denoted by α^α_{hk} and α^β_{hk}, and T and T_R are the actual temperature and the reference temperature, respectively. We assume T and T_R to be two fixed constants since our analysis is confined to the isothermal case.

Next, we consider the diffusion flux which is given by the extended Cahn–Hilliard form

$$J_i = -M_{ij}(\chi)\,\frac{\partial \hat{w}}{\partial x_j}$$

where the potential $\hat{w}$ is defined according to

$$\begin{aligned} \hat{w} = {} & \frac{\partial \psi(\chi)}{\partial \chi} - a_{ij}(\chi)\,\frac{\partial^2 \chi}{\partial x_i \partial x_j} \\ & + \frac{1}{2}\frac{\partial}{\partial \chi}\Big((\varepsilon_{ij} - \varepsilon^*_{ij}(\chi))\, C_{ijhk}(\chi)\,(\varepsilon_{hk} - \varepsilon^*_{hk}(\chi)) \Big). \end{aligned}$$

The function $\psi(\chi)$ is the non-convex combined free energy of the phases, the matrix $a_{ij}(\chi)$ contains the gradient coefficients that can be related to interface surface tensions and the mobility appears also as a matrix here, i.e. $M_{ij}(\chi)$, in order to reflect the anisotropy of the diffusion process. The matrices $M_{ij}(\chi)$ and $a_{ij}(\chi)$ are constructed in the same way as the stiffness matrix and the eigenstrains, namely

$$\begin{aligned} M_{ij}(\chi) &= \Theta(\chi)\,M^\alpha_{ij} + (1 - \Theta(\chi))\,M^\beta_{ij} \\ a_{ij}(\chi) &= \Theta(\chi)\,a^\alpha_{ij} + (1 - \Theta(\chi))\,a^\beta_{ij}. \end{aligned}$$

Putting all the previous equations together and adding appropriate initial and boundary conditions, we obtain the problem we would like to discuss. However, it contains too many complexities and no definitely significant result from the viewpoint of mathematical analysis is published, as far as we know.

Our aim is to state some existence and uniqueness theorems for a model obtained by making some simplifications concerning the tensors involved in the model and including a relaxation term. We follow [2], whose results are just a little more general. Our first simplification and changes lead to the following setting

M_{ij} is the identity matrix
the matrix $a_{hk}(\chi)$ is replaced by a scalar $a(\chi)$
we modify ψ and force χ to attain only values within $[0, 1]$
we add the term $\mu\partial_t\chi$ to $\hat{w}$.

In the sequel, a is assumed to be Lipschitz continuous and strictly positive on $[0, 1]$.

As far as the last modification is concerned, we quote [7], where a contribution like $\mu\partial_t\chi$ in the chemical potential is considered in a more general framework. However, [7] just deals with modeling. On the contrary, results concerning the analytical viewpoint can be found in [4], [8], and especially in [6], where a similar problem is considered in a more general form, but with some simplification in a different direction. In particular, Garcke's matrix a_{ij} does not depend on χ.

Coming back to our framework, the new constitutive relation w-χ has to be properly read as a differential inclusion. Hence, we are led to the following system

$$\partial_{x_j}\sigma_{ij} = 0 \tag{1}$$

$$\sigma_{ij} = C_{ijhk}(\chi)(\varepsilon_{hk}(\mathbf{u}) - \varepsilon^*_{hk}(\chi)) \tag{2}$$

$$\varepsilon_{hk}(\mathbf{u}) = \frac{1}{2}\left(\partial_{x_h}u_k + \partial_{x_k}u_h\right) \tag{3}$$

$$\partial_t\chi - \Delta w = 0 \tag{4}$$

$$\begin{aligned} w \in \mu\,\partial_t\chi - a(\chi)\Delta\chi + \frac{\partial\psi(\chi)}{\partial\chi} + \partial I(\chi) - \sigma_{hk}\frac{\partial\varepsilon^*_{hk}(\chi)}{\partial\chi} \\ +\frac{1}{2}(\varepsilon_{ij}(\mathbf{u}) - \varepsilon^*_{ij}(\chi))C'_{ijhk}(\varepsilon_{hk}(\mathbf{u}) - \varepsilon^*_{hk}(\chi)). \end{aligned} \tag{5}$$

where I is the indicator function of the interval $[0,1]$ and

$$C'_{ijhk} := \frac{\partial C_{ijhk}(\chi)}{\partial\chi} \tag{6}$$

is assumed to be independent of χ. We set

$$\eta := \sup_{ijhk} |C'_{ijhk}|. \tag{7}$$

As said before, the above system has to be complemented by appropriate initial and boundary conditions. This will be done in the next section, where existence and uniqueness results are stated for the case $N = 1$, and for the case $N = 2$ provided that $\eta = 0$, i.e., the tensor C is independent of χ.

2. – Statement of the problem and results

Accounting also for the boundary and initial conditions, we can state the full problem, at least formally, as described below. To this aim, we explain our notation.

In the sequel, Ω denotes a bounded connected open set in $\mathbb{R}^N$ whose boundary Γ consists of two smooth and nonempty parts Γ_u and Γ_σ. We term $\mathbf{n}$ the outward unit normal on Γ and set

$$Q := \Omega \times (0,T), \quad \Sigma := \Gamma \times (0,T), \quad \Sigma_i := \Gamma_i \times (0,T) \quad \text{for } i = u, \sigma$$

where T is a given final time. We look for a quadruplet $(\mathbf{u}, \chi, \xi, w)$ defined in Q, where the displacement $\mathbf{u}$ is a vector valued function while χ, ξ, w are scalar valued functions, satisfying the couple of systems described below. The first one consists in the linear elasticity system for $\mathbf{u}$ with mixed boundary conditions, namely

$$\partial_{x_j}(C_{ijhk}(\chi)\,\varepsilon_{hk}(\mathbf{u}) + y_{ij}(\chi)) = 0 \qquad \text{in } Q \tag{8}$$
$$\mathbf{u} = 0 \qquad \text{on } \Sigma_u \tag{9}$$
$$(C_{ijhk}(\chi)\,\varepsilon_{hk}(\mathbf{u}) + y_{ij}(\chi))n_j = 0 \qquad \text{on } \Sigma_\sigma \tag{10}$$

where we have set

$$y_{ij}(\chi) := -C_{ijhk}(\chi)\,\varepsilon^*_{hk}(\chi) \qquad \text{for } \chi \in [0,1].$$

In the sequel, we assume y_{ij} to be Lipschitz continuous on $[0,1]$.

The second system is an initial–boundary value problem for a Cahn–Hilliard type equation for χ, namely

$$\partial_t\chi - \Delta w = 0 \qquad \text{in } Q \tag{11}$$
$$w = \mu\,\partial_t\chi - a(\chi)\Delta\chi + \xi + \gamma(\chi, \varepsilon(\mathbf{u})) \qquad \text{in } Q \tag{12}$$
$$\xi \in \beta(\chi) \qquad \text{in } Q \tag{13}$$
$$\nabla\chi \cdot \mathbf{n} = \nabla w \cdot \mathbf{n} = 0 \qquad \text{on } \Sigma \tag{14}$$
$$\chi(0) = \chi_0 \qquad \text{in } \Omega \tag{15}$$

where β is the subdifferential of the indicator function of $[0,1]$ and $\gamma = \gamma(\chi, \varepsilon)$ is a real function suitably related to the previous ones. Its variables χ and ε vary in the interval $[0,1]$ and in the space of second order symmetric tensors, respectively. Moreover, χ_0 is a prescribed initial datum.

More generally, β could be any maximal monotone operator in $\mathbb{R}^2$ with domain $[0,1]$ and account also for the monotone part of $\partial\psi/\partial\chi$. Clearly, γ has the form

$$\gamma(\chi, \varepsilon) = z_0(\chi) + z_{ij}(\chi)\,\varepsilon_{ij} + \frac{1}{2}\varepsilon_{ij}\,C'_{ijhk}\,\varepsilon_{hk}$$

for some functions z_0 and z_{ij} which we assume to be Lipschitz continuous. Finally, we ask the tensor C to satisfy the symmetry and ellipticity conditions

$$C_{ijhk} = C_{hkij} \qquad \text{and} \qquad C_{ijhk}\varepsilon_{hk}\varepsilon_{ij} \geq \alpha_0\varepsilon_{ij}\varepsilon_{ij}$$

for some positive constant α_0 and any symmetric tensor ε and assume that the variation of a is small enough, namely

$$\sup|a'| < \inf a.$$

Note that such an inequality is trivially satisfied if a is a positive constant.

Assuming that all the conditions we have introduced on the structure of system (8–15) are fulfilled, we state our results.

Existence theorem. *Assume either $N = 2$ and $\eta = 0$ or $N = 1$. Assume moreover*

$$\chi_0 \in H^1(\Omega), \qquad 0 \leq \chi_0 \leq 1, \qquad 0 < \frac{1}{|\Omega|}\int_\Omega \chi_0 < 1. \tag{16}$$

Then, there exists a quadruplet $(\mathbf{u}, \chi, \xi, w)$ *such that*

$$\text{(17)} \qquad \mathbf{u} \in L^\infty(0,T;H^1(\Omega)^N)$$
$$\text{(18)} \qquad \chi \in H^1(0,T;L^2(\Omega)) \cap C^0([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(Q)$$
$$\text{(19)} \qquad \xi \in L^2(Q)$$
$$\text{(20)} \qquad w \in L^2(0,T;H^2(\Omega))$$

which solves problem (8–15).

As far as uniqueness is concerned, we observe that the components ξ and w of a solution would be uniquely determined by $\mathbf{u}$ and χ if β were single valued. However, this is not the case in our framework. Hence, we look for a unique pair $(\mathbf{u}, \chi)$, only.

Uniqueness theorem. *Assume* (16) *and let* $(\mathbf{u}_i, \chi_i, \xi_i, w_i)$, $i = 1, 2$, *be two solutions to problem* (8–15) *satisfying* (17–20). *Then* $\mathbf{u}_1 = \mathbf{u}_2$ *and* $\chi_1 = \chi_2$ *provided that one of the following assumptions is fulfilled:* (*i*) $N = 1$; (*ii*) $N = 2$ *and the supplementary regularity condition*

$$\text{(21)} \qquad \mathbf{u}_i \in L^4(0,T;W^{1,4}(\Omega)^N)$$

holds for $i = 1, 2$; (*iii*) $N = 2$, $\eta = 0$, *and* (21) *holds for either* $i = 1$ *or* $i = 2$.

3. – Comments

This section is devoted to give some ideas on the proofs of our results and to explain why we are forced to make the restrictions that appear in the above statements. We follow [2] but we avoid technicalities if possible.

The main idea for the existence proof is to present system (8–15) as a fixed point problem. Roughly speaking, we construct two maps $\mathcal{F}_1$ and $\mathcal{F}_2$ as follows.

For a given phase parameter χ, we solve the elasticity system (8–10) for $\mathbf{u}$ and term $\mathcal{F}_1(\chi)$ its solution. Next, for a given displacement $\mathbf{u}$, we solve the Cahn-Hilliard system (11–15) for the triplet (χ, ξ, w) and term $\mathcal{F}_2(\mathbf{u})$ the first component χ of its solution. However, in order to be rigorous, we should choose the precise domains of the maps $\mathcal{F}_i$ and prove that they actually are well defined.

Once this has been done, it is clear that a fixed point χ for the composed map $\mathcal{F} := \mathcal{F}_2 \circ \mathcal{F}_1$ yields a solution to the full problem (8–15). Indeed, it is sufficient to take $\mathcal{F}_1(\chi)$ as $\mathbf{u}$ and the solution to (11–15) as (χ, ξ, w). Therefore, existence can be proved by checking that $\mathcal{F}$ fulfilles all the assumption of the Schauder fixed point theorem.

As far as the complete definition of $\mathcal{F}_1$ is concerned, we can sketch the procedure as follows. Denoting by $\|\,.\,\|$ the norm in $L^2(0,T;H^1(\Omega))$, we set

$$D_1 := \left\{ \chi \in L^2(0,T;H^1(\Omega)) : \quad \|\chi\| \le R, \quad 0 \le \chi \le 1 \right\}$$

where R is a positive parameter. Then we define $\mathcal{F}_1$ as described above on the domain D_1. Later on we choose R in order that the range of the composed map $\mathcal{F}$ is contained in D_1.

More precisely, the function $\mathbf{u} := \mathcal{F}_1(\chi)$ is defined through the variational formulation of (8–10), which reads

$$\mathbf{u}(t) \in H^1(\Omega), \qquad \mathbf{u}(t) = 0 \quad \text{on } \Gamma_u \tag{22}$$

$$\int_\Omega C_{ijhk}(\chi(t))\, \varepsilon_{hk}(\mathbf{u}(t))\, \varepsilon_{ij}(\mathbf{v}) = \int_\Omega y_{ij}(\chi(t))\, \varepsilon_{ij}(\mathbf{v}) \tag{23}$$

for any $\mathbf{v} \in H^1(\Omega)^N$ vanishing on Γ_u (here t is considered like a parameter).

Hence, $\mathbf{u}(t)$ is well defined for almost every t. Indeed, the Korn and Poincaré inequalities ensure that the formula

$$\|\mathbf{v}\|_1^2 := \int_\Omega \varepsilon_{ij}(\mathbf{v})\, \varepsilon_{ij}(\mathbf{v})$$

defines an equivalent norm on the subspace of the $\mathbf{v}$'s we are dealing with.

In order to define $\mathcal{F}_2$, we need firstly a basic a priori bound for the function $\mathbf{u} := \mathcal{F}_1(\chi)$. We take $\mathbf{v} = \mathbf{u}(t)$ in (23) and obtain

$$\|\mathbf{u}(t)\|_1^2 = \int_\Omega y_{ij}(\chi(t))\varepsilon_{ij}(\mathbf{u}(t)).$$

This yields immediately

$$\|\mathbf{u}\| \le c_* \tag{24}$$

where $\|\,.\,\|$ is now the norm in $L^\infty(0,T;H^1(\Omega)^N)$ and c_* is a constant independent of χ and R.

Note that the map $\mathcal{F}_1$ would be well defined whenever χ were measurable and bounded and that the condition $\chi \in D_1$ does not help in proving further space regularity for the corresponding solution $\mathbf{u}$. Indeed, one can just say that $\mathbf{u}$ satisfies some kind of Meyers estimates, i.e., that it is bounded in $L^\infty(0,T;L^p(\Omega)^N)$ for some $p > 2$ thanks to the general results of [9]. In particular, a requirement like (21) cannot be obtained, unless $N = 1$.

Next, we proceed in constructing $\mathcal{F}_2$. We define its domain D_2 using estimate (24) as follows

$$D_2 := \left\{\mathbf{u} \in L^2(0,T;H^1(\Omega)^N) : \ \|\mathbf{u}\| \le c_*\right\}$$

so that the range of $\mathcal{F}_1$ is contained in D_2 and it will make sense to define the composed map $\mathcal{F}$. However, making the definition of $\mathcal{F}_2$ rigorous corresponds to prove that problem (11–15) is well posed for a given $\mathbf{u}$ in an appropriate functional framework. Already the existence proof is not trivial and can be done by solving suitable and easier approximating problems and passing to the limit, provided that the last term on the right hand side of (12) belongs to $L^2(Q)$. For that reason one has to assume either $N = 1$ or $\eta = 0$ again.

In [2] this assumption is made and the maximal monotone operator β is replaced by its Yosida approximation β_λ, which is defined on the whole real line. Moreover, the functions a and γ are suitably extended to allow even an argument χ not belonging to $[0,1]$. The resulting approximating problem is the following system

$$\partial_t \chi - \Delta w = 0 \quad \text{in } Q \tag{25}$$
$$w = \mu\, \partial_t \chi - a(\chi)\Delta\chi + \xi + \gamma(\chi, \varepsilon(\mathbf{u})) \quad \text{in } Q \tag{26}$$
$$\xi = \beta_\lambda(\chi) \quad \text{in } Q \tag{27}$$
$$\nabla\chi \cdot \mathbf{n} = \nabla w \cdot \mathbf{n} = 0 \quad \text{on } \Sigma \tag{28}$$
$$\chi(0) = \chi_0 \quad \text{in } \Omega \tag{29}$$

which differs from (11–15) in equation (27) and has at least a solution $(\chi_\lambda, \xi_\lambda, w_\lambda)$.

In order to pass to the limit as λ tends to zero, one has to establish a number of a priori estimates on the approximate solution and use compactness results. The basic observation (this holds also for system (11–15)) is that the mean value of χ_λ does not depend on time, as one easily sees from equation (25) and the second boundary condition in (28). This allows us to choose a number of useful test functions, in particular the solution to the Neumann problem

$$-\Delta v = f \quad \text{in } \Omega, \qquad \nabla v \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \qquad \int_\Omega v = 0$$

where f is either $\chi(t) - \chi_0^*$ or $\partial_t\chi(t)$ and χ_0^* is the mean value of the initial datum χ_0.

The list of a priori estimate obtained in such a way ensures that the norms of χ_λ, ξ_λ, and w_λ remain bounded in the spaces occurring in (18–20) with one exception: as β_λ is everywhere defined, the inequalities $0 \le \chi_\lambda \le 1$ are false and χ_λ is not bounded in $L^\infty(Q)$. Nevertheless, we can deduce some strong convergence for χ_λ thanks to strong compactness results in Sobolev spaces. This allows us to deal with the nonlinear terms. In particular, we can apply the general theory of maximal monotone operators (see, e.g., [1] and [3]) and obtain the inequalities $0 \le \chi \le 1$ in the limit.

As the a priori estimates holds also in the limit, one shows that the χ component of the solution belongs to D_1 for a suitable choice of R, so that the domain of $\mathcal{F}_1$ can be made precise at this point and the composed map $\mathcal{F}$ actually maps D_1 into itself. The obtained a priori estimates on χ are also sufficient to show the continuity and compactness properties for $\mathcal{F}$ that are required to apply the Schauder theorem.

However, this can be used only once one knows that $\mathcal{F}_2$ actually is one valued, i.e., one has proved a uniqueness theorem for system (11–15) with a given $\mathbf{u}$. A standard approach to such a uniqueness result consists in writing the equations for two solutions and taking the difference. Next, one multiplies the obtained equalities for suitable test functions depending on the solutions themselves and starts estimating. The main troubles arise in dealing with the nonlinear terms and some Gagliardo-Nirenberg inequalities have to be used. That is why we cannot deal with the case $N > 2$. Similar difficulties occur in proving the uniqueness result for the whole problem (8–15).

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The Minimum Free Energy of Compressible Viscoelastic Fluids *

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1. – Introduction

Recently, explicit formulae have been given for the maximum recoverable work from a specified viscoelastic state, under isothermal conditions, for a scalar [8] and a general tensor [1] constitutive relation. These formulae also represent the minimum free energy associated with a given viscoelastic state, by virtue of general theorems identifying this quantity with the maximum recoverable work [3].

This paper is based on [2], the aim of which is to derive an expression for the minimum free energy of a compressible viscoelastic fluid with linear dependence on the history of strain. Essentially the same techniques apply, though there are significant differences with respect to the earlier work [8, 1]. Firstly, there is the equilibrium pressure term, with a non-linear dependence on the density, in the constitutive relations; and secondly, there is the fact that the constitutive relations are those for an isotropic material. This is of course a special case of the full anisotropic tensorial treatment in [1]. However, it is particularly interesting in that a factorization, which is fundamental to the methodology, can be carried out as explicitly as in the scalar case [8] for general viscoelastic response. In the full anisotropic case treated in [1], it can be proved that the required factorization exists, but no general method for determining the factors has yet been given; though of course, this can be done for specific material responses.

In order to obtain the process which provides the maximum recoverable work, it is necessary to solve an integral equation of the Wiener-Hopf type. An existence and uniqueness theorem is given for this integral equation, using conditions which follow from thermodynamics.

In section 2, constitutive equations are given for a particular class of compressible viscoelastic fluids with linear dependence on strain history. Also, thermodynamic

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states and processes are defined, and the notion of equivalent states is introduced. In section 3, certain thermodynamic concepts and results are presented, with application to the particular types of material under consideration.

In section 4, the crucial factorization is carried out, while in the following section, the Wiener-Hopf integral equation for the process yielding the maximum recoverable work is derived and shown to have a unique solution. In section 6, an explicit formula for the minimum free energy is constructed and discussed; and a function on the equivalence class of states is presented.

Certain notational usages are defined and basic formulae listed in an appendix.

In the current summary version, proofs are omitted and also some of the discussion.

2. – A Particular Class of Compressible Viscoelastic Fluids

The state σ of a compressible viscoelastic fluid can be described [5, 10] by means of the mass density $\rho(\mathbf{x},t)$ and the history of strain $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\top})$ ($\mathbf{u}$ is the desplacement) relative to its present value, i.e. by means of the couple $(\rho, \mathbf{E}_r^t)$ where the relative strain history $\mathbf{E}_r^t$ is defined by $\mathbf{E}_r^t(\mathbf{x},s) = \mathbf{E}^t(\mathbf{x},s) - \mathbf{E}(\mathbf{x},t)$ $s \in \mathcal{R}^{++}$ using the notation $\mathbf{E}^t(\mathbf{x},s) = \mathbf{E}(\mathbf{x},t-s)$ for the strain history. The dependence on the spatial variable $\mathbf{x}$ will henceforth be omitted.

A fluid is necessarily isotropic. We will also assume for simplicity that it is homogeneous. The constitutive equation for the stress is given by

$$\mathbf{T}(\rho,\mathbf{E}_r^t) = -p(\rho)\mathbf{I} + \tilde{\mathbf{T}}(\rho,\mathbf{E}_r^t) \tag{1}$$

where $\mathbf{I}$ is the identity second order tensor and p denotes the pressure. The quantity $\tilde{\mathbf{T}}$ is referred to as the extra stress and is given by (see (A.2)):

$$\tilde{\mathbf{T}}(\rho,\mathbf{E}_r^t) = \rho\int_0^\infty \lambda'(s)E_r^t(s)\,ds\mathbf{I} + 2\int_0^\infty \mu'(s)\mathbf{E}_r^t(s)\,ds\,. \tag{2}$$

where the memory kernels λ', μ' have the property (the prime dnoting differentiation)

$$\lambda', \mu' \in (L^1 \cap L^2)(\mathcal{R}^+)\,. \tag{3}$$

Moreover we assume that

$$\lambda'(0), \mu'(0) \in \mathcal{R}^{--}\,. \tag{4}$$

and

$$\lambda, \mu \in (L^1 \cap L^2)(\mathcal{R}^+)\,. \tag{5}$$

where $\lambda(t) = -\int_t^\infty \lambda'(s)\,ds$ and $\mu(t) = -\int_t^\infty \mu'(s)\,ds$.

Let $\mathbf{E}^\dagger$ denote a constant history *i.e.* $\mathbf{E}^t(s) = \mathbf{E}^\dagger$ $\forall s \in \mathcal{R}^{++}$. If $\mathbf{E}^t = \mathbf{E}^\dagger$, the relative strain history $\mathbf{E}_r^t = \mathbf{0}^\dagger$ where $\mathbf{0}^\dagger$ is the zero strain history. In this case, the extra stress is zero:

$$\tilde{\mathbf{T}}(\rho,\mathbf{0}^\dagger) = \mathbf{0} \tag{6}$$

Moreover the *relaxation property* ensures that, for a static continuation $\mathbf{E}^{t+\tau}_{r\,c}$, defined as

$$\mathbf{E}^{t+\tau}_{r\,c}(s) = \begin{cases} \mathbf{0} & \text{for } s \leq \tau \\ \mathbf{E}^t_r(s-\tau) & \text{for } s > \tau \end{cases} , \tag{7}$$

the extra stress vanishes as τ diverges, viz.

$$\lim_{\tau\to\infty} \tilde{\mathbf{T}}(\rho, \mathbf{E}^{t+\tau}_{r\,c}) = \tilde{\mathbf{T}}(\rho, \mathbf{0}^\dagger) = \mathbf{0} \tag{8}$$

Given the state $\sigma(t) = (\rho(t), \mathbf{E}^t_r)$ the stress is uniquely determined by (1-2).

A process P of duration d_p will be described by a funtion $\mathbf{D}^P : [0, d_p) \to Sym$ where $\mathbf{D}^P(\tau) = \dot{\mathbf{E}}_P(\tau)$, the derivative of a strain tensor specified over a time segment of duration d_P. For any process $\mathbf{D}^P$ the evolution of $\sigma(t+\tau) = (\rho(t+\tau), \mathbf{E}^{t+\tau}_r)$, $\tau \in [0, d_p)$, is determined as the solution of the differential equations given by

$$\frac{d}{d\tau}\mathbf{E}^{t+\tau}_r(s) = \mathbf{D}^P(\tau - s) - \mathbf{D}^P(\tau) , \quad 0 < s < \tau \tag{9}$$

and the balance of mass[1]

$$\frac{d}{d\tau}\rho(t+\tau) = -\rho(t+\tau)\nabla \cdot \mathbf{v}(t+\tau) = -\rho(t+\tau)D^P(\tau) . \tag{10}$$

The effect of a given process on a particular strain history is described in more detail in [1]. Note that the solution of (10) is given by

$$\rho(t+\tau) = \rho(t) \exp\left[-\int_0^\tau D^P(s)\, ds\right] \tag{11}$$

Henceforth, Π will denote the set of all admissible process $\mathbf{D}^P$ of finite duration whereas the set of all admissible states will be denoted by Σ: [2]

$$\Sigma = \{\sigma = (\rho, \mathbf{E}^t_r) : |\mathbf{T}(\rho, \mathbf{E}^{t+\tau}_{r\,c})| < \infty , \ \forall \tau \geq 0\} . \tag{12}$$

Actually two different states $\sigma_1 = (\rho_1(t), \mathbf{E}^t_{1r})$ and $\sigma_1 = (\rho_2(t), \mathbf{E}^t_{2r})$ may yield the same stress $\mathbf{T}$. We recall the following definition [11]:

DEFINITION 1 Two states $\sigma_1(t) = (\rho_1(t), \mathbf{E}^t_{1r})$ and $\sigma_2(t) = (\rho_2(t), \mathbf{E}^t_{2r})$ are said to be equivalent if, for every process $\mathbf{D} : [0, d_p) \to Sym$, the subsequent states, $\sigma_1(t+\tau)$ and $\sigma_2(t+\tau)$, $\tau \in [0, d_p)$, obtained by (9-10), satisfy

$$\mathbf{T}(\rho_1(t+\tau), \mathbf{E}^{t+\tau}_{1r}) = \mathbf{T}(\rho_2(t+\tau), \mathbf{E}^{t+\tau}_{2r}) , \quad \forall \tau \in [0, d_p) \tag{13}$$

[1]Note that $D^P = tr(\dot{\mathbf{E}})$.

[2]In the sequel we make use of a slightly more restrictive definition, in order to render Σ compatible with a linear thermodynamic theory.

Such a definition introduces an equivalence relation, whose equivalence classes are named the minimal states $\sigma_{(m)}$ of the material. In other words, if σ_1 and σ_2 are equivalent in the sense of Definition 1, they represent the same state $\sigma_{(m)}$.

Thus the space of the minimal states $\Sigma_{(m)}$ is the space of the equivalent classes of Σ induced by Definition 1.

Observe that, by virtue of decomposition (A.2), the constitutive equations (1-2) may be rewritten as

$$\mathbf{T}(\rho, \mathbf{E}_r^t) = -p(\rho)\mathbf{I} + \rho \int_0^\infty \kappa'(s) E_r^t(s)\, ds \mathbf{I} + 2\rho \int_0^\infty \mu'(s) \mathring{\mathbf{E}}_r^t(s)\, ds \tag{14}$$

where $\kappa_1 = \lambda_1 + \frac{2}{3}\mu_1$. Equation (14) may be written in compact form as

$$\begin{aligned} \mathbf{T}(\rho, \mathbf{E}_r^t) &= -p(\rho)\mathbf{I} + \rho \mathbf{V}(\mathbf{E}_r^t) \\ \mathbf{V}(\mathbf{E}_r^t) &= \textstyle\int_0^\infty \mathbb{G}'(s) \mathbf{E}_r^t(s)\, ds \end{aligned} \tag{15}$$

where the relaxation function $\mathbb{G}'$ is a fouth-order tensor valued function $\mathbb{G}' : \mathcal{R}^+ \to \mathcal{D}(Sym)$. The first element of $\mathbb{G}'(s)$ in the expansion (A.6) is given by $\kappa'(s)$ whereas the other non-vanishing (*i.e.* diagonal) elements are equal to $2\mu'(s)$.

For materials of type (15), under the assumption that any finite density ρ yields a finite pressure $p(\rho)$, it is easy to check that the space of admissible states Σ, given by (12), may be written as $\Sigma = \mathcal{R}^+ \times \Gamma$ where

$$\Gamma = \left\{ \mathbf{E}_r^t : \left| \int_0^\infty \mathbb{G}'(s+\tau) \mathbf{E}_r^t(s)\, ds \right| < \infty \,, \ \forall \tau \geq 0 \right\}$$

Moreover the state space $\Sigma_{(m)}$ can be characterised by means of the following property:

THEOREM 1 *For a viscoelastic fluid of type (2), two states* $\sigma_1 = (\rho_1, \mathbf{E}_{1r}^t)$ *and* $\sigma_2 = (\rho_2, \mathbf{E}_{2r}^t)$ *are equivalent in the sense of Definition 1 if and only if*

$$\rho_1(t) = \rho_2(t), \ \int_0^\infty \mu'(s+\tau) \mathring{\mathbf{E}}_r^t(s)\, ds = 0, \ \int_0^\infty \kappa'(s+\tau) E_r^t(s)\, ds = 0, \ \forall \tau \geq 0 \tag{16}$$

where $\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2$.

Denoting by Γ_0 the set of all the histories $\mathbf{E}_r^t \in \Gamma$ satisfying $(16)_2$ and $(16)_3$, and by Γ/Γ_0 the usual quotient space, Theorem 1 implies that the minimal state of a linear viscoelastic material is an element of

$$\Sigma_{(m)} := \mathcal{R}^+ \times (\Gamma/\Gamma_0) \tag{17}$$

We also view a process as a function $P : \Sigma \to \Sigma$ which associates with an initial state $\sigma^i \in \Sigma$ a final state $P\sigma^i = \sigma^f \in \Sigma$. Such an evolution is governed by the differential equations (9–10). Considering P as a function $P : \Gamma \to \Gamma$ associating with any initial history $\gamma^i \in \Gamma$, a final history $P\gamma^i = \gamma^f \in \Gamma$, the evolution related to P is governed only by (9).

3. – Thermodynamics

We confine our attention to isothermal processes, so that the Second Law of Thermodynamics reduces to the Dissipation Principle stating that the work on a cycle is non-negative. A stronger principle is adopted here for reasons given below.

A set $\mathcal{S} \subset \Sigma$ is said to be *invariant* if for every $\sigma_1 \in \mathcal{S}$, and $P \in \Pi$, the state $\sigma = P\sigma_1 \in \mathcal{S}$.

DEFINITION 2 *A function $\psi : \mathcal{S} \to \mathcal{R}^+$ is a* free energy density[3] *if :*
i. the domain $\mathcal{S}$ is invariant,
ii. for any pair $\sigma_1, \sigma_2 \in \mathcal{S}$ and $P \in \Pi$, such that $P\sigma_1 = \sigma_2$ we have

$$\psi(\sigma_2) - \psi(\sigma_1) \leq W(\sigma_1, P) \tag{18}$$

A state $\sigma \in \Sigma$ is referred to as *attainable* from all of Σ if, for any initial state σ^i, there exists a process $P \in \Pi$ such that $P\sigma^i = \sigma$. A simple material system is attainable if any state σ is attainable from every other state $\sigma' \in \Sigma$

However, for a simple material with fading memory, not all states are attainable. In particular, cyclic processes constitute a very narrow class of processes related to a very narrow set of states. For this reason the Dissipation Principle is not restrictive enough and we need a stronger formulation of the Second Law.

To this aim we denote by

$$\mathcal{W}(\sigma) := \{W(\sigma, P)\,;\, P \in \Pi\} \tag{19}$$

the set of the works done by all possible processes $P \in \Pi$ acting on a given state σ. The following principle shall be adopted

Strong Dissipation Principle The set $\mathcal{W}(\sigma)$ is bounded from below for all $\sigma \in \Sigma$. Furthermore, there is a state $\sigma^\dagger$, which we refer to as the zero state, such that

$$\inf\Big\{W(\sigma^\dagger, P)\,;\; P \in \Pi\Big\} = 0 \tag{20}$$

The Strong Dissipation Principle requires that we redefine the set of all admissible states Σ, modifying (12) as follows

$$\Sigma^T = \{\sigma = (\rho, \mathbf{E}_r^t) : |\mathbf{T}(\rho, \mathbf{E}_{r\,c}^{t+\tau})| < \infty\;,\; \forall\tau \geq 0\}\,.$$

$$\Sigma := \Big\{\sigma \in \Sigma^T : \inf \mathcal{W}(\sigma) > -\infty\Big\} \tag{21}$$

If $(21)_2$ were not true, we would have a contradiction with the Second Law. In fact, $-W(\sigma, P)$ is the work yielded by the material. If it were unbounded from above, as P varies, we could extract infinite energy from the material, and then generate a perpetual motion.

For a material of type (1-2), the zero state is $\sigma^\dagger = (\rho_0, \mathbf{0}^\dagger)$, where ρ_0 is the equilibrium mass density and $\mathbf{0}^\dagger$ is the zero history introduced before (6).

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[3]Henceforth the free energy density will be referred to simply as the free energy.

DEFINITION 3 A functional ψ_m is the *minimum free energy* if:

i) ψ_m is a free energy in the sense of Definition 2 with domain $\mathcal{S} = \Sigma$.

ii) the zero state $\sigma^\dagger \in \Sigma$ is such that $\psi_m(\sigma^\dagger) = 0$

iii) for any free energy $\psi : \mathcal{S} \to \mathcal{R}^+$ such that $\sigma^\dagger \in \mathcal{S}$, and $\psi(\sigma^\dagger) = 0$, we have:

$$\psi(\sigma) \geq \psi_m(\sigma) \quad , \quad \text{forall } \sigma \in \mathcal{S} \tag{22}$$

THEOREM 2 *The functional*

$$\psi_m(\sigma) := -\inf \mathcal{W}(\sigma)$$

is the minimum free energy.

The proof of this theorem has been given in [3].

It is always possible to represent the minimum free energy as a function of the minimal state σ_m. In fact, if two equivalent states σ_1 and σ_2, continued with the same process P, yield the same stress, then they also yield the same work, giving $W(\sigma_1, P) = W(\sigma_2, P)$. Therefore $W(\sigma, P) = \hat{W}(\sigma_{(m)}, P)$ so that $\mathcal{W}(\sigma) = \hat{\mathcal{W}}(\sigma_m)$ and

$$\inf \mathcal{W}(\sigma) = \inf \hat{\mathcal{W}}(\sigma_m) \; .$$

As a consequence

$$\psi_m(\sigma) = \hat{\psi}_m(\sigma_m) \tag{23}$$

Hence the minimum free energy is independent of the definition of state that is used.

We conclude the section by stating an important property of the free energy of a material described by (1) and (2).

THEOREM 3 *For materials described by (1) and (2), every free energy may be written as the sum of two terms*

$$\psi(\sigma) = \phi(\rho) + \varphi(\gamma) \tag{24}$$

where

$$\phi(\rho) = \int_{\rho_0}^{\rho} \frac{1}{\xi^2}\, p(\xi)\, d\xi \tag{25}$$

ρ_0 being the equilibrium mass density and $\varphi : \mathcal{S}_\Gamma \to \mathcal{R}$ is defined on a set $\mathcal{S}_\Gamma$ that is Γ-invariant, (namely, if $\gamma \in \mathcal{S}_\Gamma$ then $P\gamma \in \mathcal{S}_\Gamma$ for every $P \in \Pi$), and satisfies:

$$\varphi(\gamma_2) - \varphi(\gamma_1) \leq \int_0^{d_p} \mathbf{V}(\mathbf{E}_r^t) \cdot \mathbf{D}(t)\, dt \tag{26}$$

where $P\gamma_1 = \gamma_2$ and we have dropped the superscript P on $\mathbf{D}$. Moreover if $\psi(\sigma^\dagger) = 0$, then

$$\varphi(0^\dagger) = 0 \tag{27}$$

Henceforth the right hand side of (26) will be termed the Γ-work and denoted by

$$W(\gamma, P) = \int_0^{d_p} \frac{1}{\rho}\, \tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t) \cdot \mathbf{D}(t)\, dt = \int_0^{d_p} \mathbf{V}(\mathbf{E}_r^t) \cdot \mathbf{D}(t)\, dt \tag{28}$$

whereas $\mathcal{W}_\Gamma(\gamma)$ will denote the set of all the Γ-works starting from $\gamma \in \Gamma$, viz.

$$\mathcal{W}_\Gamma(\gamma) = \{W(\gamma, P), P \in \Pi\}$$

It is easy to check that Theorems 2 and 3 imply that the minimum free energy can be written as

$$\psi_m(\sigma) = \phi(\rho) + \varphi_m(\gamma) \tag{29}$$

with φ_m given by:

$$\varphi_m(\gamma) := -\inf \mathcal{W}_\Gamma(\gamma) \tag{30}$$

The right hand side of (30) represents the maximum recoverable Γ-work.

4. – Thermodynamic restrictions and factorization

Before determining the "optimal" process maximizing the recoverable Γ-work, we recall some properties of the relaxation function $\mathbb{G}$ required by thermodynamics.

In particular, as a consequence of the Dissipation Principle, for materials described by constitutive equations (1-15) the memory kernels κ' and μ' must satisfy [6]

$$\frac{1}{\omega}\mu'_s(\omega) < 0\,, \qquad \frac{1}{\omega}\kappa'_s(\omega) < 0\,, \qquad \forall\omega \in \mathcal{R}\,, \tag{31}$$

using the notation defined by (A.7). Since

$$\mathbb{G}_c(\omega) = -\frac{1}{\omega}\mathbb{G}'_s(\omega), \tag{32}$$

the thermodynamic restrictions (31) ensure that $\mathbb{G}_c(\omega)$ is positive definite for every $\omega \in \mathcal{R}$. Moreover $\mathbb{G}_c(\omega)$ vanishes as ω^{-2} as ω tends to infinity. In fact it is easy to check that, if $\mathbb{G}''$ is integrable, we have

$$\lim_{\omega\to\infty} \omega^2\mathbb{G}_c(\omega) = -\mathbb{G}'(0) \tag{33}$$

where $\mathbb{G}'(0)$ is negative definite by virtue of (4).

The above thermodynamic properties ensure that the function $\mathbb{G}_c(\omega)$ may be factorized by [1, 7]

$$\mathbb{G}_c(\omega) = \mathbb{G}_{(+)}(\omega)\mathbb{G}_{(-)}(\omega) \tag{34}$$

where the singularities of $\mathbb{G}_{(\pm)}$ on the complex plane, and the zeros of its determinant, are all in $\Omega^{(\pm)}$ respectively.

Because of the property that $\mathbb{G}_c(\omega) \in \mathcal{D}(Sym)$, an explicit general expression can be given for the factors [2], generalizing a result derived for the scalar case in [8].

5. – Maximum Recoverable Γ-Work

Results are now summarized which prove that the problem of finding the "optimal" process maximizing the recoverable Γ-work has one and only one solution by virtue of the thermodynamic properties of the relaxation function.

From here on, we denote by $\mathbb{G}(|s|)$ the extension of $\mathbb{G}(s)$ to an even function on $\mathcal{R}$. The process which maximizes the recoverable work can be shown [2] to obey the integral equation

$$\int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}^{(m)}(\tau)\,d\tau \;= \mathbf{I}_0(t,\mathbf{E}_r^0) \quad t \in \mathcal{R}^+. \tag{35}$$

where

$$\mathbf{I}_0(t,\mathbf{E}_r^0) = -\int_0^\infty \mathbb{G}'(t+\tau)\mathbf{E}_r^0(\tau)\,d\tau. \tag{36}$$

Equation (35) is a Wiener-Hopf equation, the solution of which maximizes the recoverable Γ-work. From

$$\varphi_m(\gamma_0) = -\inf\{W(\gamma_0, P),\ \forall\ P \in \Pi\}$$

it can be shown that [2]

$$\varphi_m(\mathbf{E}_r^0) = \frac{1}{2}\int_0^\infty\int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}^{(m)}(t)\cdot\mathbf{D}^{(m)}(\tau)d\tau dt \tag{37}$$

where $\mathbf{D}^{(m)}$ is now the solution of the equation (35). For this reason it is important to prove the existence and uniqueness of the solutions of Wiener-Hopf equation (35). We denote with $\mathcal{G}$ the completion of the set $\tilde{\mathcal{G}}$ defined as

$$\tilde{\mathcal{G}} = \left\{\mathbf{D} : [0,\infty) \to Sym;\ \int_0^\infty\int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}(t)\cdot\mathbf{D}(\tau)d\tau dt < \infty .\right\} \tag{38}$$

with respect to the norm $\|\cdot\|_{\mathcal{G}}$ defined by

$$\|\cdot\|_{\mathcal{G}} = \int_0^\infty\int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}(t)\cdot\mathbf{D}(\tau)d\tau dt \tag{39}$$

The thermodynamic restrictions (31) imply that the kernel $\mathbb{G}(|t|)$ is positive definite), as may be seen from the frequency domain representation of (39) [4]. Then we can introduce an inner product on $\mathcal{G}$ defined by

$$(\mathbf{D}_1\cdot\mathbf{D}_2) = \int_0^\infty\int_0^\infty \mathbb{G}(|t-\tau|)\mathbf{D}_1(t)\cdot\mathbf{D}_2(\tau)d\tau dt$$

which makes $\mathcal{G}$ a Hilbert space. The set of processes Π is a subset of $\mathcal{G}$.

REMARK 1 By means of the norm of $\mathcal{G}$, it is possible to provide the set of the processes Π with a topology. In particular, the closure of Π using the norm (39) is the Hilbert space $\mathcal{G}$.

The equation (35) can be written as

$$\mathcal{A}\mathbf{D} = \mathbf{I}_0 \tag{40}$$

where $\mathcal{A}$ is an operator from $\mathcal{G}$ to its dual $\mathcal{G}'$. It is bounded and coercive. Then, from the Lax-Milgram theorem, we can give the following

THEOREM 4 *For any* $\mathbf{I}_0 \in \mathcal{G}'$, *equation (35) has a unique solution* $\mathbf{D} \in \mathcal{G}$. *such that*

$$\|\mathbf{D}\|_{\mathcal{G}} \leq K \,\|\mathbf{I}_0\|_{\mathcal{G}'}$$

In other words, there exists an isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. Moreover, we have from Definition 3.1 the following

PROPOSITION 1 *Two histories* $\mathbf{E}_1^0$, $\mathbf{E}_2^0$ *correspond to two equivalent states in the sense of* (16_2-16_3) *if and only if*

$$\mathbf{I}_0(t, \mathbf{E}_1^0) = \mathbf{I}_0(t, \mathbf{E}_2^0) \quad \forall t \in \mathcal{R}^+ \tag{41}$$

REMARK 2 Proposition 1 yields a bijective map between $\mathcal{G}'$ and the quotient space $\Gamma_{(m)} = \Gamma/\Gamma_0$. In other words it is possible to identify any class of equivalent histories with a function $\mathbf{I}_0$.

This result allows us to represent the minimum free energy as a function defined on the set $\Gamma_{(m)}$ of equivalent histories.

6. – Construction of the minimum free energy

Rewriting the Wiener Hopf equation (35) at any time t (rather than $t = 0$), we obtain

$$\int_0^\infty \mathbb{G}(|\tau - s|)\mathbf{D}^{(m)}(s)\, ds \;= \mathbf{I}_0(\tau, \mathbf{E}_r^t)\ , \ \tau > 0 \tag{42}$$

with

$$\mathbf{I}_0(\tau, \mathbf{E}_r^t) = -\int_0^\infty \dot{\mathbb{G}}(\tau + s)\mathbf{E}_r^t(s)\, ds\ , \qquad \tau \geq 0 \tag{43}$$

and where $\mathbf{D}^{(m)}$ is the optimal process acting on γ. The maximum recoverable work gives the minimum free energy $\psi_m(\rho), \mathbf{E}_r^t) = \phi(\rho) + \varphi_m(\mathbf{E}_r^t)$ with ϕ defined by (25) and φ_m, the maximum recoverable Γ-work, given by

$$\varphi_m(\mathbf{E}_r^t) = \frac{1}{2}\int_0^\infty \int_0^\infty \mathbb{G}(|\tau - s|)\mathbf{D}^{(m)}(\tau) \cdot \mathbf{D}^{(m)}(s) d\tau ds \tag{44}$$

Wiener-Hopf equations of the first kind are not solvable in the general case. Nevertheless the thermodynamic properties of the integral kernel $\mathbb{G}$ allow us to determine the solution $\mathbf{D}^{(m)}$ of (42). Details of the derivation are given in [2]. One obtains

$$\begin{aligned} \mathbf{D}_+^{(m)}(\omega) &= -\mathbb{G}_{(+)}(\omega)^{-1}\mathbf{p}_{(+)}^t(\omega) \\ \mathbf{p}_{(-)}^t(\omega) &= -\frac{1}{2}\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{r}_-(\omega) \end{aligned} \tag{45}$$

using the notation for Fourier transforms defined in (A.7) and where $\mathbf{p}_{(\pm)}^t(z)$ are analytic respectively for $z \in \Omega^\mp$ (see (A.10)) and are defined by:

$$\begin{aligned} \mathbf{p}^t(z) &:= \frac{1}{4\pi i}\int_{-\infty}^\infty \frac{\mathbb{G}_{(-)}^{-1}(\omega)\mathbf{I}_+^t(\omega)}{\omega - z} d\omega, \quad z \in \Omega \setminus \mathcal{R}, \\ \mathbf{p}_{(\pm)}^t(\omega) &:= \lim_{\alpha \to 0^\mp} \mathbf{p}^t(\omega + i\alpha) \end{aligned} \tag{46}$$

REMARK 3 : It follows that

$$\lim_{\omega\to\infty} \mathbf{D}_+^{(m)}(\omega) \neq \mathbf{0} \,,$$

so that $\mathbf{D}^{(m)}(\tau)$ has an initial delta-function type singularity as $\tau \to 0^+$. Thus the optimal continuation $\mathbf{E}^{(m)}$, where $\mathbf{D}^{(m)} = \dot{\mathbf{E}}^{(m)}$, has an initial discontinuity as $\tau \to 0^+$ so that $\mathbf{E}(0^+) \neq \mathbf{E}(t)$.

REMARK 4 : Since $\det \mathbb{G}_{(+)}(0) \neq 0$, it follows that

$$\mathbf{E}^{(m)}(\infty) - \mathbf{E}^{(m)}(0^-) = \int_{0^-}^{\infty} \mathbf{D}^{(m)}(\tau)d\tau = \mathbf{D}_+^{(m)}(0) = -\mathbb{G}_{(+)}(0)^{-1}\mathbf{p}_{(+)}^t(0)$$

where we have emphasized that the integral includes the discontinuity. Therefore, the optimal continuation tends to the finite limit

$$\lim_{\tau\to\infty} \mathbf{E}^{(m)}(\tau) = \mathbf{E}^{(m)}(\infty) = \mathbf{E}(t) - \mathbb{G}_{(+)}(0)^{-1}\mathbf{p}_{(+)}^t(0)$$

The substitution of (45) into (44), yields

$$\begin{aligned}
\varphi_m(\mathbf{E}_r^t) &= \frac{1}{2}\int_0^\infty \int_0^\infty \mathbb{G}(|\tau - s|)\mathbf{D}^{(m)}(s)\cdot\mathbf{D}^{(m)}(\tau)\,ds\,d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{G}_c(\omega)\mathbf{D}_+^{(m)}(\omega)\cdot\overline{\mathbf{D}_+^{(m)}(\omega)}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}_{(+)}^t(\omega)|^2\,d\omega
\end{aligned} \tag{47}$$

Therefore the minimum free energy takes the form

$$\psi_m(\rho, \mathbf{E}_r^t) = \phi(\rho) + \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}_{(+)}^t(\omega)|^2\,d\omega \tag{48}$$

Actually, in view of equations (42) and (44), Proposition 1 and Remark 2, it is clear that φ_m is a function of the element γ_m of $\Gamma_{(m)}$, namely

$$\hat{\varphi}_m(\gamma_m) = \varphi_m(\mathbf{E}_r^t)$$

and hence $\mathbf{p}_{(+)}^t$ provides an explicit representation of equivalence class γ_m, as explicitely shown by the following theorem.

THEOREM 5 *For every viscoelastic material with a symmetric relaxation function, a given strain history $\mathbf{E}_r^t$ is equivalent to the zero history $\mathbf{0}^\dagger$ in the sense of (16)$_2$ and (16)$_3$ if and only if the quantity $\mathbf{p}_{(+)}^t$ is such that*

$$\mathbf{p}_{(+)}^t(\omega) = 0 \,, \quad \forall\omega \in \mathcal{R} \tag{49}$$

As a consequence of expression (47) and Theorem 5, we have that $\hat{\varphi}_m$ provides a norm in $\Gamma_{(m)}$, namely

$$\|\gamma_m\|^2 = \hat{\varphi}_m(\gamma_m) \,.$$

Thus the minimum free energy ψ_m induces a norm in the space of the minimal states $\Sigma_{(m)}$. In fact, if $\sigma_m = (\rho, \gamma_m)$ and $\hat{\psi}(\sigma_m) = \psi(\rho, \mathbf{E}_r^t)$, then equations (47–48) yield

$$\|\sigma_m\|^2 = \hat{\psi}(\sigma_m) = \phi(\rho) + \hat{\varphi}_m(\gamma_m) \,.$$

7. – Concluding remarks

The results obtained above are entirely consistent with those in [8, 1] for a linear viscoelastic solid. Also, the explicit results for materials with relaxation functions given by a sum of exponentials presented in [8, 1] are equally applicable in the present case, on taking account of the notational equivalences specified in the last paragraph and on putting $\mathbb{G}(\infty) = 0$.

In obtaining these explicit results in [1], it is assumed that all fourth order tensors are simultaneously diagonalizable; and if the results are to be used, it must be possible to find the diagonal forms explicitly. Observe that in the present work, the diagonal form for the relaxation tensor is achieved without difficulty by (14) and the factors $\mathbb{G}_\pm$ are diagonal, so the no assumptions are necessary.

The results presented here apply also to isotropic linear viscoelastic solids, with the minor modification of introducing a non-zero $\mathbb{G}(\infty)$.

Appendix: Notation and basic formulae

The space of symmetric second order tensors acting on $\mathcal{R}^3$ is denoted by Sym and is isomorphic to $\mathcal{R}^6$. Operating on Sym is the space of fourth order tensors $Lin(Sym)$. We shall often write a second order tensor $\mathbf{A}$ in terms of its trace A and its trace-free part $\overset{\circ}{\mathbf{A}}$,

$$(A.1) \qquad A = tr(\mathbf{A}), \qquad \overset{\circ}{\mathbf{A}} = \mathbf{A} - \frac{1}{3}A\mathbf{I}\,,$$

where $\mathbf{I}$ is the identity tensor in Sym.

We introduce a scalar product on Sym as follows: if $\mathbf{A}, \mathbf{B} \in Sym$ then $\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{AB})$. The associated norm is defined as $|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A}$. Since $\mathbf{I}$ and $\overset{\circ}{\mathbf{A}}$ are orthogonal, *i.e.* $tr(\mathbf{I}\overset{\circ}{\mathbf{A}}) = 0$, for every $\mathbf{A} \in Sym$, the decomposition

$$(A.2) \qquad \mathbf{A} = \frac{1}{3}A\,\mathbf{I} + \overset{\circ}{\mathbf{A}}$$

is unique, namely for any tensor $\mathbf{A} \in Sym$ there exist a unique scalar A and a unique trace free tensor $\overset{\circ}{\mathbf{A}} \in Sym$ satisfying (A.2). As a consequence, the decomposition (A.2) allows us to introduce an orthonormal basis of Sym

$$(A.3) \qquad \mathbf{N}_1, ..., \mathbf{N}_6 \ : \ tr(\mathbf{N}_h\mathbf{N}_k) = \delta_{hk}\,, \qquad \mathbf{N}_1 = \frac{1}{\sqrt{3}}\mathbf{I}\,, \quad \mathbf{N}_i = \overset{\circ}{\mathbf{N}}_h\,, \ h = 2, ..., 6\,,$$

δ_{hk} being the Kronecker symbol.

Henceforth we treat each tensor $\mathbf{A} \in Sym$ as a vector in $\mathcal{R}^6$ whose first component is $\frac{1}{\sqrt{3}}A = \frac{1}{\sqrt{3}}tr(\mathbf{A})$. Observe that if

$$(A.4) \qquad \mathbf{A} = \sum_{i=1}^{6} A_i\mathbf{N}_i\,, \qquad \mathbf{B} = \sum_{i=1}^{6} B_i\mathbf{N}_i\,,$$

then

$$\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{AB}) = \sum_{i=1}^{6} A_iB_i\,,$$

Consequently [9] any fourth order tensor $\mathbb{K} \in Lin(Sym)$ will be identified with an element of $Lin(\mathcal{R}^6)$ by the representation

$$\text{(A.5)} \qquad \mathbb{K} = \sum_{i,i=1}^{6} K_{ij}\mathbf{N}_i \otimes \mathbf{N}_j$$

and $\mathbb{K}^\top$ means the transpose of $\mathbb{K}$ as an element of $Lin(\mathcal{R}^6)$. In the sequel we deal with the space $\mathcal{D}(Sym)$ of fourth order tensors whose representation in $Lin(\mathcal{R}^6)$ is a diagonal matrix, viz

$$\text{(A.6)} \qquad \mathbb{K} = \sum_{i,i=1}^{6} K_i\delta_{ij}\mathbf{N}_i \otimes \mathbf{N}_j \ ,$$

and also with complex valued tensors. Then, denoting by Ω the complex plane, $Sym(\Omega)$ and $\mathcal{D}(Sym(\Omega))$ denote respectively the tensors described by (A.4) and (A.6) with $A_i, B_i, K_i \in \Omega$.

The symbols $\mathcal{R}, \mathcal{R}^+$ and $\mathcal{R}^{++}$ denote the reals, the non-negative reals and the strictly positive reals, respectively, while $\mathcal{R}^-$ and $\mathcal{R}^{--}$ denote the non-positive and strictly negative reals.

For every function $f : \mathcal{R} \to \mathcal{V}$, where $\mathcal{V}$ is any finite-dimensional vector space, in particular in the present context Sym or $\mathcal{D}(Sym)$, let f_F, denote its *Fourier transform* viz. $f_F(\omega) = \int_{-\infty}^{\infty} f(s)e^{-i\omega s}ds$. Also, we define

$$\text{(A.7)} \qquad \begin{array}{ll} f_+(\omega) = \int_0^\infty f(s)e^{-i\omega s}ds, & f_-(\omega) = \int_{-\infty}^0 f(s)e^{-i\omega s}ds \\ f_s(\omega) = \int_0^\infty f(s)\sin\omega s\, ds, & f_c(\omega) = \int_0^\infty f(s)\cos\omega s\, ds \end{array}$$

The relations defining f_F and (A.7) are to be understood as applying to each component of the tensor quantities involved. Some constraint must be placed on these components to ensure that the Fourier transforms exist. Unless otherwise stated, it is assumed that all components of tensors in the time domain belong to $L^1(R) \cap L^2(R)$ (or $L^1(R^\pm) \cap L^2(R^\pm)$ in the case of $f_\pm$) so that in the frequency domain, they belong to $L^2(R)$ (or $L^2(R^\pm)$) [13, 12].

For $f : \mathcal{R}^+ \to \mathcal{V}$ we can always extend the domain of f to $\mathcal{R}$, by considering its *causal* extension *viz.*

$$\text{(A.8)} \qquad f(s) = \begin{cases} f(s) & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases}$$

in which case

$$\text{(A.9)} \qquad f_F(\omega) = f_+(\omega) = f_c(\omega) - if_s(\omega)$$

The complex ω plane, denoted earlier by Ω, will play an important role in our discussions. We define the following sets:

$$\text{(A.10)} \qquad \Omega^+ = \{\zeta \in \Omega : \Im_m\zeta \geq 0\} \ , \quad \Omega^{(+)} = \{\zeta \in \Omega : \Im_m\zeta > 0\} \, .$$

Analogous meanings are assigned to Ω^- and $\Omega^{(-)}$.

The quantities $f_\pm$ defined by (A.7) are analytic in $\Omega^{(\mp)}$ respectively. This analyticity is extended by assumption to $\Omega^\mp$. The function f_+ may be defined by (A.7) and analytic on a portion of Ω^+ if for example f decays exponentially at large times. Over that portion of Ω^+ for which the integral definition is meaningless, f_+ is defined by analytic continuation.

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KAM Methods for Nonautonomous Schrödinger Operators

Sandro Graffi *

1. – Introduction and statement of the results

In this lecture a result will be reported[1] concerning the spectral analysis of quasi-periodically non autonomous Schrödinger operators through recent advances in KAM theory[12, 13, 15]. The presentation covers the statement of the results and the basic ideas underlying the proof. The reader is referred to [1] for all technical details.

It is convenient to formulate the basic technical proposition in the language of abstract differential equations in Hilbert spaces.

Consider the non-autonomous, linear differential equation in a separable Hilbert space $\mathcal{H}$

$$\mathrm{i}\dot{\psi}(t) = (A + \epsilon P(\omega_1 t, \omega_2 t, ..., \omega_n t))\psi(t), \quad \psi(t) \in \mathcal{H},\ \epsilon \in \mathbb{R} \tag{1}$$

under the following conditions:

A1 The operator A is positive self-adjoint. Spec(A) is discrete, and all eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3, ...$ are simple. There is $d > 1$ such that

$$\lambda_i \sim i^d\ , \quad i \to \infty. \tag{2}$$

A2 $P(\phi_1, ..., \phi_n) \equiv P(\phi)$ is a function from the n-dimensional torus $\mathbb{T}^n \equiv \mathbb{R}^n/2\pi Z^n$ into the symmetric operators in $\mathcal{H}$, $\omega := (\omega_1, \ldots, \omega_n) \in [0,1]^n$ is a frequency vector.

A3 For $\delta \geq 0$, denote $\mathcal{B}^\delta$ the Banach space of all closed operators T in $\mathcal{H}$ such that $A^{-\delta/d}T$ is bounded (remark that $\mathcal{B}^0 = \mathcal{L}(\mathcal{H})$), with norm

$$\|T\|_\delta := \sup_{\|x\|_{\mathcal{H}}=1} \|A^{-\delta/d}Tx\|_{\mathcal{H}} \tag{3}$$

Then the map $\mathbb{T}^n \ni \phi \to P(\phi) \in \mathcal{B}^\delta$ is analytic for some $\delta < d - 1$.

Our purpose is to prove the following

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THEOREM 1 *There exist $\epsilon_* > 0$, a subset $\Pi^\epsilon \subset \Pi := [0,1]^n$ and, if $|\epsilon| < \epsilon_*$ and $\omega \in \Pi^\epsilon$, a unitary operator $U_\epsilon(\omega t) \equiv U_\epsilon(\omega_1 t, \omega_2 t, ..., \omega_n t)$ in $\mathcal{H}$ with the following properties:*

T1 $U_\epsilon(\omega t)$ is analytic in t and quasiperiodic with frequencies ω;

T2 $U_\epsilon(\omega t)$ transforms equation (1) into a system of the form

$$\text{(4)} \qquad \mathrm{i}\dot{\chi}(t) = A_\infty(\omega t)\chi(t)$$

$$\text{(5)} \qquad A_\infty := \mathrm{diag}(\lambda_1^\infty + \mu_1^\infty(\omega t), \lambda_2^\infty + \mu_2^\infty(\omega t), \lambda_3^\infty + \mu_3^\infty(\omega t), ...)$$

Here $\{\lambda_i^\infty\}_{i=1}^\infty \in \mathbb{R}$ and any function $\mu_i^\infty(\phi) : \mathbb{T}^n \to \mathbb{R}$ is analytic with zero average;

T3 There exists $C > 0$ such that:

$$\|1 - U_\epsilon(\omega t)\|_0 \le C\epsilon\ , \quad |\lambda_i^\infty - \lambda_i| \le C i^\delta \epsilon\ , \quad |\mu_i(\omega t)| \le C i^\delta \epsilon\ , \quad |\Pi - \Pi^\epsilon| \stackrel{\epsilon \to 0}{\to} 0\ .$$

Straightforward integration of (4) reduces (1) to an autonomous system which makes the almost-periodic nature of all its solutions evident.

COROLLARY 1

1. *If $|\epsilon| < \epsilon_*$, $\omega \in \Pi^\epsilon$ there exists a unitary transformation $U_F(\omega t)$, quasiperiodic with frequency ω and such that $\|1 - U_F(\omega t)\|_\delta \le C\epsilon$, which transforms (1) into the system*

   $$\text{(6)} \qquad \mathrm{i}\dot{x} = A_F x, \quad A_F := \mathrm{diag}(\lambda_1^\infty, \lambda_2^\infty, \lambda_3^\infty, ...)\ ;$$

2. *For any initial datum ψ_0 the solution $\psi(t)$ of (1) is almost-periodic with frequencies $2\pi/\lambda_1^\infty, 2\pi/\lambda_2^\infty, \ldots; \omega_1, \ldots, \omega_n$, i.e. has the form*

   $$\text{(7)} \qquad \psi(t) = \sum_{i=0}^\infty \phi_i^0(\omega t) e^{i\lambda_i^\infty t}$$

   where $\{\phi_i^0(\omega t)\}_{i=1}^\infty$ are the components of $U_\epsilon(\omega t)\psi_0$ along the eigenvector basis of A.

The above result can be equivalently formulated in terms of Floquet spectrum ([9] for the quasi-periodic case). Consider indeed on $\mathcal{K} := \mathcal{H} \otimes L^2(\mathbb{T}^n)$ the Floquet Hamiltonian operator

$$\text{(8)} \qquad K_F := -\mathrm{i}\sum_{l=1}^n \omega_l \frac{\partial}{\partial \phi_l} + A + \epsilon P(\phi)\ .$$

The maximal operator in $\mathcal{K}$ generated by the differential expression (8), still denoted K_F, is self-adjoint by A3, which makes $A + \epsilon P(\omega t)$ self-adjoint on $D(A)$ for all t. Then:

COROLLARY 2 *For* $|\epsilon| \leq \epsilon_*$ *and* $\omega \in \Pi^\epsilon$ *the spectrum of* K_F *is pure point; its eigenvalues are* $\nu_{j,k} := \lambda_j^\infty + k \cdot \omega$, $j = 0, 1, 2 \ldots$, $k \in \mathbb{Z}^n$.

As in [3, 4, 5, 7, 11, 14, 8] the main motivation for this corollary is the (Floquet) spectral analysis for the time dependent Schrödinger equation in dimension one, namely:

THEOREM 2 *Consider the time dependent Schrödinger equation*

$$(9)\quad H(t)\psi(x,t) = i\partial_t \psi(x,t),\ x \in \mathbb{R}; \qquad H(t) := -\frac{d^2}{dx^2} + Q(x) + \epsilon V(x, \omega t),\ \epsilon \in \mathbb{R}$$

and the corresponding Floquet Hamiltonian (8) under the following conditions:

1. $Q(x) \in C^\infty(\mathbb{R};\mathbb{R})$, $Q(x) \sim |x|^\alpha$ *for some* $\alpha > 2$ *as* $|x| \to \infty$*;*
2. $V(x,\phi)$ *is a* $C^\infty(\mathbb{R};\mathbb{R})$*-valued holomorphic function of* $\phi \in \mathbb{T}^n$, *with* $|V(x,\phi)||x|^{-\beta}$ *bounded as* $|x| \to \infty$ *for some* $\beta < \dfrac{\alpha - 2}{2}$.

Then there is $\epsilon^* > 0$ *such that the spectrum of* K_F *is pure point for all* $|\epsilon| < \epsilon^*$, $\omega \in \Pi^\epsilon$.

2. – The formal construction

Without loss of generality equation (1) can be written as a first-order system in ℓ^2:

$$(10)\qquad \mathrm{i}\dot{x} = (A + \epsilon P(\omega t))x\ , \qquad x \in \ell^2$$

$$(11)\qquad A = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots)\ , \quad \lambda_i \in \mathbb{R}\ , \quad \lambda_i > 0$$

where λ_i and $P(\omega t) \equiv P(\omega_1 t, \omega_2 t, \ldots, \omega_n t)$ fulfill conditions A1-A3.

The key point of any KAM method is the construction of a coordinate transformation mapping the original problem into a new one of the same form with a much smaller size of the perturbation, typically the square of the original one. Here we construct and estimate, by an algorithm very close to that of [8], a unitary operator which maps (10) into an equation of the same form but with a perturbation of order ϵ^2.

In this Section we describe the procedure; in Sect. 3 we work out the estimates, and in Sect.4 we set up the iterative scheme and prove its convergence.

Let $B(\phi_1, \ldots, \phi_n) \in \mathcal{B}^0$ be anti-selfadjoint $\forall \phi \in \mathbb{T}^n$. Given the unitary operator $e^{\epsilon B(\phi)}$, for fixed $\omega \in \Pi$ perform the change of basis $x = e^{\epsilon B(\omega t)} y$. Substitution in (10) yields

$$(12)\qquad \mathrm{i}\dot{y} = (A + \tilde{P}^1(\omega t))y$$

The new perturbation $\tilde{P}^1$ is (the explicit dependence of B on t is omitted):

$$(13)\qquad \tilde{P}^1 := \epsilon\left\{[A, B] - \mathrm{i}\dot{B} + P\right\}$$
$$+ \left(e^{-\epsilon B} A e^{\epsilon B} - A - \epsilon[A, B]\right) + \epsilon\left(e^{-\epsilon B} P e^{\epsilon B} - P\right) - \mathrm{i}\epsilon\left(e^{-\epsilon B} \dot{B} e^{\epsilon B} - \dot{B}\right).$$

If B makes the curly bracket vanish $\tilde{P}^1$ becomes of order ϵ^2. Hence we study the equation

(14) $$[A, B] - \mathrm{i}\dot{B} + P = 0 \; .$$

Taking its matrix elements between the eigenvectors of A this equation becomes

(15) $$-\mathrm{i}\sum_{l=1}^{n} \omega_l \frac{\partial}{\partial \phi_l} B_{ij} + (\lambda_i - \lambda_j) B_{ij} = P_{ij} \; ,$$

Expand both sides in Fourier series, i.e. write

$$B_{ij} = \sum_{k \in Z^n} \hat{B}_{ijk} \mathrm{e}^{\mathrm{i}k\cdot\phi}, \qquad P_{ij} = \sum_{k \in Z^n} \hat{P}_{ijk} \mathrm{e}^{\mathrm{i}k\cdot\phi}.$$

Equating the Fourier coefficients of both sides (15) becomes

$$(\omega \cdot k + \lambda_i - \lambda_j)\hat{B}_{ijk} = \hat{P}_{ijk} \; .$$

Clearly this equation cannot be solved when $i = j$ and $k = 0$. Assuming now ω such that $\omega \cdot k + \lambda_i - \lambda_j \neq 0$ when $i \neq j$ or $k \neq 0$, the natural definition of B would be the operator with matrix elements defined as

(16) $$\begin{aligned} B_{ij} &:= \sum_{k \in Z^n} \frac{\hat{P}_{ijk}}{\omega \cdot k + \lambda_i - \lambda_j} \mathrm{e}^{\mathrm{i}k\cdot\phi} \; , \quad i \neq j \\ B_{ii} &:= \sum_{k \in Z^n - \{0\}} \frac{\hat{P}_{iik}}{\omega \cdot k} \mathrm{e}^{\mathrm{i}k\cdot\phi} \end{aligned}$$

The second line in (13) is of order ϵ^2 only if the operator B is bounded. However P is not bounded; as a consequence the operator $\mathrm{diag}(B_{ii})$ is in general unbounded, and the above definition cannot yield the desired result. The idea is therefore to define B by the first of (16) with $B_{ii} = 0$; one can guess that, since the denominators $\omega \cdot k + \lambda_i - \lambda_j$ tend to infinity as i or j diverge, it should be possible to generate a bounded B even if P is unbounded. In the next section we will prove that this is actually the case.

With the above definition of B the curly bracket in (13) turns out to be the operator $\epsilon\, \mathrm{diag}(P_{ii})$, and hence in terms of the variables y the equation takes the form.

$$\mathrm{i}\dot{y} = (A^1 + \epsilon^2 P^1(\omega t))y \; ,$$

with $A^1 = A + \epsilon \mathrm{diag}(P_{ii}(\omega t))$. This system is defined only for ω in the subset of Π where the denominators in (16) do not vanish. In the next section we will assume a diophantine type condition also for such denominators, to be valid on a Cantor subset of Π. Then it will turn out that P^1 depends in a Lipschitz way on ω in such a subset.

Iterating the construction, we see that the operator A is replaced by the operator A^1 which depends also on the angles ϕ. As we shall see, this is precisely the point where Kuksin's result[13] enters in a critical way.

3. – *Squaring the order of the perturbation*

Keeping in mind the discussion of the preceding section we first set some notation, and then construct and estimate the transformation squaring the order of the perturbation.

Let $\mathbb{T}^n_s$ be the complexified torus with $|\mathrm{Im}\phi_i| \leq s$. If f is an analytic function from $\mathbb{T}^n_s$ to a Banach space (in what follows $\mathbb{C}$ or the complexification of $\mathcal{B}^\delta$), we denote

$$\|f\|_s = \sup_{\phi\in\mathbb{T}^n_s} \|f(\phi)\|$$

For $\mathcal{B}^\delta$-valued functions we use the particular symbol

$$\|f\|_{\delta,s} := \sup_{\phi\in\mathbb{T}^n_s} \|f(\phi)\|_\delta \ .$$

Let Π^- be a closed nonempty subset of Π of positive measure. If f has an additional (Lipschitz continuous) dependence on $\omega \in \Pi^-$ we define the norm

$$\|f\|^{\mathcal{L}}_s := \|f\|_s + \sup_{\phi\in\mathbb{T}^n} \sup_{\omega,\omega'\in\Pi^-} \frac{\|f(\phi,\omega) - f(\phi,\omega')\|}{|\omega-\omega'|} \ .$$

In particular for $\mathcal{B}^\delta$-valued functions we use the notation $\|.\|^{\mathcal{L}}_{\delta,s}$.

Let us now include our system into a more general framework, which, by the above discussion, is convenient for the iteration scheme. Consider in ℓ^2 the equation

$$\mathrm{i}\dot{x} = (A^- + P^-(\omega t))x \tag{17}$$

under the following conditions

H1)

$$A^- = \mathrm{diag}(\lambda_1^-(\omega) + \mu_1^-(\omega t,\omega), \lambda_2^-(\omega) + \mu_2^-(\omega t,\omega), \lambda_3^-(\omega) + \mu_3^-(\omega t,\omega), \ldots) \ , \tag{18}$$

Here:

H1.a) $\forall i = 1,\ldots$ $\lambda_i^-(\omega)$ is positive and Lipschitz continuous w.r.t. $\omega \in \Pi^-$; moreover

$$\lambda_i^- \sim i^d \ ,$$

uniformly in $\omega \in \Pi^-$. Hence there is $C_\lambda^- > 0$ independent of ω such that

$$\left|\lambda_i^- - \lambda_j^-\right| \geq C_\lambda^- |i^d - j^d| \ . \tag{19}$$

H1.b) There is $C_\omega^- > 0$ suitably small and $\delta < d-1$ such that

$$\sup_{\omega,\omega'\in\Pi^-} \frac{|\lambda_i^-(\omega) - \lambda_i^-(\omega')|}{|\omega-\omega'|} \leq C_\omega^- i^\delta \tag{20}$$

H1.c) $\forall i = 1, \dots\ \mu_i^-(\omega) : \mathbb{T}_s^n \times \Pi^- \to \mathcal{R}$ is analytic w.r.t. ϕ, Lipschitz continuous w.r.t. ω, and has zero average, i.e.

$$\int_{\mathbb{T}^n} \mu_i(\phi, \omega)\, d\phi = 0.$$

Moreover it fulfills the estimates

$$\|\mu_i\|_s \leq C_\mu^- i^\delta \tag{21}$$

$$\sup_{\phi \in \mathbb{T}_s^n} \sup_{\omega, \omega' \in \Pi^-} \frac{|\mu_i^-(\omega, \phi) - \mu_i^-(\omega', \phi)|}{|\omega - \omega'|} \leq C_\omega^- i^\delta \tag{22}$$

H2) The operator valued function $P^- : \mathbb{T}_s^n \times \Pi^- \to \mathcal{B}^\delta$ is analytic with respect to $\phi \in \mathbb{T}_s^n$ and Lipschitz continuous w.r.t. $\omega \in \Pi^-$.

H3) there exist $\gamma^- > 0$ and $\tau > n + 2/(d-1)$ such that, for any $\omega \in \Pi^-$, one has

$$|\omega \cdot k| \geq \frac{\gamma^-}{|k|^\tau}\ , \quad \forall k \in \mathbb{Z}^n - \{0\}\ , \tag{23}$$

$$|\lambda_i - \lambda_j + \omega \cdot k| \geq \frac{\gamma^- |i^d - j^d|}{1 + |k|^\tau}\ , \forall k \in \mathbb{Z}^n\ , \quad i \neq j \tag{24}$$

REMARK 1 In the next section we will prove that it is possible to construct a set Π^- of positive measure such that also the original system (1) fulfills the above assumption.

Let now

$$B : \mathbb{T}_s^n \ni (\phi_1, \dots, \phi_n) \mapsto B(\phi_1, \dots, \phi_n) \in \mathcal{B}^0 \tag{25}$$

be an analytic map with $B(\phi_1, \dots, \phi_n)$ anti-selfadjoint for each real value of $(\phi_1, \dots, \phi_n)$. Consider the corresponding unitary operator $e^{B(\phi_1, \dots, \phi_n)}$, and (as above) for any $\omega \in \Pi^-$ consider the unitary change of basis $x = e^{B(\omega t)} y$. Substitution in equation (17) yields

$$\mathrm{i}\dot{y} = (A^+ + P^+(\omega t)) y \tag{26}$$

$$A^+ := A^- + \mathrm{diag}(P^-). \tag{27}$$

Here $\mathrm{diag}(P^-)$ is the diagonal matrix formed by the diagonal elements of P^-, that is $\mathrm{diag}(P^-) := \mathrm{diag}(P_{11}^-(\omega t), P_{22}^-(\omega t), P_{33}^-(\omega t) \dots)$.

The new perturbation P^+ is given by (the explicit dependence of B on t is omitted):

$$\begin{aligned} P^+ &:= \left\{ [A^-, B] - \mathrm{i}\dot{B} + (P^- - \mathrm{diag}(P^-)) \right\} + \\ &+ \left(e^{-B} A^- e^B - A^- - [A^-, B] \right) + \left(e^{-B} P^- e^B - P^- \right) - \mathrm{i} \left(e^{-B} \dot{B} e^B - \dot{B} \right). \end{aligned} \tag{28}$$

According to the standard procedure we subtract the mean of the perturbation. Namely, we write $A^+ = \mathrm{diag}(\lambda_i^+ + \mu_i^+(\omega t))$ where $\lambda_i^+ = \lambda_i^- + \overline{P_{ii}(\phi)}$ (the overline denotes angular average). Hence the functions $\mu^+(\phi)$ have zero average; the quantities λ_i^+ are independent of ϕ and by A3 fulfill the estimate $|\lambda_i^+ - \lambda_i^-| \le C_\mu^- i^\delta$.

The main step of the proof is to construct B so as to make the curly bracket in (28) vanish, i.e. to solve for the unknwon B the equation

$$[A^-, B] - \mathrm{i}\dot{B} + (P^- - \mathrm{diag}(P^-)) = 0 \; , \tag{29}$$

The procedure explained in the previous section has to be modified since now the eigenvalues of A^- depend also on the angles ϕ. The construction is based on a lemma by Kuksin [13] that we now summarize.

On the n-dimensional torus consider the equation

$$-\mathrm{i}\sum_{k=1}^{n} \omega_k \frac{\partial}{\partial \phi_k}\chi(\phi) + E_1\chi(\phi) + E_2 h(\phi)\chi(\phi) = b(\phi) \; . \tag{30}$$

Here χ denotes the unknown, while b, h denote given analytic functions on $\mathbb{T}_s^n$. h has zero average; E_1, E_2 are positive constants and $\|h\|_s \le 1$. Concerning the frequency vector $\omega = (\omega_1, ..., \omega_n)$ the assumptions are:

$$|\omega \cdot k| \ge \frac{\gamma_2}{|k|^\tau} \; , \forall k \in \mathbb{Z}^n - \{0\} \; , \quad |\omega \cdot k + E_1| \ge \frac{\gamma_1}{1 + |k|^\tau} \; , \forall k \in \mathbb{Z}^n \; . \tag{31}$$

The final hypothesis is an order assumption on the magnitude of the different parameters, namely: given $0 < \theta < 1$ and $C > 0$ we assume

$$E_1^\theta \ge C E_2 \tag{32}$$

LEMMA 1 *(Kuksin) Under the above assumptions equation (30) has a unique analytic solution χ which for any $0 < \sigma < s$ fulfills the estimate*

$$\|\chi\|_{s-\sigma} \le C_1 \frac{1}{\gamma_1 \sigma^{a_1}} \exp\left(\frac{C_2}{\gamma_2^{a_2}\sigma^{a_3}}\right) \|b\|_s \, . \tag{33}$$

Here a_1, a_2, a_3, C_1, C_2 constants independent of $E_1, E_2, \sigma, s, \gamma_1, \gamma_2, \omega$.

To apply this lemma to the construction and estimation of B, denote $\mathcal{G}$ the Banach space of all bounded operators B in ℓ^2 such that $A^{-\delta/d} B A^{\delta/d}$ extends to a bounded linear operator. The norm in $\mathcal{G}$ is denoted

$$\|B\|^{\mathcal{G}} := \max\left\{\|B\|_0, \|A^{-\delta/d} B A^{\delta/d}\|_0\right\} . \tag{34}$$

Moreover for the $s-$ norms of an analytic function on the torus taking values in $\mathcal{G}$ (possibly Lipschitz-continuous on $\omega \in \Pi^-$) we will use the notations

$$\|B\|_s^{\mathcal{G}} \; , \qquad \|B\|_s^{\mathcal{G},\mathcal{L}} \; .$$

In what follows the notation $a \lesssim b$ stands for "there exists a constant C independent of $C_\omega^\pm, C_\mu^\pm, \gamma^\pm, s, \sigma, i, j, K$ (some of these parameters will be defined later on) such that $a \le Cb$. Equivalently we will use the notation $b \gtrsim a$.

LEMMA 2 *Let* $\frac{\delta}{d-1} < \theta < 1$, $\gamma_* > 0$, $C^*_\omega > 0$, *and* $C^* > 0$ *be fixed. Assume that*

$$C^* > \frac{C^-_\mu}{C^-_\lambda}\ , \quad \gamma \geq \gamma_*\ , \quad C_\omega \leq C^*_\omega\ . \tag{35}$$

Then for any $0 < \sigma < s$ *equation (29) has a unique solution* $B \in \mathcal{G}$ *analytic on* $\mathbb{T}^n_{s-\sigma}$, *fulfilling the estimate*

$$\|B\|^{\mathcal{G},\mathcal{L}}_{s-\sigma} \leq\cdot \frac{1}{\sigma^{b_1}} \exp\left(\frac{c}{\sigma^{b_2}}\right) \left\|P^-\right\|^{\mathcal{L}}_{\delta,s}\ . \tag{36}$$

Here c, b_1, b_2 *are constants depending only* $\theta, n, \tau, \delta, C^*, \gamma_*, C^*_\omega$.

PROOF. See [1]. ∎

We are now ready to state the main result of this section.

LEMMA 3 *Consider the system (17) within the stated assumptions. Assume furthermore that also (35) holds. Then there exists an anti-selfadjoint operator* $B \in \mathcal{G}$ *analytically depending on* $\phi \in \mathbb{T}^n_{s-\sigma}$, *and Lipschitz continuous in* $\omega \in \Pi^-$ *such that*

1. *B fulfills the estimate (36);*
2. *For any* $\omega \in \Pi^-$ *the unitary operator* $e^{B(\omega t)}$ *transforms the system (17) into the system (26);*
3. *The new perturbation* P^+ *fulfills the estimate*

$$\left\|P^+\right\|^{\mathcal{L}}_{\delta,s-\sigma} \leq\cdot \left(\left\|P^-\right\|^{\mathcal{L}}_{\delta,s}\right)^2 \exp\left(\frac{c}{\sigma^{b_1}}\right) \tag{37}$$

4. *For any positive* K *such that* $(1 + K^\tau) < \dfrac{\gamma^-}{\|P^-\|_{\delta,s}}$, *there exists a closed set* $\Pi^+ \subset \Pi^-$ *and a* $d_4 > 1$ *(independent of* K*) fulfilling*

$$\left|\Pi^- - \Pi^+\right| \leq\cdot \gamma^- \left(1 + \frac{1}{K^{d_4}}\right) \tag{38}$$

5. *If* $\omega \in \Pi^+$ *then assumptions H1-H3 above are fulfilled also by* A^+ *provided the constants are replaced by the new ones defined by*

$$\gamma^+ = \gamma^- - \left\|P^-\right\|_{\delta,s}(1 + K^\tau)\ , \quad C^+_\mu = C^-_\mu + \left\|P^-\right\|_{\delta,s}\ , \tag{39}$$

$$C^+_\omega = C^-_\omega + \left\|P^-\right\|^{\mathcal{L}}_{\delta,s}\ , \quad C^+_\lambda = C^-_\lambda - 2\left\|P^-\right\|_{\delta,s}\ . \tag{40}$$

PROOF. See [1]. ∎

4. – Iteration

In this section we set up the iteration needed to prove the stated results. First we preassign the values of the various constants occurring in the iterative estimates. Hence we keep ϵ, K, s and γ fixed and define, for $l \geq 1$,

$$\epsilon_l := \epsilon^{(4/3)^l} \ , \quad \sigma_l := \frac{s}{4l^2} \ , \quad s_l = s_{l-1} - \sigma_l \ , \quad K_l := lK \tag{41}$$

$$\gamma_l = \gamma_{l-1} - 4\epsilon_l(1 + K_l^\tau) \ , \quad C_{\mu,l} = C_{\mu,l-1} + \epsilon_l \ , \tag{42}$$

$$C_{\lambda,l} = C_{\lambda,l-1} - 2\epsilon_l \ , \quad C_{\omega,l} = C_{\omega,l-1} + \epsilon_l \ . \tag{43}$$

The initial values of the sequences are chosen as follows:

$$\gamma_0 := \gamma \ , s_0 = s, \quad C_{\mu,0} := 0 \ , \quad C_{\lambda,0} := C_\lambda \ , \quad C_{\omega,0} := 0 \ .$$

PROPOSITION 1 *There exist $\epsilon_* = \epsilon_*(\gamma) > 0$ and, for any $l \geq 1$, a closed set $\Pi_l^\gamma \subset \Pi$ such that, if $|\epsilon| < \epsilon_*$, one can construct for $\omega \in \Pi_l^\gamma$ a unitary transformation U_ϵ^l, analytic and quasiperiodic in t with frequencies ω, mapping the system (10) into the system*

$$\mathrm{i}\dot{x} = (A^l + P^l(\omega t))x \tag{44}$$

where:

1. *$U_\epsilon^l(\omega t)$ is as follows: $U_\epsilon^l(\phi) = \mathrm{e}^{B_\epsilon^1(\phi)}\mathrm{e}^{B_\epsilon^2(\phi)}...\mathrm{e}^{B_\epsilon^l(\phi)}$, and the anti-selfadjoint operators $B_\epsilon^j \in \mathcal{G}$, j=1,...,l depend analytically on $\phi \in \mathbb{T}^n_{s-\sigma_l}$, are Lipschitz continuous in $\omega \in \Pi_l^\gamma$ and fulfill (36) with P_{l-1}, σ_l in place of P^-, σ, respectively.*

2. *A^l has the form of (18) with the upper index "minus" replaced by l, i.e.*

   $$A^l = \mathrm{diag}(\lambda_1^l(\omega) + \mu_1^l(\omega t, \omega), \lambda_2^l(\omega) + \mu_2^l(\omega t, \omega), \lambda_3^l(\omega) + \mu_3^l(\omega t, \omega), ...) \ , \tag{45}$$

3. *The corresponding λ_i^l and μ_i^l fulfill conditions H1, H2, H3 of the previous section, provided λ_i^-, μ_i^- are replaced by λ_i^l, μ_i^l, respectively.*

4. *The following estimates hold*

   $$\left\|P^l\right\|_{\delta,s_l} \leq \epsilon_l \ , \quad \left\|B_\epsilon^l\right\|_{\delta,s_{l+1}}^{\mathcal{G},\mathcal{L}} \leq \epsilon_l \ , \quad \left|\Pi_l^\gamma - \Pi_{l+1}^\gamma\right| \leq \gamma_l\left(1 + \frac{1}{(lK)^{d_4}}\right) . \tag{46}$$

The proof of Theorem 1, Corollaries 1 and 2 and Theorem 2 is a direct consequence of the above Lemams. See [1] for details.

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Phase-field systems with memory effects in the order parameter dynamics

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1. – Introduction

A phase-field system of Caginalp type (see, e.g., Ch.4 in [2]) describes the evolution of the (relative) temperature ϑ and of the order parameter (or phase-field) χ in a material undergoing two different phases (e.g., solid and liquid). Supposing that the material occupies a bounded domain $\Omega \subset \mathbb{R}^3$, this system has the form (setting all the physical constants equal to 1)

$$\partial_t(\vartheta + \lambda(\chi)) + \nabla \cdot \mathbf{q} = f \tag{1}$$

$$\chi_t = -w \tag{2}$$

$$w = -\Delta\chi + \beta(\chi) - \lambda'(\chi)\vartheta \tag{3}$$

in $\Omega \times (\tau, \infty)$, τ being some fixed initial time. Here λ is a smooth function, $\mathbf{q}$ stands for the heat flux, f represents the heat supply and β is the derivative of a double well potential, e.g., $\beta(\chi) = \chi^3 - \chi$. If $\mathbf{q}$ is given by the Fourier law

$$\mathbf{q} = -\kappa\nabla\vartheta \tag{4}$$

where $\kappa > 0$ is the heat conductivity, we have the well-known Caginalp model (see [3], cf. also [2]). On the other hand, we can consider an alternative model by assuming the Gurtin-Pipkin law (see [16])

$$\mathbf{q} = -\int_0^\infty k(\sigma)\nabla\vartheta(t-\sigma)d\sigma \tag{5}$$

being k the heat conductivity relaxation kernel. The latter model has also been investigated in details from the mathematical viewpoint (see [1, 4, 5, 6, 7, 9, 11]).

Let us consider the order parameter dynamics, i.e., equations (2)-(3), more closely. The quantity w may be viewed as a generalized force that drives the system towards equilibrium acting instantaneously. In a series of recent papers (see [19] and references therein), it has been proposed and phenomenologically justified a variant of (2)-(3) characterized by a delayed response to w; that is,

$$\chi_t = -\int_0^\infty h(\sigma)w(t-\sigma)d\sigma \tag{6}$$

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where h is a suitable memory kernel. This delay effect can take place even at standard temperature regimes, therefore it is quite interesting to analyze the corresponding phase-field systems both in the Fourier case and in the Gurtin-Pipkin's. The latter one has been the first to be considered and studied (cf. [10, 14, 15, 18, 19]), while the former has been just recently investigated (see [8, 12]).

Summing up, the systems we want to examine are given by equations (3) and (6) coupled with

$$\partial_t(\vartheta + \lambda(\chi)) - \kappa\Delta\vartheta(t-\sigma)\, d\sigma = f \tag{7}$$

or

$$\partial_t(\vartheta + \lambda(\chi)) - \int_0^\infty k(\sigma)\Delta\vartheta(t-\sigma)\, d\sigma = f \tag{8}$$

respectively, in $\Omega \times (\tau, \infty)$.

Here we want to give an overview and present further developments of some recent results on both the systems (see [12, 14]) in the framework of the history space approach developed by the authors (see [13] and its references). The main goal of this theory is to interpret evolution systems with memory as dissipative dynamical systems in suitable phase spaces which accounts for the past histories of the state variables. In many cases, this formulation allows to perform an analysis that leads to show, in particular, the existence of a (universal) attractor. Following this path, we first show that our systems can be viewed as dynamical systems; that is, they generate a strongly continuous (nonlinear) semigroup on a certain phase space. Then, we shall deal with the existence of absorbing sets and, finally, we shall state the existence theorems for the universal attractors.

To introduce the history space formulations, we first introduce the initial conditions. As far as system (3), (6)-(7) is concerned, we suppose

$$\vartheta(\tau) = \vartheta_0, \qquad \chi(t) = \chi^0(t) \qquad \forall\, t \leq \tau \tag{9}$$

while for system (3), (6), (8), we assume

$$\vartheta(t) = \vartheta^0(t), \qquad \chi(t) = \chi^0(t) \qquad \forall\, t \leq \tau. \tag{10}$$

Here ϑ^0 and χ^0 are given past histories.

For the sake of simplicity, we consider the same boundary conditions for both systems; namely,

$$\vartheta = 0 \quad \text{on } \partial\Omega \times (\tau, \infty) \tag{11}$$

$$\partial_{\mathbf{n}}\chi = 0 \quad \text{on } \partial\Omega \times (\tau, \infty) \tag{12}$$

where $\mathbf{n}$ denotes the outward normal to the boundary $\partial\Omega$.

Consider first problem (3), (6)-(7), (9), (11)-(12) and introduce the additional variable

$$\xi^t(s) = \int_0^s w(t-y)\, dy$$

for any $s > 0$ and any $t > \tau$. Note that ξ fulfills the initial and boundary conditions

$$\begin{aligned} &\xi^\tau = \xi_0 \qquad \text{in } \Omega \times (0, \infty) \\ &\xi^t(0) = 0 \qquad \text{in } \Omega \times (\tau, \infty) \end{aligned}$$

where

$$\xi_0 = \int_0^s w^0(\tau - y)\, dy \qquad \text{in } \Omega,\ \forall\, s \geq 0$$

with

$$w^0 = -\Delta\chi^0 + \beta(\chi^0) - \lambda'(\chi^0)\vartheta^0 \qquad \text{in } \Omega,\ \forall\, t \leq 0. \tag{13}$$

Moreover, ξ also solves the hyperbolic first-order partial differential equations

$$\partial_t \xi + \partial_s \xi = w$$

in $\Omega \times (\tau, \infty) \times (0, \infty)$.

On the other hand, if h is smooth enough and vanishes at ∞ along its first derivative, then an integration by parts in time of the convolution product appearing in equation (6) yields

$$\partial_t \chi + \int_0^\infty \nu(\sigma)\xi^t(\sigma)\, d\sigma = 0$$

where we have set

$$\nu = -h'.$$

Consequently, the reformulation of our original problem (3), (6)-(7), (9), (11)-(12) in a space history setting reads

Problem $\mathbf{P}_1$. *Find (ϑ, χ, ξ) solution to the system*

$$\begin{aligned} &\partial_t(\vartheta + \lambda(\chi)) - \kappa\Delta\vartheta = f \\ &\partial_t \chi + \int_0^\infty \nu(\sigma)\xi^t(\sigma)\, d\sigma = 0 \\ &\partial_t \xi + \partial_s \xi = w \\ &w = -\Delta\chi + \beta(\chi) - \lambda'(\chi)\vartheta \end{aligned}$$

in $\Omega \times (\tau, \infty)$, with boundary and initial conditions

$$\begin{aligned} &\vartheta = 0 \qquad \text{on } \partial\Omega \times (\tau, \infty) \times (0, \infty) \\ &\partial_{\mathbf{n}} \chi = 0 \qquad \text{on } \partial\Omega \times (\tau, \infty) \\ &\xi^t(0) = 0 \qquad \text{in } \Omega \times (\tau, \infty) \\ &\vartheta(\tau) = \vartheta_0, \quad \chi(\tau) = \chi_0 \qquad \text{in } \Omega \\ &\xi^\tau = \xi_0 \qquad \text{in } \Omega \times (0, \infty) \end{aligned}$$

where

$$\chi^0(\tau) = \chi_0.$$

Consider now problem (3), (6), (8), (10)-(12). Arguing similarly, we set (cf. [14])

$$\eta^t(s) = \int_0^s \vartheta(t - y)\, dy, \qquad \xi^t(s) = \int_0^s w(t - y)\, dy$$

for any $s > 0$ and any $t > \tau$. This additional variables satisfy the initial and boundary conditions

$$\begin{aligned} &\eta^\tau = \eta_0, \quad \xi^\tau = \xi_0 && \text{in } \Omega \times (0,\infty) \\ &\eta^t(0) = \xi^t(0) = 0 && \text{in } \Omega \times (\tau,\infty) \end{aligned}$$

where

$$\eta_0 = \int_0^s \vartheta^0(\tau - y)\,dy, \qquad \dot{\xi}_0 = \int_0^s w^0(\tau - y)\,dy \qquad \text{in } \Omega,\ \forall s \geq 0$$

with w^0 is given by (13).

Besides, η and ξ are solutions to the hyperbolic first-order partial differential equations

$$\partial_t \eta + \partial_s \eta = \vartheta, \qquad \partial_t \xi + \partial_s \xi = w$$

in $\Omega \times (\tau,\infty) \times (0,\infty)$.

Then, supposing that k and h are smooth enough and vanishing at ∞ along their first derivatives, we can integrate by parts in time the convolution products appearing in equations (6) and (8). This gives

$$\begin{aligned} &\partial_t(\vartheta + \lambda(\chi)) - \int_0^\infty \mu(\sigma)\Delta\eta^t(\sigma)\,d\sigma = f \\ &\partial_t \chi + \int_0^\infty \nu(\sigma)\xi^t(\sigma)\,d\sigma = 0 \end{aligned}$$

with the positions

$$\mu = -k', \qquad \nu = -h'.$$

This procedure leads to

Problem $\mathbf{P}_2$. *Find $(\vartheta, \chi, \eta, \xi)$ solution to the system*

$$\begin{aligned} &\partial_t(\vartheta + \lambda(\chi)) - \int_0^\infty \mu(\sigma)\Delta\eta^t(\sigma)\,d\sigma = f \\ &\partial_t \chi + \int_0^\infty \nu(\sigma)\xi^t(\sigma)\,d\sigma = 0 \\ &\partial_t \eta + \partial_s \eta = \vartheta \\ &\partial_t \xi + \partial_s \xi = w \\ &w = -\Delta\chi + \beta(\chi) - \lambda'(\chi)\vartheta \end{aligned}$$

in $\Omega \times (\tau,\infty)$, with boundary and initial conditions

$$\begin{aligned} &\int_0^\infty \mu(\sigma)\eta^t(\sigma)\,d\sigma = 0 \qquad \textit{on } \partial\Omega \times (\tau,\infty) \times (0,\infty) \\ &\partial_{\mathbf{n}} \chi = 0 \qquad \textit{on } \partial\Omega \times (\tau,\infty) \\ &\eta^t(0) = \xi^t(0) = 0 \qquad \textit{in } \Omega \times (\tau,\infty) \\ &\vartheta(\tau) = \vartheta_0, \quad \chi(\tau) = \chi_0 \qquad \textit{in } \Omega \\ &\eta^\tau = \eta_0, \quad \xi^\tau = \xi_0 \qquad \textit{in } \Omega \times (0,\infty) \end{aligned}$$

where

$$\vartheta^0(\tau) = \vartheta_0, \qquad \chi^0(\tau) = \chi_0.$$

In the next section we shall introduce the notion of (weak) solution for Problems $\mathbf{P}_1$ and $\mathbf{P}_2$. We shall consider the autonomous case (i.e., f independent of time), just for the sake of simplicity. Then, we shall present the assumptions which ensure $\mathbf{P}_1$ and $\mathbf{P}_2$ to be well posed. This fact entails that our systems generate strongly continuous semigroups on suitable phase spaces. The final section will be devoted to the longterm behavior; that is, the existence of absorbing sets and, especially, of the universal attractors. We point out that the results about $\mathbf{P}_1$ generalize those of [12] where only the exponential kernel case is analyzed. More details on the related proofs will be given in a forthcoming paper.

2. – Well-posedness

Before introducing the weak formulations of our problems some notation is in order. Let $\Omega \subset \mathbb{R}^3$ be a bounded open and connected set with smooth boundary $\partial\Omega$. Then we define

$$H = L^2(\Omega), \qquad V_0 = H_0^1(\Omega), \qquad V = H^1(\Omega).$$

Adopting the usual identification of H with its dual H^* (dual space), we recall the compact and dense embeddings

$$V_0 \hookrightarrow H \equiv H^* \hookrightarrow V_0^* \qquad \text{and} \qquad V \hookrightarrow H \equiv H^* \hookrightarrow V^*.$$

The norm and the inner product on a (real) vector space X are indicated by $\langle \cdot, \cdot \rangle_X$ and $\| \cdot \|_X$, respectively. The symbol $\langle \cdot, \cdot \rangle$ denotes the duality pairing either between V_0 and V_0^* or between V and V^*.

Given a positive function α defined on $\mathbb{R}^+ = (0, \infty)$ and a real Hilbert space X, the symbol $L^2_\alpha(\mathbb{R}^+, X)$ stands for the Hilbert space of X-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$\langle \psi_1, \psi_2 \rangle_{L^2_\alpha(\mathbb{R}^+,X)} = \int_0^\infty \alpha(\sigma) \langle \psi_1(\sigma), \psi_2(\sigma) \rangle_X \, d\sigma.$$

The assumptions on the memory kernels needed in this section are

(K1) $\quad \mu, \nu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$

(K2) $\quad \mu(s) \geq 0, \;\; \nu(s) \geq 0 \quad \forall\, s \in \mathbb{R}^+$

(K3) $\quad \mu'(s) \leq 0, \;\; \nu'(s) \leq 0 \quad \forall\, s \in \mathbb{R}^+$

(K4) $\quad \int_0^\infty \mu(\sigma)\, d\sigma > 0$ and $\int_0^\infty \nu(\sigma)\, d\sigma > 0$.

To avoid unessential complications, we assume that (see [14] for more general assumptions)

$$\beta(r) = r^3 - r, \qquad r \in \mathbb{R}.$$

For the same reason, we suppose that f is time independent.

Consider now Problem $\mathbf{P}_1$ and let

$$
\begin{array}{ll}
\textbf{(H1)} & \lambda \in C^2(\mathbb{R}) \quad \text{and} \quad \lambda'' \in L^\infty(\mathbb{R}) \\
\textbf{(H2)} & f \in H \\
\textbf{(H3)} & \vartheta_0 \in H \\
\textbf{(H4)} & \chi_0 \in V \\
\textbf{(H5)} & \xi_0 \in L^2_\nu(\mathbb{R}^+, H).
\end{array}
$$

Then the definition of weak solution is

DEFINITION 1 Let ν satisfy (K1)-(K4) and let (H1)-(H5) hold. For $T > \tau \in \mathbb{R}$, set $I = [\tau, T]$. A triplet (ϑ, χ, ξ) which fulfills

$$
\begin{aligned}
&\vartheta \in C^0(I, H) \cap L^2(I, V_0) \\
&\chi \in W^{1,\infty}(I, H) \cap C^0(I, V) \\
&\xi \in C^0(I, L^2_\nu(\mathbb{R}^+, H)) \\
&\partial_t(\vartheta + \lambda(\chi)) \in L^2(I, V^*) \\
&w \in L^\infty(I, V^*)
\end{aligned}
$$

is a solution to problem $\mathbf{P}_1$ in the time interval I with initial data $(\vartheta_0, \chi_0, \xi_0)$ provided that, for almost every $t \in I$,

$$\langle \partial_t(\vartheta + \lambda(\chi)), u\rangle + \kappa\langle \nabla\vartheta, \nabla u\rangle_{H^3} = \langle f, u\rangle_H \qquad \forall u \in V_0 \tag{14}$$

$$\partial_t\chi + \int_0^\infty \nu(\sigma)\xi(\sigma)\, d\sigma = 0 \qquad \text{a.e. in } \Omega \tag{15}$$

$$\langle \partial_t\xi + \partial_s\xi, \zeta\rangle_{L^2_\nu(\mathbb{R}^+,V^*)} = \langle w, \zeta\rangle_{L^2_\nu(\mathbb{R}^+,V^*)} \qquad \forall \zeta \in L^2_\nu(\mathbb{R}^+, V^*) \tag{16}$$

$$\langle w, v\rangle = \langle \nabla\chi, \nabla v\rangle_{H^3} + \langle \chi^3 - \chi - \lambda'(\chi)\vartheta, v\rangle \qquad \forall v \in V \tag{17}$$

with initial conditions

$$
\begin{aligned}
&\vartheta(\tau) = \vartheta_0 && \text{a.e. in } \Omega \\
&\chi(\tau) = \chi_0 && \text{a.e. in } \Omega \\
&\xi^\tau = \xi_0 && \text{a.e. in } \Omega \times \mathbb{R}^+.
\end{aligned}
$$

Well-posedness of $\mathbf{P}_1$ can be recovered by the techniques developed in [14] (see [12] for the exponential kernel case).

THEOREM 1 *For any fixed initial time $\tau \in \mathbb{R}$, any $T > \tau$, and any*

$$(\vartheta_0, \chi_0, \xi_0) \in \mathcal{H}_1 = H \times V \times L^2_\nu(\mathbb{R}^+, H)$$

there exists a unique solution (ϑ, χ, ξ) to Problem $\mathbf{P}_1$ according to Definition 1. Moreover, this solution is given by a strongly continuous semigroup $S_1(t)$ acting on $\mathcal{H}_1$.

As far as Problem $\mathbf{P}_2$ is concerned, since we are able to prove uniqueness only when λ is linear (see [14]), we need to replace (H1). Besides, we must introduce a further initial datum, namely,

$$\begin{array}{lll} \textbf{(H6)} & \lambda(r) = r, & r \in \mathbb{R} \\ \textbf{(H7)} & \eta_0 \in L^2_\mu(\mathbb{R}^+, V_0). & \end{array}$$

Our definition of weak solution now reads

DEFINITION 2 Let (K1)-(K4), (H2)-(H7) hold. For $T > \tau \in \mathbb{R}$, set $I = [\tau, T]$. A quadruplet $(\vartheta, \chi, \eta, \xi)$ which fulfills

$$\begin{aligned} &\vartheta \in C^0(I, H) \\ &\chi \in W^{1,\infty}(I, H) \cap C^0(I, V) \\ &\eta \in C^0(I, L^2_\mu(\mathbb{R}^+, V_0)) \\ &\xi \in C^0(I, L^2_\nu(\mathbb{R}^+, H)) \\ &\partial_t \vartheta \in L^\infty(I, V^*) \\ &w \in L^\infty(I, V^*) \end{aligned}$$

is a solution to problem $\mathbf{P}_2$ in the time interval I with initial data $(\vartheta_0, \chi_0, \eta_0, \chi_0)$ provided that, for almost every $t \in I$,

$$\langle \partial_t(\vartheta + \chi), u\rangle + \int_0^\infty \mu(\sigma)\langle \nabla\eta(\sigma), \nabla u\rangle_{H^3}\, d\sigma = \langle f, u\rangle_H \qquad \forall u \in V_0 \tag{18}$$

$$\partial_t \chi + \int_0^\infty \nu(\sigma)\xi(\sigma)\, d\sigma = 0 \qquad \text{a.e. in } \Omega \tag{19}$$

$$\langle \partial_t \eta + \partial_s \eta, \psi\rangle_{L^2_\mu(\mathbb{R}^+, H)} = \langle \vartheta, \psi\rangle_{L^2_\mu(\mathbb{R}^+, H)} \qquad \forall \psi \in L^2_\mu(\mathbb{R}^+, H) \tag{20}$$

$$\langle \partial_t \xi + \partial_s \xi, \zeta\rangle_{L^2_\nu(\mathbb{R}^+, V^*)} = \langle w, \zeta\rangle_{L^2_\nu(\mathbb{R}^+, V^*)} \qquad \forall \zeta \in L^2_\nu(\mathbb{R}^+, V^*) \tag{21}$$

$$\langle w, v\rangle = \langle \nabla\chi, \nabla v\rangle_{H^3} + \langle \chi^3 - \chi - \vartheta, v\rangle \qquad \forall v \in V \tag{22}$$

with initial conditions

$$\begin{array}{ll} \vartheta(\tau) = \vartheta_0 & \text{a.e. in } \Omega \\ \chi(\tau) = \chi_0 & \text{a.e. in } \Omega \\ \eta^\tau = \eta_0 & \text{a.e. in } \Omega \times \mathbb{R}^+ \\ \xi^\tau = \xi_0 & \text{a.e. in } \Omega \times \mathbb{R}^+. \end{array}$$

REMARK 1 In the above first-order equations ruling the integrated past histories, $-\partial_s$ has to be understood as the infinitesimal generator of the right-translation semigroup on $L^2_\nu(\mathbb{R}^+, H)$ or $L^2_\mu(\mathbb{R}^+, V_0)$, accordingly (see [13] for more details).

Problem $\mathbf{P}_2$ is well posed due to (see [14])

THEOREM 2 *For any fixed initial time $\tau \in \mathbb{R}$, any $T > \tau$, and any*

$$(\vartheta_0, \chi_0, \eta_0, \xi_0) \in \mathcal{H}_2 = H \times V \times L^2_\mu(\mathbb{R}^+, V_0) \times L^2_\nu(\mathbb{R}^+, H)$$

there exists a unique solution $(\vartheta, \chi, \eta, \xi)$ to Problem $\mathbf{P}_2$ according to Definition 2. Moreover, this solution is given by a strongly continuous semigroup $S_2(t)$ acting on $\mathcal{H}_2$.

REMARK 2 The above results concerning the modified problem $\mathbf{P}_1$ and $\mathbf{P}_2$ can be translated back in terms of the original problems (3), (6)-(7), (9), (11)-(12) or (3), (6), (8), (10)-(12), respectively, provided that suitable assumptions on the past histories are made (see Section 4 in [13]).

3. – Longtime behavior

Here we require additional assumptions on the memory kernels; that is,

(K5) $\lim_{s\to 0}\mu(s) < \infty$ and $\lim_{s\to 0}\nu(s) < \infty$

(K6) $\mu'(s) + \delta\mu(s) \leq 0,\ \ \nu'(s) + \delta\nu(s) \leq 0$ for some $\delta > 0$, $\forall\, s \in \mathbb{R}^+$.

Observe that (K6) entails the exponential decay of both the kernels. This condition is also invoked in the stability analysis of linear systems with memory (see, e.g., [17]) and it is also crucial to prove the dissipativity of our dynamical systems or, more precisely, the existence of an absorbing set for the semigroups $S_1(t)$ and $S_2(t)$ as stated by the following theorems (cf. [12, 14]).

THEOREM 3 *Let the assumptions of Definition* 1 *hold. Suppose that* ν *fulfills* (K5)-(K6). *Then, there exists* $R_1 > 0$ *such that, given any* $R > 0$ *and any initial datum* $z_0 = (\vartheta_0, \chi_0, \xi_0) \in \mathcal{H}_1$ *satisfying* $\|z_0\|_{\mathcal{H}_1} \leq R$, *there is* $t_R \geq 0$ *such that*

$$\|S_1(t)z_0\|_{\mathcal{H}_1} \leq R_1 \qquad \forall\, t \geq t_R.$$

THEOREM 4 *Let the assumptions of Definition* 2 *hold. Moreover, assume* (K5)-(K6). *Then, there exists* $R_2 > 0$ *such that, given any* $R > 0$ *and any initial datum* $z_0 = (\vartheta_0, \chi_0, \eta_0, \xi_0) \in \mathcal{H}_2$ *satisfying* $\|z_0\|_{\mathcal{H}_2} \leq R$, *there is* $t_R \geq 0$ *such that*

$$\|S_2(t)z_0\|_{\mathcal{H}_2} \leq R_2 \qquad \forall\, t \geq t_R.$$

We recall that the balls of radii R_1 and R_2 are called absorbing sets for the semigroups $S_1(t)$ and $S_2(t)$, respectively. Note that the trajectories departing from any bounded set of initial data eventually enter the absorbing set uniformly in time.

As explained in [13] (cf. also its references), the existence of a uniform absorbing set is a preliminary step in order to show the existence of the universal attractor. Let us recall its definition for the reader's convenience (see, e.g., [20]).

DEFINITION 3 Let $(\mathcal{X}, d_{\mathcal{X}})$ be a complete metric space. A compact set $\mathcal{A} \subset \mathcal{X}$ is the *universal attractor* for a strongly continuous semigroup $S(t)$ acting on $\mathcal{X}$ if it is invariant and

$$\lim_{t\to\infty} \delta_{\mathcal{X}}(S(t)\mathcal{B}, \mathcal{A}) = 0$$

for any bounded set $B \subset \mathcal{X}$. Here $\delta_{\mathcal{X}}$ denotes the *Hausdorff semidistance* in $\mathcal{X}$, defined as

$$\delta_{\mathcal{X}}(\mathcal{B}_1, \mathcal{B}_2) = \sup_{z_1\in\mathcal{B}_1} \inf_{z_2\in\mathcal{B}_2} d_{\mathcal{X}}(z_1, z_2).$$

We thus can conclude by stating our main result

THEOREM 5 *Let the assumptions of Theorem* 3.i, i=1,2, *hold. Then the semigroup* $S_i(t)$ *on* $\mathcal{H}_i$ *generated by* $\mathbf{P}_i$ *possesses a (unique) connected universal attractor* $\mathcal{A}_i \subset \mathcal{H}_i$.

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Elliptic problems depending on a parameter in plane curvilinear polygons

Davide Guidetti *

1. – Introduction

The aim of this paper is to survey certain recent results obtained by the author in collaboration with F. Colombo and A. Lorenzi, concerning the existence and regularity of solutions of elliptic problems of Dirichlet type in plane curvilinear polygons. As we are also interested in applications to parabolic equations, we shall consider problems of the form

$$\begin{cases} (\lambda - A(z,\partial))u(z) = f(z), & z \in O, \\ u(z) = 0, & z \in \partial O, \end{cases} \tag{1}$$

where λ is a complex parameter. In particular, we are concerned with estimates of the solution u (when existing) depending on λ.

For related questions, we recall that P. Grisvard considered elliptic problems in traditional L^p or Hölder continuous function spaces in plane domains,while the Russian school, on the lines of Kondratiev, developed a rich theory in spaces of functions with weights. Coming back to more traditional spaces, J. O. Adeyeye studied the generation of an analytic semigroup by the Laplace operator with various boundary conditions in L^p spaces $(1 < p < +\infty)$ in a polygon. Under the previous conditions, he characterized also the real interpolation spaces between the domain with Dirichlet boundary conditions and L^p. For some bibliography concerning these results, we refer to [3].

In the paper [3] the classical mixed Cauchy-Dirichlet problem for the heat equation in a plane angle was studied. To this aim, the author considered also estimates depending on a parameter and characterized real interpolation spaces for the Poisson equation in a plane angle, working in the framework of continuous and Hölder-continuous functions, even of negative order.

The results of [3] were developed in [1], where they were extended to the case of operators with nonconstant coefficients in a plane curvilinear polygon.

Our aim is precisely to give some account of the results of [1] and [3].

We conclude this introduction with some notation we shall use in the sequel.

If $A \subseteq \mathbf{R}^n$, we shall indicate with ∂A its boundary and with $\overline{A}$ its closure.

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If O is an open subset of $\mathbf{R}^n$, $C(O)$ will indicate the space of complex valued uniformly continuous and bounded functions of domain O. We shall always identify each of them with its continuous extension to the closure $\overline{O}$ of O.

If $\sigma \in \mathbf{N}$ and O is open in $\mathbf{R}^n$, we put

$$C^\sigma(O) := \{f \in C(O) : \partial^\alpha f \in C(O) \forall \alpha \text{ with } |\alpha| \leq \sigma\}.$$

Here $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{N}_0^n$, where $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$, and $|\alpha| := \alpha_1 + ... + \alpha_n$.

If $\sigma > 0$, we put

$$C^\sigma(O) := \{f \in C^{[\sigma]}(O) : \partial^\alpha f \text{ is Hölder-continuous of exponent } \{\sigma\} \ \forall \alpha \text{ with } |\alpha| \leq \sigma\},$$

where $[\sigma]$ stands for the integer part of σ and $\{\sigma\} := \sigma - [\sigma]$. If $f \in C^\sigma(O)$ for some $\sigma \geq 0$, we can consider its trace γf on ∂O. Under reasonable assumptions on ∂O, γf belongs to $C^\sigma(\partial O)$, which can be definied by local charts.

Each of the previous spaces will be endowed with a natural norm $\|.\|_{\sigma,O}$ or $\|.\|_{\sigma,\partial O}$ respectively.

We shall indicate with $C^\infty(O)$ the set $\bigcap_{\sigma \geq 0} C^\sigma(O)$.

We introduce also the spaces $C^\sigma(O)$ with $\sigma < 0$. The definition is the following: let $\sigma \in \mathbf{R}$, $\sigma = -m + \alpha$, with $m \in \mathbf{N}$ and $\alpha \in [0, 1)$. We set:

$$C^\sigma(O) := \{\sum_{|\beta| \leq m} \partial^\beta f_\beta : f_\beta \in C^\alpha(O), |\beta| \leq m\}. \tag{2}$$

The derivatives in (2) are intended in the sense of distributions. If $f \in C^\sigma(O)$, with $\sigma = -m + \alpha$, ($m \in \mathbf{N}$, $\alpha \in [0, 1)$), we define its norm $\|.\|_{\sigma,\partial O}$ in the following way:

$$\|f\|_{\sigma,O} := \inf\{\sum_{|\beta| \leq m} \|f_\beta\|_{\alpha,O} : f_\beta \in C^\alpha(O), f = \sum_{|\beta| \leq m} \partial^\beta f_\beta\}. \tag{3}$$

If A is a linear operator in a Banach space X, we shall indicate with $\rho(A)$ the resolvent set of A.

If X and Y are Banach spaces, we shall use the notation $(X, Y)_{\theta,p}$ ($\theta \in (0, 1)$, $p \in [1, +\infty]$) to indicate the real interpolation space between X and Y. For its definition and properties, we refer to [5].

We shall consider also spaces of the form $C^\sigma([0, T]; X)$, with X Banach space. The definitions are analogous to the previous, taking $(0, T)$ instead of O.

2. – The Poisson equation depending on a parameter in a plane angle

In this section we shall describe some of the results of [3], which are preliminary to the case of nonconstant coefficients in a curvilinear polygon.

First of all, we set

$$\Omega_\omega := \{z \in \mathbf{C} \setminus \{0\} : z = \rho e^{i\theta}, \rho > 0, 0 < \theta < \omega\}. \tag{4}$$

(Whenever it is convenient, we shall identify $\mathbf{R}^2$ with $\mathbf{C}$). We consider the problem

$$
(5) \qquad \begin{cases} (\lambda - \Delta)u(z) = f(z), & z \in \Omega_\omega, \\ u(z) = 0, & z \in \partial\Omega_\omega, \end{cases}
$$

where λ is a complex parameter and Δ stands for the Laplace operator.

In case $\omega = \pi$,the following result holds (for a proof see [2] and theorem 3.2 in [3]):

THEOREM 1 *Consider the problem (5) with $\omega = \pi$, $|\lambda| \geq 1$, $Arg(\lambda) \neq \pi$,$\sigma > -2$, $\sigma \notin \mathbf{Z}$, $f \in C^\sigma(\Omega_\pi)$. Then there exists a unique solution $u \in C^{\sigma+2}(\Omega_\pi)$. Moreover, if $\sigma < 2$ and $|Arg(\lambda)| \leq \omega_0 < \pi$, there exists $C(\omega_0) > 0$, such that one of the following estimates holds:*

$$
(6) \qquad |\lambda| \|u\|_{\sigma,\Omega_\pi} + \|u\|_{\sigma+2,\Omega_\pi} \leq C(\omega_0)\|f\|_{\sigma,\Omega_\pi} \quad \text{if } \sigma < 0,
$$

$$
(7) \qquad \begin{aligned} |\lambda| \|u\|_{\sigma,\Omega_\pi} + \|u\|_{\sigma+2,\Omega_\pi} &\leq C(\omega_0)(\|f\|_{\sigma,\Omega_\pi} \\ &+ |\lambda|^{\frac{\sigma}{2}} \|\gamma f\|_{0,\partial\Omega_\pi}) \text{ if } 0 < \sigma < 2, \sigma \neq 1. \end{aligned}
$$

In case $\omega \neq \pi$, the situation is more complicated, in the sense that the singularity of the boundary in $(0,0)$ causes the appearence of singular solutions. But we shall see that, apart certain isolated values of σ, estimates like (6) and (7) continue to hold, if we replace $C^{2+\sigma}(\Omega_\omega)$ with a new space $\tilde{C}^{2+\sigma}(\Omega_\omega)$, which we are going to describe. Such a space can be obtained adding to $C^{2+\sigma}(\Omega_\omega)$ a certain finite dimensional space $v(\{S_{\omega,k} : k \in \mathbf{N}, \frac{k\pi}{\omega} < 2 + \sigma\})$, generated by the functions $S_{\omega,k}$, with $k \in \mathbf{N}$ and $\frac{k\pi}{\omega} < 2+\sigma$. These functions $S_{\omega,k}$ can be constructed in two different ways, depending on whether $\frac{k\pi}{\omega}$ belongs to $\mathbf{N}$ or not.

Assume first that $\frac{k\pi}{\omega} \notin \mathbf{N}$. Set, for $z \in \Omega_\omega$,

$$
(8) \qquad \phi_k(z) := Im\{z^{\frac{k\pi}{\omega}}\}.
$$

Observe that ϕ_k is harmonic in Ω_ω and vanishes in $\partial\Omega_\omega$. Fix χ of class C^∞ and with compact support in $\mathbf{R}^2$ and set

$$
(9) \qquad S_{\omega,k} := \phi_k \chi.
$$

Then $S_{\omega,k} \notin C^{2+\sigma}(\Omega_\omega)$ if $\frac{k\pi}{\omega} < 2 + \sigma$, but $\Delta S_{\omega,k} \in C^\infty(\Omega_\omega)$.

Assume now that $\frac{k\pi}{\omega} \in \mathbf{N}$. In this case $z \to Im\{z^{\frac{k\pi}{\omega}}\}$ is a polynomial,so that it is infinitely smooth. Let $l := \frac{k\pi}{\omega}$. Observe that, as $\omega \neq \pi$, $l \geq 2$. Then one can show that, given $f \in C^{l-2}(\Omega_\omega)$,in order that there exist a solution $u \in C^l(\Omega_\omega)$ of (5), it is necessary that f satisfies a certain compatibility condition. To give some flavour of this fact, consider, for example, the case $\omega = \frac{\pi}{2}$ and $k = 1$, so that $l = 2$. Then it is clear that, if $f \in C(\Omega_\omega)$, in order that there exist a solution $u \in C^2(\Omega_\omega)$ of (5), it is necessary that $f(0,0) = 0$. So, for example, if $f(z) \equiv 1$, we cannot expect the existence of a solution in $C^2(\Omega_\omega)$. In general, if $\frac{k\pi}{\omega} = l \in \mathbf{N}$, one can choose a certain function $S_{\omega,k} \in \bigcap_{\epsilon>0} C^{\frac{k\pi}{\omega}-\epsilon}(\Omega_\omega)$ such that $\Delta S_{\omega,k} \in C^\infty(\Omega_\omega)$. For an explicit expression of $S_{\omega,k}$ in this second case, see [3](3.21) and (3.24).

We are now in position to give the following extension of Theorem 1 (for a proof see [3]):

THEOREM 2 *Consider the problem (5) with* $\omega \neq \pi$, $|\lambda| \geq 1$, $Arg(\lambda) \neq \pi$. *Let* $\sigma \in]-2, \frac{\pi}{\omega} \wedge 2[$, $\sigma \notin \mathbf{Z}$, $\frac{\omega(2+\sigma)}{\pi} \notin \mathbf{Z}$, $f \in C^{\sigma}(\Omega_{\omega})$. *Then there exists a unique solution* $u \in \tilde{C}^{\sigma+2}(\Omega_{\omega})$. *Moreover, if* $\sigma < 2$, *an estimate like (6) or (7) holds, if we replace* $\|u\|_{2+\sigma,\omega}$ *with* $\|u\|^{*}_{2+\sigma,\omega}$, *where* $\|.\|^{*}_{2+\sigma,\omega}$ *is a natural norm in* $\tilde{C}^{2+\sigma}(\Omega_{\omega})$.

REMARK 1 If $2+\sigma < \frac{\pi}{\omega}$, then $\tilde{C}^{2+\sigma}(\Omega_{\omega}) = C^{2+\sigma}(\Omega_{\omega})$ and we get the same result as in case $\omega = \pi$.

We consider now the case of $f \in C(\Omega_{\omega})$. The following result holds:

THEOREM 3 *Let* A *be the operator defined as follows:*

$$(10) \qquad \begin{aligned} &D(A) = \cap_{\sigma<0}\{u \in \tilde{C}^{2+\sigma}(\Omega_{\omega}) : \Delta u \in C(\Omega_{\omega}), \gamma u = 0\}, \\ &Au = \Delta u, u \in D(A). \end{aligned}$$

Then $\rho(A)$ *contains* $\{\lambda \in \mathbf{C} \setminus \{0\} : Arg(\lambda) \neq \pi\}$. *Moreover,* $\forall R > 0, \forall \omega_0 < \pi$ *there exists* $C(R, \omega_0) > 0$ *such that,* $\forall \lambda \in \mathbf{C} \setminus \{0\}$ *with* $|\lambda| \geq R, |Arg(\lambda)| \leq \omega_0$,

$$(11) \qquad \|(\lambda - A)^{-1}\|_{\mathcal{L}(C(\Omega_{\omega}))} \leq C(R, \omega_0)|\lambda|^{-1}.$$

Finally, $\forall \theta \in (0,1)$, *such that* $\theta \neq \frac{1}{2}$ *and* $\frac{(2+\theta)\omega}{\pi} \notin \mathbf{N}$, *one has*

$$(12) \qquad (C(\Omega_{\omega}), D(A))_{\theta,\infty} = \{f \in \tilde{C}^{2\theta}(\Omega_{\omega}) : \gamma f = 0\}.$$

REMARK 2 In Theorem 3 we can replace $C(\Omega_{\omega})$ with $L^{\infty}(\Omega_{\omega})$.

We conclude this section with an application of the previous results to the mixed Cauchy-Dirichlet problem for the heat equation. The result we are going to state can be obtained from theorems 1, 2 and 3, using techniques which are inspired by the theory of analytic semigroups in Banach spaces (see [3], section 4). The corresponding version in case of a smooth boundary is well known (see [4] IV, theorem 5.1).

Consider the problem

$$(13) \qquad \begin{cases} D_t u(t,z) = \Delta u(t,z) + f(t,z), & (t,z) \in [0,T] \times \Omega_{\omega}, \\ u(t,z) = 0, & (t,z) \in [0,T] \times \partial\Omega_{\omega}, \\ u(0,z) = u_0(z), & z \in \Omega_{\omega}. \end{cases}$$

Let $\alpha \geq 0$. We set

$$(14) \qquad \begin{aligned} \tilde{C}^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \Omega_{\omega}) := \{&u \in C^{1+\frac{\alpha}{2}}([0,T]; C(\Omega_{\omega})) \cap B([0,T]; \tilde{C}^{2+\alpha}(\Omega_{\omega})) : \\ &D_t u \in B([0,T]; C^{\alpha}(\Omega_{\omega}))\}. \end{aligned}$$

The following theorem holds:

THEOREM 4 *Let* $\omega \in (0, 2\pi)$, $\alpha \in (0, \min\{\frac{\pi}{\omega}, 2\}) \backslash \{1\}$. *Then the following conditions are necessary and sufficient in order that the problem (13) have a solution* $u \in \tilde{C}^{1+\frac{\alpha}{2},2+\alpha}([0,T] \times \Omega_{\omega})$:

(I) $f \in C^{\frac{\alpha}{2}}([O,T] : C(\Omega_{\omega})) \cap B([0,T]; C^{\alpha}(\Omega_{\omega}))$;

(II) $u_0 \in \tilde{C}^{2+\alpha}(\Omega_{\omega})$;

(III) $\gamma u_0 = 0$;

(IV) $\gamma(\Delta u_0 + f(0,.)) = 0$.

3. – Elliptic problems in plane curvilinear polygons with nonconstant coefficients

We pass to the case of elliptic problems in plane curvilinear polygons with nonconstant coefficients. The results we are going to describe are contained in [1].

We start by specifying what we mean with plane curvilinear polygon.

DEFINITION 1 Let O be a bounded open subset of $\mathbf{R}^2$. We shall say that O is a *plane curvilinear polygon* if, $\forall P \in \partial O$ there exists a neighbourhood W of P in $\mathbf{R}^2$ and a diffeomorphism $\psi : W \to \mathcal{B}$ of class $C^{1,1}$ (where we have indicated with $\mathcal{B}$ the unit ball in $\mathbf{R}^2$), such that $\psi(P) = (0,0)$ and $\psi(W \cap O) = \Omega_\omega \cap \mathcal{B}$, for some $\omega \in (0, 2\pi)$.

It is easily seen, applying a simple compactness argument, that $\omega = \pi$ for all points of ∂O, with the exception of a finite number of them. We shall call these points the *vertexes* of O. Let

$$\mathcal{A}(z,\partial) := \sum_{i,j=1}^{2} a_{i,j}(z)\partial_{ij}^2 + \sum_{j=1}^{"} a_j(z)\partial_j + a_0(z). \tag{15}$$

with real valued coefficients of domain $\overline{O}$. We shall assume that $a_{1,2} = a_{2,1}$ and that there exists $\nu > 0$ such that, $\forall z \in \overline{O}, \forall \xi \in \mathbf{R}^2$,

$$\sum_{i,j=1}^{2} a_{i,j}(z)\xi_i\xi_j \geq \nu|\xi|^2. \tag{16}$$

Indicate now with $P_1, ..., P_m$ the vertexes of O. Then, for each of them, we can choose properly the diffeomorphism ψ (see definition 1) in such a way that the transformed operator in Ω_ω has as a principal part in $(0,0)$ Δ. We shall indicate with $\omega(P_r, \mathcal{A})$ the value of ω corresponding to P_r.

By these local changes of coordinates and a suitable partition of unity, we can construct the spaces $\tilde{C}^\sigma(O, \mathcal{A})$, starting from the spaces $\tilde{C}^\sigma(\Omega_\omega)$ (see definition 3.2 in [1]) . Then one can prove the following result (see [1]):

THEOREM 5 *Concerning operator (15), assume the following:*

(a) $\forall i, j \in \{1,2\}$ $a_{i,j}, a_j, a_0 \in C^{\theta'}(O)$, *for some* $\theta' > \max\{0, 1 - \frac{\pi}{\max_r \omega(P_r, \mathcal{A})}\}$;

(b) if, for a certain $r \in \{1, ..., m\}$, $\frac{\pi}{\omega(P_r, \mathcal{A})} < 2$, *then the coefficients* $a_{i,j}$ $(i,j \in \{1,2\})$ *are of the form*

$$a_{i,j}(z) - a_{i,j}(P_r) = (b_{i,j}(z), z - P_r),$$

with $b_{i,j} \in C^{\theta'}(O, \mathbf{R}^2)$.

Then, $\forall \omega_0 \in [0, \pi)$, *there exists* $R > 0$, *such that, if* $|\lambda| \geq R$ *and* $|Arg(\lambda)| \leq \omega_0$, *the problem (1) has,* $\forall f \in C(O)$, *a unique solution* $u \in \bigcap_{\alpha < \alpha_0} \tilde{C}^\alpha(O, \mathcal{A})$, *with*

$$\alpha_0 := \min\{2, 1 + \frac{\pi}{\max_r \omega(P_r, \mathcal{A})}\}. \tag{17}$$

If $0 \leq \alpha < \alpha_0$ and, either $\alpha < \frac{\pi}{\max_r \omega(P_r, \mathcal{A})}$, or $\alpha \neq 1$, there exists $C > 0$, depending on R, ω_0, α, such that, $\forall f \in C(O)$,

$$\|u\|_{\tilde{C}^\alpha(O,\mathcal{A})} \leq C|\lambda|^{\alpha-1}\|f\|_{C(O)}. \tag{18}$$

REMARK **3** If $u \in \bigcap_{\alpha<\alpha_0} \tilde{C}^\alpha(O, \mathcal{A})$, it is possible to give a generalized meaning to $\mathcal{A}(z, \partial)u$ (see [1]).

REMARK **4** If O is "convex at the vertexes",that is, $\max_r \omega(P_r, \mathcal{A}) < \pi$, then $\alpha_0 = 2$.

Define now the following operator A:

$$\begin{aligned} D(A) &= \textstyle\bigcap_{\alpha<\alpha_0}\{u \in \tilde{C}^\alpha(O) : \mathcal{A}(z,\partial)u \in C(\Omega_\omega), \gamma u = 0\}, \\ Au &= \mathcal{A}(z,\partial)u, u \in D(A). \end{aligned} \tag{19}$$

Then, for certain values of $\theta \in (0,1)$, we are able to characterize the real interpolation space $(C(O), D(A))_{\theta,\infty}$:

THEOREM **6** *Assume that the assumptions of Theorem 5 are satisfied. Then $\forall\theta \in (0,1)$, such that $\theta \neq \frac{1}{2}$, $\frac{2\theta\pi}{\omega(P_r,\mathcal{A})} \notin \mathbf{N}$ for every vertex P_r of O, and $2\theta < \alpha_0$, one has*

$$(C(O), D(A))_{\theta,\infty} = \{f \in \tilde{C}^{2\theta}(O, \mathcal{A}) : \gamma f = 0\}. \tag{20}$$

REMARK **5** Theorems 5 and 6 can be extended fro $C(O)$ to $L^\infty(O)$, with obvious changes (essentially replacing everywhere $C(O)$ with $L^\infty(O)$).

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Fractional diffusion and wave equations

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1. – Introduction

The diffusion equation $\mathrm{D}\, u = \nabla^2 u$ and the wave equation $\mathrm{D}^2 u = \nabla^2 u$ belong to a one-parameter family of time-fractional equations $\mathrm{D}^\alpha u = \nabla^2 u$, $0 < \alpha \leq 2$, where D^α is the Caputo fractional derivative and D denotes the time derivative $\partial/\partial t$. The Laplacian $\Delta := \nabla^2$ can be replaced by a fractional Laplacian $-(-\overline{\Delta})^\nu$ of order 2ν, $0 < \nu \leq 1$, giving rise to the class of space-time fractional equations

$$\mathrm{D}^\alpha u = -(-\overline{\Delta})^\nu u \tag{1}$$

where $\overline{\Delta}$ denotes the closure of the Laplacian in an appropriate Banach space. For $0 < \alpha \leq 1$ these equations represent a diffusion process. For $\nu = 1$ such a diffusion process falls into the subdiffusion category [21]. Subdiffusion is is characterized by the relation $\langle[\mathbf{x}(T + t) - \mathbf{x}(t)]^2\rangle = A\, T^\gamma$, $\gamma < 1$, $A = \text{const} > 0$, as opposed to the normal diffusion (Brownian motion), for which $\gamma = 1$. This means that in subdiffusion particles tend to move slower than in the ordinary diffusion. At the level of individual particle motions subdiffusion is modeled by a class of generalized random walks known as Continuous Time Random Walks, with a random waiting time between two successive jumps [23, 22]. Solutions of fractional diffusion equations represent asymptotic properties of the probability densities derived from the CTRW model for large times and propagation distances [33, 18]. Another phenomenon often associated with subdiffusion is diffusion on fractals, in view of the sublinear dependence of the mean square displacement on time lapse. The correct formulation is however $\mathrm{D}\, u = L\, u$ where L denotes the Laplacian on a fractal [15, 4]. Some *ad hoc* formulations of diffusion on fractals [24, 8] involve a local operator generalizing the Laplacian.

Another kind of anomalous diffusion, known as superdiffusion, corresponds to $\gamma > 1$. It is sometimes associated with space-fractional diffusion equations $0 < \nu < 1$ [35, 36, 37], but the solutions of space-fractional diffusion equations (including space-time fractional diffusion equations with $\alpha \leq 1$) have divergent second-order moments. Nevertheless a relation $\langle[\mathbf{x}(T + t) - \mathbf{x}(t)]^\mu\rangle = D\, T^{\mu\gamma}$ for some $\mu < 2$ and $\gamma > 1$ remains valid. In order to avoid the controversial issue of a correct definition of superdiffusion we shall call the diffusion satisfying the last relation a generalized superdiffusion. In superdiffusion and in generalized superdiffusion a particle tends to move faster than in the Brownian motion.

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Superdiffusion is alternatively modeled by CTRWs. In order to obtain a model of superdiffusion with a finite second-order moment one has to assume that the two random variables involved, the waiting time and the jump length, are not independent [17]. Such CTRW are called coupled. It is however known that asymptotic properties of a coupled CTRW are very different from the solutions of space-time fractional diffusion equations [18]. In this paper we shall consider generalized superdiffusion represented by space-fractional equations and by uncoupled CTRWs.

Solutions of the space-fractional equation $\mathrm{D}\,u = -(-\overline{\Delta})^\nu\, u$ can be expressed in terms of a symmetric multivariate β-stable probability density, $\beta = 2\nu$ [11]. The operator $-(-\overline{\Delta})^\nu$, the fractional power of the closed linear operator $-\overline{\Delta}$ [34], is the generator of a semigroup associated with the corresponding isotropic stable Lévy process [30]. A natural generalization of fractional diffusion equations consists in replacing the operator $-(-\overline{\Delta})^\nu$ by an infinitesimal generator L of a semigroup associated with an arbitrary stable Lévy process [11, 12]. In this case the solution of $\mathrm{D}u = Lu$ is a time-dependent probability distribution of a Lévy process. A further generalization of the diffusion equation is obtained by substituting the Caputo fractional derivative D^α, $0 < \alpha \leq 1$, for D.

For $1 < \alpha \leq 2$ the solutions of time- and space-time fractional equations (2) exhibit a wave-like behavior, with propagating crests and troughs. In three dimensions the solutions are no longer non-negative. For $\alpha < 2$ the precursor disturbances propagate at an infinite speed, but the dominating crests and troughs propagate according to the law $r = \text{const} \times t^{\alpha/\beta}$. For $\alpha \leq 1.5$ the dominating signal spreads (diffuses) rather fast, but for $\alpha \cong 2$ its amplitude is strongly attenuated while its shape is remarkably stable. For $\nu = 1$ and $\alpha \cong 2$ time fractional wave equations have been proposed as a model of constant Q wave propagation, where Q represents an attenuation per unit wavelength [2, 16].

Solutions of time- and space-time fractional diffusion and wave equations in one-dimensional space have been constructed in several papers [31, 7, 10, 19, 20]. Multi-dimensional time-, space- and space-time fractional equations were discussed in [31, 13, 11, 12]. The transition from diffusion to wave propagation in one-dimension was discussed by Mainardi and Gorenflo [9], by Schneider and Wyss [31] and Fujita [7] for multi-dimensional time-fractional diffusion-wave equations and by Hanyga [13, 12] in the general case. In [12] the connection between unimodality and non-negativity was discovered and used to define diffusion as opposed to wave propagation.

2. – Fundamental solutions

We shall consider the initial-value problem in the d-dimensional space R^d

$$\begin{aligned} \mathrm{D}^\alpha\, u - L\,u &= f(t, \mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{R}^d, t > 0 \\ u(t, \mathbf{x}) &= 0 \quad \text{for } \mathbf{x} \in \mathbb{R}^d, t < 0 \\ u(0+\mathbf{x}) = u_0(\mathbf{x}), &\quad \mathrm{D}\,u(0+, \mathbf{x}) = v_0(\mathbf{x}) \end{aligned} \tag{2}$$

with $0 < \alpha \leq 2$. The initial condition for $\mathrm{D}\,u(0+)$ is necessary for $1 < \alpha \leq 2$ and has to be dropped for $0 < \alpha \leq 1$. The fractional derivative D^α is defined in the sense of Caputo (other definitions would not be consistent with the well-posedness of the IVP eq. (2)):

$$\mathrm{D}^{\alpha} f = \mathrm{I}^{m-\alpha}\, \mathrm{D}^{m} f \tag{3}$$

where m denotes an integer such that $m-1 < \alpha \leq m$ and $\mathrm{I}^{\gamma} g$, $\gamma \in \mathbb{C}$ denotes the fractional integral

$$\mathrm{I}^{\gamma}\, g(t) := \int_0^t \frac{\tau^{\gamma-1}}{\Gamma(\gamma)} g(t-\tau)\, \mathrm{d}\tau \tag{4}$$

For $0 < \alpha \leq 1$ the operator L is assumed to be an infinitesimal generator of a β-stable Lévy process [30] on the Banach space $\mathcal{C}_0\left(\mathbb{R}^d\right)$ of continuous functions vanishing at infinity, endowed with the sup norm. For $1 < \alpha \leq 2$ the operator L is the fractional power $-(-\overline{\Delta})^{\nu}$, where $\overline{\Delta}$ denotes the closure of the Laplacian on $\mathcal{L}^2\left(\mathbb{R}^d\right)$ and $\nu = \beta/2$, $0 < \beta \leq 2$.

An arbitrary solution of the IVP (2) can be expressed in terms of two or three fundamental solutions $G_t^{(d)}, P_t^{(d)}$ and $Q_t^{(d)}$, $t \geq 0$, of the problem (2) with $u_0 = v_0 = 0$, $f = \delta(t)\, \delta(\mathbf{x})$; $u_0 = \delta(\mathbf{x})$, $v_0 = f = 0$ and $u_0 = f = 0$, $v_0 = \delta(\mathbf{x})$, respectively (the third fundamental solution is defined for $\alpha > 1$ only). In the diffusive case $0 < \alpha \leq 1$ the fundamental solution $Q_t^{(d)}$ is irrelevant while $P_t^{(d)}$ is a probability distribution generated by the original probability concentrated at the origin:

$$\begin{gathered} P_t^{(d)}(\mathbf{x}) \geq 0 \\ \int P_t^{(d)}(\mathbf{x})\, \mathrm{d}_d x = 1 \end{gathered} \tag{5}$$

3. – Formulation in terms of abstract Volterra equations and well-posedness

Well-posedness of the IVP for the space-fractional equations in the Banach space $\mathcal{C}_0\left(\mathbb{R}^d\right)$ is an easy consequence of the fact that L is a generator of a strongly continuous semigroup. On the other hand time- (and space-time-) fractional equations can be expressed as abstract Volterra equations [27]. We now demonstrate the corresponding transformations.

For $0 < \alpha \leq 1$ we shall apply the operator I^{α} to both sides of the differential equation in (2) using the definition of the Caputo derivative and the semigroup property $\mathrm{I}^{\gamma}\, \mathrm{I}^{\beta} = \mathrm{I}^{\gamma+\beta}$ of the fractional integrals:

$$\mathrm{I}\, \mathrm{D}\, u = \mathrm{I}^{\alpha}\left[L\, u + F\right]$$

The Volterra equation formulation is now obtained by working out the left-hand side and applying the initial conditions:

$$u(t) = u_0 + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left[L\, u(\tau) + F(\tau)\right] \mathrm{d}\tau \tag{6}$$

For $1 < \alpha \leq 2$ the result is

$$\mathrm{I}^2\, \mathrm{D}\, u = \mathrm{I}^{\alpha}\left[L\, u + F(t)\right]$$

hence

$$u(t) = u_0 + t\,v_0 + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,[L\,u(\tau) + F(\tau)]\,\mathrm{d}\tau \tag{7}$$

Both eq. (6) and (7) are specific instances of the more general Volterra equation

$$u(t) = g(t) + \int_0^t k(t-\tau)\,L\,u(\tau)\,\mathrm{d}\tau \tag{8}$$

with $g, u : [0,T] \to X$, where X is a Banach space and the function g is given.

The properties of eq. (8) can be restated in terms of a simpler equation

$$u(t) = x + \int_0^t k(t-\tau)\,L\,u(\tau)\,\mathrm{d}\tau, \quad t \in [0,T] \tag{9}$$

with a constant $x \in X$. Following [27] we define well-posedness in terms of the existence of strong solutions of the corresponding Volterra equations, with additional continuous dependence provisions:

DEFINITION 1 Let L be a closed linear operator defined on a dense domain $\mathcal{D}(L)$ in a Banach space X.
The problem (8) is said to be *well-posed* if for every $x \in \mathcal{D}(L)$ there is a unique solution $u(t;x) \in \mathcal{D}(L)$ of eq. (9) such that (i) $u(\cdot;x)$ is continuous in the graph norm $\|y\|_L := \|y\| + \|L\,y\|$ on $\mathcal{D}(L)$, (ii) $u(t;\cdot)$ is continuous at 0 in the graph norm.

Well-posedness is equivalent to the existence of a resolvent $\{S(t) \mid t \geq 0\}$, defined as a strongly continuous one-parameter family of bounded linear operators on X with the following properties:

(i). $S(t)\,\mathcal{D}(L) \subset \mathcal{D}(L)$;

(ii). $L\,S(t)\,x = S(t)\,L\,x$ for every $x \in \mathcal{D}(L)$;

(iii). $S(t)\,x = x + \int_0^t k(t-\tau)\,S(\tau)\,Lx\,\mathrm{d}\tau$ for all $t \geq 0$

[27]. The solution of a well-posed problem (8) with a continuous $g : [0,T] \to X$ and a resolvent $\{S(t) \mid t \geq 0\}$ is given by the formula [27]

$$u(t) = \mathrm{D} \int_0^t S(t-\tau)\,g(\tau)\,\mathrm{d}\tau \tag{10}$$

THEOREM 1 *Let L be the infinitesimal generator L of a stable Lévy process and* $0 < \alpha \leq 1$.
The problem (6) *is well-posed in the Banach space* $C_0\left(\mathbb{R}^d\right)$.

PROOF. Let the function ϑ_α be defined by the formula

$$\vartheta_\alpha(t) = \begin{cases} t^{\alpha-1}/\Gamma(\alpha) & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{11}$$

The Laplace transform of ϑ_α is $s^{-\alpha}$. For $0 < \alpha \leq 1$ and s in the closed right half of the complex plane $\arg s^{-\alpha} \in [-\pi/2, \pi/2]$. Furthermore, $\forall k \in \mathbb{Z}_+$, $k \leq n$ $|s^k \, \mathrm{d}^k \, s^{-\alpha}/\mathrm{d}s^k| \leq K(n) \, |s^{-\alpha}|$. By Theorem 3.1 of [27] eq. (6) has a $\mathcal{C}^\infty$-smooth bounded resolvent and hence the problem (6) is well-posed. ∎

For $\alpha = 2$ the resolvent of (7) is the cosine family of the operator L [27]. We recall the basic definitions:

Definition 2 A strongly continuous family of bounded linear operators $\{S(t) \mid t \in \mathbb{R}\}$ on a Banach space X is said to be a *cosine family* if it satisfies the equations

$$S(t+\tau) + S(t-\tau) = 2S(t)\, S(\tau); \quad S(0) = \mathrm{Id}$$

Let $\{S(t) \mid t \in \mathbb{R}\}$ be a cosine family. The limit $L\, x = \lim_{t \to 0} 2t^{-2} \, [S(t)\, x - x]$ exists for x in a dense subset $\mathcal{D}$ of X. The operator L thus defined is termed the infinitesimal generator of the cosine family [32, 25, 6]. The above observation suggests that we must require that L is an infinitesimal generator of a cosine family. This condition is certainly satisfied if we choose L to be a closed linear extension of the Laplacian on $X = \mathcal{L}^2\left(\mathbb{R}^d\right)$, but the theorem below shows that fractional powers of the negative Laplacian are also generators of cosine families on X.

Theorem 2 *Let $1 < \alpha \leq 2$. Let $X = \mathcal{L}^2\left(\mathbb{R}^d\right)$ and let $L = -(-\overline{\Delta})^\nu$, where $0 < \nu \leq 1$ and $\overline{\Delta}$ is a the closed linear extension of the Laplacian on X.*
The problem (7) is well-posed on X and the resolvent is analytic.

Proof. Step 1. We show that the operator L is self-adjoint on $X = \mathcal{L}^2\left(\mathbb{R}^d\right)$.
Let $X\hat{} = \mathcal{F}X$ denote the Fourier image of X, a complex Banach space. The operator $\hat{L} = \mathcal{F}^{-1}\, L\mathcal{F}$ is a multiplication operator

$$\hat{L}\, f(\mathbf{k}) = -|\mathbf{k}|^{2\nu}\, f(\mathbf{k})$$

defined on

$$\mathcal{D}(\hat{L}) := \{f \in X\hat{} \mid -|\mathbf{k}|^{2\nu}\, f(\mathbf{k}) \text{ is square integrable}\}$$

It is obviously a symmetric operator. It is therefore sufficient to show that $\hat{L} = \hat{L}^\dagger$, or, equivalently, that every element of the domain of the ajoint $\hat{L}^\dagger$ lies in $\mathcal{D}(\hat{L})$.
Let θ_R be the characteristic function of the ball of radius R centered at the origin and let $||\cdot||$ denote the $\mathcal{L}^2$ norm.

$$||\hat{L}^\dagger f|| = \lim_{R\to\infty} ||\theta_R\, \hat{L}^\dagger f|| = \lim_{R\to\infty} \sup_{||g||=1} \left| \left(g, \theta_R\, \hat{L}^\dagger f\right) \right| = \lim_{R\to\infty} \sup_{||g||=1} |(\hat{L}\, \theta_R\, g, f)|$$

since $\theta_R\, g \in \mathcal{D}(L)$. Hence

$$\lim_{R\to\infty} ||\theta_R\, \hat{L}\, f|| = \lim_{R\to\infty} \sup_{||g||=1} \left| \int g(\mathbf{k}) |\mathbf{k}|^{2\nu} f(\mathbf{k})\, \mathrm{d}k \right|$$

$$= \lim_{R\to\infty} \sup_{||g||=1} |(\hat{L}\, \theta_R\, g, f)| = ||\hat{L}^\dagger f|| < \infty$$

By the Fatou lemma, $\hat{L} f \in X$ and $f \in \mathcal{D}(\hat{L})$, hence the operators $\hat{L}$ and L are self-adjoint, q.e.d.

Step 2. We now show that L is an infinitesimal generator of a cosine family. The spectrum of L is $\mathbb{R}_- \cup \{0\}$. Using the spectral theory we define the operator family

$$C(t)\, f := \int_{-\infty}^{0} \cosh\left(t\, \xi^{1/2}\right)\, P(\mathrm{d}\xi)\, f \tag{12}$$

where $\{P_\xi \mid \xi \in \mathbb{R}\}$ is the spectral operator family of L. Note that $\cosh(x)$ is a function of x^2, hence the square root of $\xi \le 0$ in the above equation is an artefact of the definition of the function cosh.
$\{C(t) \mid t \ge 0\}$ is a strongly continuous cosine family ([6], p. 41). Its infinitesimal generator is L:

(13)

$$\lim_{t \to 0} \|2t^{-2}\left[C(t)\, f - f\right] - L\, f\|^2 = \int_{-\infty}^{0} \left[\frac{2\cosh\left(t\xi^{1/2}\right) - 1}{t^2\, \xi} - 1\right]^2 \xi^2\, (f, P(\mathrm{d}\xi)\, f) = 0$$

$$\forall f \in \mathcal{D}(L)$$

Indeed, there is a continuously differentiable function ϕ such that $\cosh(x) \equiv: \phi(x^2)$, $\phi'(x^2) = \sinh(x)/x$ and $(\cosh(x) - 1)/x^2 = \phi'(\zeta\, x^2)$, for some $\zeta \in [0, 1]$, hence the dominated Lebesgue theorem applies for $f \in \mathcal{D}(L)$.

Step 3. We now use the above property of L in the proof of well-posedness. For $1 < \alpha \le 2$ the function ϑ_α is a Bernstein function:

$$\vartheta_\alpha, \mathrm{D}\vartheta_\alpha \ge 0; \qquad \forall n \in \mathbb{Z}_+ \quad n \ge 2 \Rightarrow (-1)^n\, \mathrm{D}^n \vartheta_\alpha \ge 0$$

Furthermore the function defined by eq. (4.64) of [27] has the property (i) stated in Theorem 4.6 of the same reference. Therefore eq. (7) has an analytic resolvent [27, 26]. ■

4. – Stable probability densities and infinitesimal generators of stable Lévy processes

We shall present explicit forms of the linear operators which generate stable Lévy processes. To this effect we need some background in stable probabilities and Lévy process [30]. A Lévy process $\{Y_t \mid t \ge 0\}$ is a stochastic process with values in $\mathbb{R}^d$, with independent identically distributed increments, such that $Y_0 = 0$ almost surely and satisfying some regularity requirements which we shall not discuss here. The above properties imply that Y_1 has an infinitely divisible distribution.

Definition **3** The probability distribution P is said to be *infinitely divisible* if for every positive integer $n \ge 2$ there is a probability distribution P_n such that P is the convolutional product of n factors P_n

$$P = P_n * P_n \ldots * P_n = P_n^{*n} \quad \forall n \in \mathbb{N} \tag{14}$$

The convolution of probability distributions is defined by the formula

$$P * Q(B) = \int_B P(X - \mathrm{d}y)\, Q(\mathrm{d}y)$$

In terms of characteristic functions $\chi_P(\mathbf{z}) := \int \mathrm{e}^{\mathrm{i}\,\mathbf{z}\cdot\mathbf{x}}\, P(\mathrm{d}x)$, we have for every n, $\chi_P = {\chi_{P_n}}^n$, where χ_{P_n} is the characteristic function of a probability distribution P_n.

THEOREM 3 *(Lévy-Khintchine representation)*
Let P be an infinitely divisible probability distribution on $\mathbb{R}^d$.
Then the characteristic function χ_P has the following form

$$\log \chi_P(\mathbf{z}) = \mathbf{iq} \cdot \mathbf{z} - \frac{1}{2}\mathbf{z} \cdot \mathbf{Az} + \int \left[\mathrm{e}^{\mathrm{i}\mathbf{z}\cdot\mathbf{y}} - 1 - \mathrm{i}\mathbf{z} \cdot \mathbf{y}\, h_{\mathcal{B}_1}(\mathbf{y})\right] \mu(\mathrm{d}\mathbf{y})$$

where $\mathbf{A}$ is a symmetric positive semi-definite $d \times d$ matrix, μ is a measure on $\mathbb{R}^d$ satisfying the conditions $\mu(\{0\}) = 0$,

$$\int \left[|\mathbf{y}|^2 \wedge 1\right] \mu(\mathrm{d}\mathbf{y}) < \infty$$

and $h_{\mathcal{B}_1}$ denotes the characteristic function of the unit ball $\mathcal{B}_1$ in $\mathbb{R}^d$.

The measure μ is called the Lévy measure of P.

The characteristic function of the probability distribution of the Lévy process Y_t is $(\chi_P)^t$, where P denotes the probability distribution of X_1 and χ_P denotes the characteristic function of P.

An important subclass of infinitely divisible probability distributions are α-stable probability distributions.

DEFINITION 4 Let $0 < \alpha \leq 2$. A Lévy process $\{Y_t \mid t \geq 0\}$ is said to be *α-stable* if if the probability distribution P of the random variable Y_1 is α-stable.

THEOREM 4 *A Lévy process $\{Y_t \mid t \geq 0\}$ is α-stable if and only if it has the following property of self-similarity: Y_{at} has the same probability distribution as $a^{1/\alpha} Y_t$.*

[30].

We recall the definition and the basic property of α-stable probability distributions relevant for our purposes.

DEFINITION 5 The probability distribution P is said to be *α-stable* if any sequence of independent random variables Y_k having the probability distribution P has the property

$$\lim_{\mathrm{d}\ n\to\infty} \left(\sum_{k=1}^{n} Y_k - a_n\right) / b_n = Y_1$$

where the limit exists in the sense of distributions.

THEOREM 5 *The characteristic function χ_P of an α-stable probability distribution has a Lévy-Khintchine representation with* $\mathrm{d}\mu = C_d r^{-\alpha-1}\,\mathrm{d}r$. *More specifically,*

$$(15)\quad \log\chi_P(\mathbf{z}) = \begin{cases} \mathrm{i}\mathbf{q}_0\cdot\mathbf{z} + \int_S \mathrm{d}\lambda(\Omega)\int_0^\infty \left[\mathrm{e}^{\mathrm{i}\mathbf{z}\cdot\mathbf{x}} - 1\right] r^{-\alpha-1}\,\mathrm{d}r, & 0<\alpha<1 \\ \mathrm{i}\mathbf{q}_1\cdot\mathbf{z} + \int_S \mathrm{d}\lambda(\Omega)\int_0^\infty \left[\mathrm{e}^{\mathrm{i}\mathbf{z}\cdot\mathbf{x}} - 1 - \mathrm{i}\mathbf{z}\cdot r\Omega\right] r^{-\alpha-1}\,\mathrm{d}r, & 1<\alpha<2 \end{cases}$$

where λ is a finite measure on the unit sphere $\mathcal{S} = \mathcal{S}^{d-1}$.

For simplicity we have omitted the special cases $\alpha = 1$, corresponding to the Cauchy distribution, and $\alpha = 2$, corresponding to the normal distribution.

For univariate probability distributions ($d = 1$) the sphere reduces to $\mathcal{S} = \{-1, 1\}$ and the characteristic function of an α-stable probability distribution can be expressed in the form

$$\log\chi_P(z) = -c|z|^\alpha\left[1 - \mathrm{i}\theta\tan(\pi\alpha/2)\,\mathrm{sgn}z + \mathrm{i}qz\right], \quad -1\le\theta\le 1$$

where the parameter $\theta\in[-1,1]$, $\theta := (\lambda(\{1\}) - \lambda(\{-1\}))/(\lambda(\{1\}) + \lambda(\{-1\}))$, is called *skewness* and $c = \lambda(\{1\}) + \lambda(\{-1\})$. A more compact parametrization of univariate α-stable probability distributions is sometimes used:

$$(16)\qquad \log\chi_P(z) = -C|z|^\alpha \mathrm{e}^{-\mathrm{i}(\pi/2)\vartheta\,\alpha\,\mathrm{sgn}z}$$

with $|\vartheta|\le 1\wedge(2-\alpha)/\alpha$. The last formula remains valid for $\alpha = 1, 2$ too. Excluding for simplicity the special case $\alpha = 1$, the asymmetry parameter ϑ is related to θ by the formula

$$(17)\qquad \vartheta = \begin{cases} (2/(\alpha\pi))\tan^{-1}(\theta\tan(\alpha\pi/2)) \\ (-2/(\alpha\pi))\tan^{-1}(\theta\tan((2-\alpha)\pi/2)) \end{cases}$$

with $\tan^{-1}$ taking values in $[-\pi/2, \pi/2]$.

For $\alpha < 1$ and $\theta = 1$ the probability distribution P is concentrated on the positive real axis and its density, denoted here by $p^+(\cdot;\alpha)$, can also be defined by the formula

$$(18)\qquad p^+(x;\alpha) = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{st-s^\alpha}\,\mathrm{d}s$$

An α-stable probability distribution is said to be centrally symmetric (c. s.) if the measure λ is symmetric with respect to reflections in the origin $\mathbf{x}\to-\mathbf{x}$. The characteristic function of a general c.s. α-stable probability distribution has the form

$$\log\chi_P(\mathbf{z}) = -\int_S |\mathbf{z}\cdot\Omega|^\alpha\,\mathrm{d}\lambda(\Omega)$$

where λ is arbitrary finite non-zero c. s. measure on $\mathcal{S}$. For a spherically symmetric α-stable probability distribution we have [30]

$$\log\chi_P(\mathbf{z}) = -c|\mathbf{z}|^\alpha\,.$$

The importance of α-stable probability distributions is explained by the following generalization of the Central Limit Theorem:

THEOREM 6 *If the random variables Y_n, $n = 0, 1, \ldots$, are independent with the same probability distribution P satisfying the conditions*

$$1 - P(x) \sim c x^{-\mu} \quad \textit{for } x \to \infty$$

$$P(x) \sim d|x|^{-\mu} \quad \textit{for } x \to -\infty$$

with $\mu \neq 1$, and

$$Z_n := (\sum_{i=1}^{n} Y_i - a_n)/b_n$$

then

$$P_{Z_n}(x) \to P(x; \alpha, \theta)$$
$$\alpha = \min\{2, \mu\}$$
$$\theta = (c-d)/(c+d)$$
$$a_n = C n^{1/\alpha} \qquad b_n = 0 \textit{ or } n\,\mathsf{E}[Y_1]$$

where $P(x; \alpha, \theta)$ denotes the α-stable probability distribution with skewness θ.

[33].

Lévy processes are Markovian. The transition probability q of a Lévy process $\{Y_t \mid t \geq 0\}$ from $\mathbf{x}$ to a Borel subset $\mathcal{B}$ of $\mathbb{R}^d$ is defined by the formula

$$q_t(\mathbf{x}, \mathcal{B}) := P(Y_{t+s} - Y_s \in \mathcal{B} - \mathbf{x}) = P_t(\mathcal{B} - \mathbf{x})$$

where $P_t := P_{Y_t}$ denotes the probability distribution of the random variable Y_t. The transition probability satisfies the Bachelier-Smoluchowski-Chapman-Kolmogorov equation

$$\int q_s(\mathbf{x}, \mathrm{d}y)\, q_t(\mathbf{y}, \mathcal{B}) = q_{t+s}(\mathbf{x}, \mathcal{B})$$

We are now in position to define a strongly continuous unitary semigroup $\{U_t \mid t \geq 0\}$ on the Banach space $\mathcal{C}_0\left(\mathbb{R}^d\right)$ of continuous functions on $\mathbb{R}^d$ vanishing at infinity (endowed with the sup norm):

$$U_t\, f(\mathbf{x}) := \int f(\mathbf{x} - \mathbf{y})\, P_t(\mathrm{d}y) \equiv \int f(\mathbf{y})\, q_t(\mathbf{x}, \mathrm{d}y)$$

Let L denote the infinitesimal generator of $\{U_t \mid t \geq 0\}$. The operator L is a closed linear extension of the non-local operator

$$(19) \quad L\, f(\mathbf{x}) = \mathbf{q} \cdot \nabla f(\mathbf{x}) + A_{ij}\, \partial^2 f / \partial x_i\, \partial x_j + \int [f(\mathbf{x} + r\Omega) - f(\mathbf{x}) - r\Omega \cdot \nabla f(\mathbf{x}) h_{\mathcal{B}_1}(\mathbf{y})]\; \mu(\mathrm{d}y)$$

where μ, $\mathbf{A}$ are as in Theorem 3 [30]. Up to an additional constant, such operators are also known as the most general infinitesimal generators of semigroups $\{U(t) \mid t \geq 0\}$ satisfying positivity [28]. For an α-stable Lévy process the operator L we either have $\mathbf{A} \neq 0$, $\mu = 0$ (the Gaussian case) or $\mathbf{A} = 0$, $\mu(\mathrm{d}y) = \lambda(\mathrm{d}\Omega)\, r^{-\alpha-1}\, dr$,

$r := |\mathbf{y}|$, $\alpha < 2$. Redefining the vector $\mathbf{q}$ one has the following explicitly scale-invariant representations of L for $\alpha < 2$, $\alpha \neq 1$

(20)

$$(Lf)(\mathbf{x}) = \begin{cases} \mathbf{q} \cdot \nabla f(\mathbf{x}) + \int \left[f(\mathbf{x} + r\Omega) - f(\mathbf{x})\right] \lambda(\mathrm{d}\Omega)\, r^{-\alpha-1}\, \mathrm{d}r \\ \qquad\qquad \text{for } \alpha < 1 \\ \mathbf{q} \cdot \nabla f(\mathbf{x}) + \int \left[f(\mathbf{x} + r\Omega) - f(\mathbf{x}) - r\,\Omega \cdot \nabla f(\mathbf{x})\right] \lambda(\mathrm{d}\Omega)\, r^{-\alpha-1}\, \mathrm{d}r \\ \qquad\qquad \text{for } 1 < \alpha < 2 \end{cases}$$

An alternative construction of the operator L on its core space can be found in [11].

Example.

For an isotropic stable Lévy process the operator L is a fractional power [34, 5] of the closure of the Laplacian in $\mathcal{C}_0\left(\mathbb{R}^d\right)$

(21)

$$L = -(-\overline{\Delta})^\nu = \begin{cases} \int_S \mathrm{d}\lambda_0(\Omega) \int_0^\infty \left[f(\mathbf{x} + r\Omega) - f(\mathbf{x})\right] r^{-2\nu-1}\, \mathrm{d}r & 0 < \nu < 1/2 \\ \int_S \mathrm{d}\lambda_0(\Omega) \int_0^\infty \left[f(\mathbf{x} + r\Omega) + f(\mathbf{x} - r\Omega) - 2f(\mathbf{x})\right] & \\ \qquad \times r^{-2\nu-1}\, \mathrm{d}r & 1/2 < \nu < 1 \end{cases}$$

where λ_0 denotes the uniform probability distribution on $\mathcal{S}^{d-1}$.

5. – Self-similarity of the fundamental solutions

We assume that L is an infinitesimal generator of a β-stable Lévy process.

For $s = a\,t$ and $\mathbf{y} = b\,\mathbf{x}$ we have $\mathrm{D}_t^\alpha = a^\alpha\, \mathrm{D}_s^\alpha$, $L_\mathbf{x} = b^\beta\, L_\mathbf{y}$. This implies that the fundamental solutions are self-similar. Indeed, let $u(t, \mathbf{x})$ be one of the fundamental solutions $G_t^{(d)}$, $P_t^{(d)}$ or $Q_t^{(d)}$. The function $v(t, \mathbf{x}) := c\,u(a\,t, b\,\mathbf{x})$ satisfies the same IVP provided $b = a^{\alpha/\beta}$ and $c = a^{1-\alpha}\, b^d = a^{1+(d/\beta-1)\alpha}$ in the first case, $c = b^d = a^{d\alpha/\beta}$ in the second case and $c = b^d/a = a^{d\alpha/\beta-1}$ in the third case.

In view of uniqueness of the IVP involved, the fundamental solutions have the following forms

$$G_t^{(d)}(\mathbf{x}) = t^{-1-(d/\beta-1)\,\alpha}\, F_1(t^{-\alpha/\beta}\,\mathbf{x}) \tag{22}$$

$$P_t^{(d)}(\mathbf{x}) = t^{-d\,\alpha/\beta}\, F_2(t^{-\alpha/\beta}\,\mathbf{x}) \tag{23}$$

$$Q_t^{(d)}(\mathbf{x}) = t^{1-d\,\alpha/\beta}\, F_3(t^{-\alpha/\beta}\,\mathbf{x}) \tag{24}$$

where F_1, F_2, F_3 are some functions on $\mathbb{R}^d$.

In the following we shall concentrate on the propagator $P_t^{(d)}$. For the other two fundamental solutions, see [13, 12].

6. – Some integral representations of the fundamental solutions

The propagator $P_t^{(d)}(\cdot;\beta) \equiv G_t^{(d)}(\cdot;\beta)$ of a space-fractional IVP (2) with $\alpha = 1$ and a centrally symmetric Lévy measure can be expressed in terms of a β-stable probability density $p^{(d)}(\cdot;\beta,\lambda)$ on $\mathbb{R}^d$ [11]:

$$P_t^{(d)}(\mathbf{x};\beta,\lambda) = t^{-d/\beta}\, p^{(d)}\left(t^{-1/\beta}\,\mathbf{x};\beta,\lambda\right) \tag{25}$$

The probability density as well as the propagator depend on the index β as well as on the measure λ on the unit sphere in $\mathbb{R}^d$, appearing in eq. (19). The propagator $P_t(\cdot;\alpha,\beta,\lambda)$ of the space-time fractional IVP (2) with $0<\alpha\le 1$, $0<\beta\le 2$ and a centrally symmetric measure λ is given by the formula [12]

$$P_t^{(d)}(\mathbf{x};\alpha,\beta,\lambda) = t^{-d\alpha/\beta}\int_0^\infty \xi^{d\alpha/\beta}\, p^+(\xi;\alpha)\, p^{(d)}((t/\xi)^{-\alpha/\beta}\,\mathbf{x};\beta,\lambda)\,\mathrm{d}\xi \tag{26}$$

which is equivalent to

$$P_t^{(d)}(\mathbf{x};\alpha,\beta,\lambda) = \int_0^\infty p^+(\xi;\alpha)\, P_{(t/\xi)^\alpha}(\mathbf{x};\beta,\lambda)\,\mathrm{d}\xi \tag{27}$$

The two formulae are valid only for $0<\alpha\le 1$. For $d=1$ the operators L can be parameterized by $\vartheta\in[-m,m]$, $m=\min\{1,(2-\alpha)/\alpha\}$ instead of the measure λ. The factor α/β entirely determines all the scaling properties of the propagator, but for $\beta<2$ the propagator has an infinite second-order moment [12].

In particular, for a time-fractional diffusion equation with $\alpha\le 1$ and $\beta=2$ the space-fractional propagator in eq. (27) is the Gaussian distribution of the Brownian motion

$$P_t^{(d)}(\mathbf{x};2) = \frac{1}{(4\pi t)^{d/2}}\exp\left(-|\mathbf{x}|^2/(4t)\right) \tag{28}$$

and eq. (27) reduces to a superposition of Gaussian distributions

$$P_t^{(d)}(\mathbf{x};\alpha,2) = \frac{1}{(2\pi t^\alpha)^{d/2}}\int_0^\infty \xi^{\alpha d/2}\, p^+(\xi;\alpha)\,\exp\left(-\xi^\alpha|\mathbf{x}|^2/\left(4t^\alpha\right)\right)\,\mathrm{d}\xi \tag{29}$$

The propagator of a spherically symmetric space-fractional diffusion equation can also be expressed in terms of a superposition of Gaussian probability densities [11]:

$$P_t^{(d)}(\mathbf{x};\beta) = \frac{1}{(2\pi)^{d/2}\,t^{d/\beta}}\int_0^\infty p^+\left(\xi;\beta/2\right)\,\exp\left(-|\mathbf{x}|^2/(4\xi\, t^{2/\beta}\right)\,\mathrm{d}\xi \tag{30}$$

Eq. (30) follows from a well-known formula linking a semigroup $\{S_\nu(t)\mid t\ge 0\}$ generated by a fractional power $-(-A)^\nu$ of an infinitesimal generator A of a uniformly bounded semigroup $S(t)$ on a Banach space X [34]. This formula, due to Bochner and Phillips, can be rephrased as follows

$$S_\nu(t)\,f = \int_0^\infty p^+(\xi;\nu)\,S(t^{1/\nu}\,\xi)\,f\,\mathrm{d}\xi \quad \forall f\in X \tag{31}$$

Eq. (30) follows from eq. (31) for $X=\mathcal{C}_0$, A - the closed linear extension of the Laplacian in X,

$$(S(t)\,f)\,(\mathbf{x}) = \int P_t^{(d)}(\mathbf{x}-\mathbf{y};2)\,f(\mathbf{y})\,\mathrm{d}_d y \tag{32}$$

The propagator for $0<\alpha\le 2$ and $L=\overline{\Delta}$ can be expressed in terms of the Mainardi function [13].

For $1 < \alpha \leq \beta$, $d = 1$ and $L = -(-\Delta)^{\nu}$ the propagator can be expressed in terms the propagators of eq. (2) with $\alpha = \beta$ [20]. Using the identity [12]

$$M(z;\alpha) = \alpha^{-1} z^{-\alpha-1} p^{+}\left(z^{-1/\alpha};\alpha\right)$$

their expression, in the second line of eq. (6.16), can be rephrased as follows

$$P_t^{(1)}(x;\alpha,\beta,\vartheta) = \int_0^{\infty} p^{+}(\xi;\alpha/\beta)\, P^{(1)}_{(t/\xi)^{\alpha/\beta}}(x;\beta,\beta,\vartheta)\, \mathrm{d}\xi \tag{33}$$

provided $1 < \alpha \leq \beta$. From the above formula propagators the propagators for $d = 3$ can be obtained by application of the operator $(2\pi\, r)^{-1}\, \partial/\partial r$. The propagator $P_t^{(1)}(\cdot;\beta,\beta,\vartheta)$ is an algebraic function of $x/t^{\beta/2}$ [20].

7. – Unimodality of diffusive processes

7.1. – *Introduction.* – Broadly speaking, the propagator $P_t^{(d)}$ represents a diffusion if for every $t \geq 0$ (i) $P_t^{(d)} \geq 0$, (ii) $\int P_t^{(d)}\, \mathrm{d}_d x = 1$ and if
(iii) $P_t^{(d)}$ either has a single maximum or if it diverges to infinity at single point (a priori a δ-type singularity need not be excluded).

For the centrally symmetric case the last condition can be formulated more precisely:
(iii$a$) $P_t^{(d)}$ either has a single maximum at the origin or it diverges to infinity for $\mathbf{x} \to 0$ and it decreases in every radial direction.

Condition (iii$a$) amounts to unimodality of $P_t^{(d)}$ with mode 0, to be defined more precisely later. In view of the self-similarity of the propagator, criterion (iii$a$) implies that every radial section of a diffusive propagator $P_t^{(d)}$ represents a gradually widening hump with a maximum or singularity at the origin. On the other hand, wave-like behavior is characterized by the maximum propagating away from the origin in all the directions. This is property is equivalent to the existence of two maxima of $P_1^{(1)}$ at two points outside the origin, since self-similarity already implies that such maxima of $P_t^{(1)}$ move away from the origin. In the wave-like case we shall consider only spherically symmetric infinitesimal generators, hence the wave-like behavior involves a crest located on an expanding sphere. Furthermore, if the maximum of $P_t^{(1)}$ is located off the origin then $P_t^{(3)}$ changes sign in the spherically symmetric case. This strengthens the case against diffusion if condition (iii$a$) is violated (note that $P_t^{(1)}$ does satisfy conditions (i) and (ii) even if condition (iii$a$) is violated.). It should however be kept in mind that in one dimension a single maximum moving in one direction is a typical of a biased diffusion, associated with a skewed stable probability.

Substitution $P_t^{(1)}(x;\beta,1)$ from eq. (25) in an extension of the formula (27) to the asymmetric case yields

$$P_t^{(1)}(x;\alpha,\beta,1) = t^{-\alpha/\beta} f\left(x t^{-\alpha/\beta}\right) \tag{34}$$

$$f(x) := \int_0^{\infty} p^{+}(\xi;\alpha)\, p^{+}(x\, \xi^{\alpha/\beta};\beta)\, \xi^{\alpha/\beta}\, \mathrm{d}\xi \tag{35}$$

For $\alpha = 1/2$, $\beta = 1/3, 2/3$ the functions $p^+(x;\alpha$, $p^+(x;\beta)$ can be expressed in terms of elementary functions and the Airy function [12].

To wit, a univariate probability density is unimodal with a mode at 0 if it has either a maximum at 0 (possibly flat) or diverges to $+\infty$ for $\mathbf{x} \to 0$ or has a delta-spiked singularity at 0. Our basic tool for proving unimodality is eq. (26), which implies that for $0 < \alpha \leq 1$ the propagator P_t lies in a closed convex hull of unimodal probability densities on $\mathbb{R}^d$. It is crucial for our argument that the second factor in the integrand of (26) is unimodal with a mode at the origin for every value of the integration variable.

7.2. – *Unimodality in one-dimensional space.* – For $d = 1$ there are more precise results on unimodality in probability theory than for higher dimensions. We begin with a definition of unimodality appropriate for one dimension.

DEFINITION 6 A probability distribution Q on R is *unimodal with mode at the origin* if either it has an atom $c\,\delta_0$ at the origin or if its density $q(x) \geq 0$ is finite except at a finite number of points and satisfies the inequalities
(i) $q(x) < q(0)$ for $a - \epsilon < x < a \leq 0$ and for $0 < b < x < b + \epsilon$ where a, b and $\epsilon > 0$ are some constants.
(ii) If $a < b$ then $q(x) = q(0) < \infty$ for all $x \in [a, b]$, otherwise $a = b = 0$ and $q(0) \leq \infty$.

By a theorem of Yamazato (Theorem 53.1 in [30]) every β-stable probability distribution is unimodal. Its mode obviously lies at the origin if the probability distribution is symmetric about the origin. The latter property does not hold if skewness is different from zero. Assuming a parametrization of univariate stable probability densities $p(\cdot;\beta,\vartheta)$ in terms of the formula

$$\log \mathsf{E}\left[e^{\mathrm{i}\,z\,x}\right] = -C\,|z|^\beta\,\mathrm{e}^{-\mathrm{i}\pi\,\beta\,\vartheta/2} \tag{36}$$

with $|\vartheta| \leq \min\{1, (2-\beta)/\beta\}$ and $C > 0$, the maximum of $p(\cdot;\beta,\vartheta)$ lies at a point $x = a$ in $\mathbb{R}_+$ ($\mathbb{R}_-$) if $\vartheta > 0$ ($\vartheta < 0$) and $\beta \neq 1$ (this probably also applies to $\beta = 1$ by numerical arguments). Consequently, the location maximum of the second factor in eq. (26) depends on the integration variable. Since $p^+(\cdot;\alpha)$ is not monotone, it is not possible to prove unimodality for $P_t^{(1)}$ if $\vartheta \neq 0$.

In particular, the case of $\alpha = 1$, $\beta = 1/2$ provides a counter-example to unimodality with mode 0 of $P_t^{(1)}$ since the maximum of $p^+(x;1/2)$ lies at $x = 1/6$. More generally, for $\theta \neq 0$ the maximum of $p^+(x;\beta,\vartheta)$ lies at a point $x_0 \neq 0$ ([29], Sec. 1.6). The maximum of $P_t^{(1)}(\cdot;1,\beta,\vartheta)$, with $\vartheta > 0$, propagates to the right due to a higher probability of diffusion to the right. On the other hand for $\alpha < 1$, $\vartheta \neq 0$ the propagator can be unimodal with mode at $x = 0$. Thus, for example

$$P_t(x;1/2,1/3,1) = t^{-3/2}\,h\left(x/t^{3/2}\right)\,\left(x/t^{3/2}\right)^{-4/3}$$

where

$$h(z) := \frac{1}{2\sqrt{\pi}}\int_0^\infty \mathrm{d}\zeta\,\mathrm{e}^{-\zeta/4}\,\mathrm{Ai}\left(\zeta^{1/2}\,z\right)$$

is a finite unimodal function with mode at $z = 0$.

We are thus forced to restrict our attention to symmetric univariate stable probability distributions. For symmetric univariate probability distributions Definition 6 is equivalent to the following definition:

DEFINITION 7 A symmetric univariate probability distribution is said to be unimodal if it lies in the closed convex hull of probability distributions which are uniform on their supports $[-a, a]$, $a > 0$.

cf [3]. Closure is meant here in the weak measure topology. Furthermore, every unimodal symmetric probability distribution Q is a *generalized mixture* of distribution functions W_a which are uniform on intervals $[-a, a]$, $a > 0$:

$$Q(\mathcal{A}) = \int_0^\infty W_a(\mathcal{A})\,\mu(\mathrm{d}a) \quad \text{for every Borel subset } \mathcal{A} \text{ of } \mathbb{R} \tag{37}$$

where μ is an univariate probability distribution with support on $\overline{\mathbb{R}_+}$ [3]. The converse of the last statement is also true.

Combining these facts with eq. (26) we have established the following theorem:

THEOREM 7 *Let* $0 < \alpha \leq 1$, $0 < \beta \leq 2$ *and* $\vartheta = 0$.
The propagator $P_t^{(1)}$ *is unimodal with mode at 0.*

7.3. – *Unimodality for* $d > 1$. – For $d > 1$ the requirement of symmetry has to be replaced by central symmetry, i.e. symmetry with respect to the reflections $\mathbf{x} \to -\mathbf{x}$ in the origin.

DEFINITION 8 A centrally symmetric probability distribution Q on $\mathbb{R}^d$ is said to be *central convex unimodal* if it lies in the closed convex hull of all probability distributions whose supports are convex cetrally symmetric subsets of $\mathbb{R}^d$ and which have constant density on their support.

cf [3]. This definition is rather understable on intuitive grounds, but the following definition of unimodality is even closer to our purposes:

DEFINITION 9 A probability distribution Q on R^d is *monotonely unimodal* if for every convex centrally symmetric set $\mathcal{A}$ and for every $\mathbf{x} \in \mathbb{R}^d$ the function $t \to Q(\mathcal{A} + t\mathbf{x})$ is non-increasing for $t \geq 0$.

By Anderson's theorem [1, 3] every central convex unimodal probability distribution is mononotonely unimodal.

The following definition, equivalent to Def. 8, involves the concept of a generalized mixture of probability distributions which are uniform on and supported by convex centrally symmetric subsets of $\mathbb{R}^d$ [14]:

DEFINITION 10 A probability distribution Q is *symmetric unimodal* if there is a family $\{W_\xi \mid \xi \in \Xi\}$ of convex centrally symmetric sets such that Ξ is a probability space with a probability distribution μ, and

$$Q(\mathcal{A}) = \int_\Xi W_\xi(\mathcal{A})\,\mu(\mathrm{d}\xi) \quad \forall \text{ Borel } \mathcal{A} \subset \mathbb{R}^d$$

where each W_ξ is a probability distribution with support $\mathcal{W}_\xi$ and a density which is constant on $\mathcal{W}_\xi$, for all $\xi \in \Xi$.

THEOREM 8 *Let* $0 < \alpha \le 1$, $0 < \beta \le 2$ *and let* λ *be a finite centrally symmetric measure on the unit sphere* $\mathcal{S}^{d-1}$.
The propagator $P_t^{(d)}(\cdot;\alpha,\beta,\lambda)$ *is finite at the origin if* $d \le \beta$ *and diverges to* $+\infty$ *for* $\mathbf{x} \to 0$ *in the other case. For* $\mathbf{y} \in \mathbb{R}^d$, $\mathbf{y} \neq 0$, *the function* $P_t^{(d)}(r\,\mathbf{y};\alpha,\beta,\lambda)$ *is finite and non-increasing for* $r > 0$.

PROOF. Every α-stable probability distribution is symmetric unimodal [14]. It can therefore be represented in terms of a generalized mixture of probability distributions W_ξ with convex centrally symmetric supports and constant densities. According to eq. (26) $P_t^{(d)}$ is a generalized mixture of α-stable probability densities. Substituting the expression for $p^{(d)}(\cdot;\beta\lambda)$ in terms of W_ξ in eq. (26) and folding together the two integrals an expression for $P_t^{(d)}$ in terms of a generalized mixture of W_ξ is obtained [12], which proves that $P_t^{(d)}$ is symmetric unimodal and, hence, central convex unimodal [3]. By theorem due to Anderson [1, 3] it is monotonely unimodal. Regularity properties of $P_t^{(d)}$ are established in [12]. Applying these properties and the Lebesgue theorem the last statement of Theorem 8 is proved [12]. ∎

7.4. – *Wave-like behavior of* $P_t^{(d)}$. – Numerical investigations in [12] and [20] show that solutions of symmetric equations 2 for $\alpha > 1$ in a one-dimensional space are bimodal, with two maxima propagating rightwards and leftwards. A spherically symmetric propagator of eq. (2) in $\mathbb{R}^3$ can be obtained by applying the formula [12]

$$P_t^{(3)}(\mathbf{x};\alpha,\beta,\lambda_0) = -\frac{1}{2\pi\, r}\frac{\partial}{\partial r}P_t^{(1)}(r;\alpha,\beta,0) \tag{38}$$

where $r = |\mathbf{x}|$ and λ_0 denotes the uniform probability on the unit sphere $\mathcal{S}^{d-1}$. If $P_t^{(1)}$ has a maximum at $x > 0$ then then the corresponding spherically symmetric propagator $P_t^{(3)}$ changes sign.

The transition from diffusive to propagative behavior in time-fractional equations in 1, 2 and 3 dimensions is discussed at length in [13].

8. – Concluding remarks

Our well-posedness results cover anisotropic fractional diffusion equations ($0 < \alpha \le 1$) as well as isotropic diffusion-wave equations ($L = -(-\overline{\Delta}$, $0 < \alpha \le 2$). Diffusive behavior, associated with unimodality of the propagator, has been demonstrated for centrally symmetric operators L and $0 < \alpha \le 1$. Dropping the assumption of central symmetry in one dimension leads to biased diffusion (higher probability for moving in one direction). The propagator is still unimodal but the mode propagates away from the origin. It is not clear what happens in the multi-dimensional case if the assumption of central symmetry is dropped.

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Exponential decay on the mean in linear viscoelasticity*

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1. – Introduction

In linear viscoelasticity, the dissipation effects due to memory lead to results of stability and decay of the energy under the hypothesis that the constitutive equations satisfy the thermodynamic restrictions (see for example [3]).

In order to prove that the energy approaches exponentially zero as time goes to infinity, further hypotheses must be required. More precisely, the memory kernel, its derivative and a suitable linear combination of them must be positive definite ([2, 5]). These requests imply the convexity and the exponential decay of the kernel and by using the theory of contraction semi-groups or suitable estimates, a point to point exponential decay of the energy is obtained.

In this paper we give a sufficient condition on the relaxation function for the exponential decay on the mean of the energy and prove that, if the memory kernel has an exponential decay on the mean, i.e. if there exists a positive constant λ such that the kernel multiplied by $\exp(\lambda t)$ is an integrable function of the time, then the energy has the same behavior.

The method used here is based on the study of the Laplace transform of the solution of the evolutive problem in linear viscoelaticity. The main result consists in proving that it is possible to define the Laplace transform of the solution in the region of complex plane $\mathcal{R}p > -\gamma$, with $\gamma > 0$, and that this function is analytic and goes to zero when p goes to infinity.

2. – Constitutive hypothesis

In the linearized theory of isothermal viscoelasticity the stress tensor $\mathbf{T}$ is determined by the history $\mathbf{E}^t$ of the infinitesimal strain tensor $\mathbf{E}$ trough the hereditary law

$$\mathbf{T}(\mathbf{x},t) = \mathbb{G}_0(\mathbf{x})\mathbf{E}(\mathbf{x},t) + \int_{\mathbb{R}^+} \dot{\mathbb{G}}(\mathbf{x},s)\mathbf{E}^t(\mathbf{x},s)\,ds, \qquad (\mathbf{x},t) \in \Omega \times \mathbb{R}^+, \tag{1}$$

where the elastic modulus $\mathbb{G}_0$ and the Boltzmann function $\dot{\mathbb{G}}$ are symmetric fourth order tensors.

*Preview of the work in collaboration with J.A.D. Applebay, M. Fabrizio and D.W. Reynolds [1].

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We consider the mixed initial boundary value problem in a bounded and regular domain Ω

$$\ddot{\mathbf{u}}(\mathbf{x},t) = \nabla\cdot\left[\mathbb{G}_0(\mathbf{x})\nabla\cdot\mathbf{u}(\mathbf{x},t) + \int_{\mathbb{R}^+}\dot{\mathbb{G}}(\mathbf{x},s)\nabla\cdot\mathbf{u}^t(\mathbf{x},s)\,ds\right], \tag{2}$$

with Dirichlet boundary condition

$$\mathbf{u}(\mathbf{x},t)_{|\partial\Omega} = \mathbf{0} \tag{3}$$

and initial condition

$$\mathbf{u}^0(\mathbf{x},s) = \mathbf{u}_0(\mathbf{x},-s). \tag{4}$$

The following assumptions on the constitutive tensors hold:

$\mathcal{P}_1$ – $\mathbb{G}_0 \in L^\infty(\Omega)$, $\quad \dot{\mathbb{G}} \in L^2(\mathbb{R}^+;L^\infty(\Omega)) \cap L^1(\mathbb{R}^+;L^\infty(\Omega))$.

$\mathcal{P}_2$ – The constitutive equation (1) satisfies the thermodynamic restrictions for a linear viscoelastic solid. In particular, the instantaneous elastic tensor $\mathbb{G}_0$ and the equilibrium elastic tensor

$$\mathbb{G}_\infty(\mathbf{x},t) = \mathbb{G}_0(\mathbf{x}) + \int_{\mathbb{R}^+}\dot{\mathbb{G}}(\mathbf{x},s)\,ds$$

are bounded and positive definite, so that there exist four positive constants G_0, g_0, G_∞, g_∞, such that for every $\mathbf{v} \in H_0^1(\Omega)$

$$g_0\|\mathbf{v}\|^2_{H_0^1} \le \int_\Omega \mathbb{G}_0(\mathbf{x})\nabla\mathbf{v}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x})\,d\mathbf{x} \le G_0\|\mathbf{v}\|^2_{H_0^1}, \tag{5}$$

$$g_\infty\|\mathbf{v}\|^2_{H_0^1} \le \int_\Omega \mathbb{G}_\infty(\mathbf{x})\nabla\mathbf{v}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x})\,d\mathbf{x} \le G_\infty\|\mathbf{v}\|^2_{H_0^1}. \tag{6}$$

Let $\widehat{\dot{\mathbb{G}}}_s$ be the half range sine Fourier transform of $\dot{\mathbb{G}}$, namely

$$\widehat{\dot{\mathbb{G}}}_s(\mathbf{x},\omega) = \int_{\mathbb{R}^+}\sin\omega s\,\dot{\mathbb{G}}(\mathbf{x},s)\,ds,$$

then there exist two functions $G_s : \{\mathbb{R}\backslash 0\} \to \mathbb{R}^{++}$, $g_s : \{\mathbb{R}\backslash 0\} \to \mathbb{R}^{++}$ such that

$$g_s(\omega)\|\mathbf{v}\|^2_{H_0^1} \le -\int_\Omega \omega\widehat{\dot{\mathbb{G}}}_s(\mathbf{x},\omega)\nabla\mathbf{v}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x})\,d\mathbf{x} \le G_s(\omega)\|\mathbf{v}\|^2_{H_0^1}. \tag{7}$$

$\mathcal{P}_3$ – The tensor $\dot{\mathbb{G}}(\cdot,0)$ exists bounded and negative definite, i.e.

$$g_1\|\mathbf{v}\|^2_{H_0^1} \le -\int_\Omega \dot{\mathbb{G}}(\mathbf{x},0)\nabla\mathbf{v}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x})\,d\mathbf{x} \le G_1\|\mathbf{v}\|^2_{H_0^1}, \tag{8}$$

with $0 < g_1 \le G_1$.

$\mathcal{P}_4$ – There exist $\lambda > 0$ and $\mu > 0$ so that

$$\left|\int_{\mathbb{R}^+}\int_\Omega e^{\lambda t}\dot{\mathbb{G}}(\mathbf{x},t)\nabla\mathbf{v}(\mathbf{x})\cdot\nabla\mathbf{v}(\mathbf{x})\,d\mathbf{x}\,dt\right| \le \mu\|\mathbf{v}\|^2_{H_0^1}. \tag{9}$$

To study the exponential stability, the Laplace transform method is used. To this end some properties on the Laplace transform are recalled.

Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function, then for any complex number $p = \sigma + i\omega$

$$\begin{aligned}\tilde{g}(p) &= \int_{\mathbb{R}^+} e^{-ps}\, g(s)\, ds = \int_{\mathbb{R}^+} e^{-\sigma s} \cos\omega s\; g(s)\, ds - i\int_{\mathbb{R}^+} e^{-\sigma s} \sin\omega s\; g(s)\, ds \\ &= \tilde{g}_c(p) - i\tilde{g}_s(p)\end{aligned}$$

denotes the Laplace transform of g.

REMARK 1 **(Relations between Laplace and Fourier transforms)** For any complex $p = \sigma + i\omega \in \mathbb{C}^{++}$, ($\sigma \in \mathbb{R}^{++}$, $\omega \in \mathbb{R}$) the Laplace transform $\tilde{\mathbb{G}}$ is related to the sine Fourier transform $\hat{\mathbb{G}}_s$ by the following relation

$$\begin{aligned}\tilde{\mathbb{G}}(\mathbf{x}, p) &= \frac{2}{\pi}\int_{\mathbb{R}^+} \frac{(\sigma^2 - \omega^2 + \tau^2)\tau\hat{\mathbb{G}}_s(\mathbf{x}, \tau)}{(\sigma^2 - \omega^2 + \tau^2)^2 + 4\sigma^2\omega^2}\, d\tau \\ &\quad -\frac{2i}{\pi}\int_{\mathbb{R}^+} \frac{2\sigma\omega\tau\hat{\mathbb{G}}_s(\mathbf{x}, \tau)}{(\sigma^2 - \omega^2 + \tau^2)^2 + 4\sigma^2\omega^2}\, d\tau. \qquad (10)\end{aligned}$$

As a consequence of (10) and under the assumptions $\mathcal{P}_1$ and $\mathcal{P}_2$, for any $\sigma > 0$ and $\omega \neq 0$, there exist two positive constants $\alpha(\sigma)$ and $\beta(\sigma, \omega)$ such that

$$\int_\Omega \left[\mathbb{G}_0(\mathbf{x}) + \tilde{\mathbb{G}}_c(\mathbf{x}, \sigma)\right] \nabla\mathbf{v}(\mathbf{x}) \cdot \nabla\mathbf{v}(\mathbf{x})\, d\mathbf{x} \geq \alpha(\sigma)\|\mathbf{v}\|^2_{H^1_0}, \qquad (11)$$

$$-\int_\Omega \omega\tilde{\mathbb{G}}_s(\mathbf{x}, \sigma + i\omega)\nabla\mathbf{v}(\mathbf{x}) \cdot \nabla\mathbf{v}(\mathbf{x})\, d\mathbf{x} \geq \beta(\sigma, \omega)\|\mathbf{v}\|^2_{H^1_0}. \qquad (12)$$

LEMMA 1 *For any $\sigma_0 \in \mathbb{R}$ such that $\tilde{\mathbb{G}}(\mathbf{x}, p)$ is well defined for $p = \sigma_0 + i\omega$*

$$\lim_{\omega\to\infty}[\omega\; \tilde{\mathbb{G}}_s(\mathbf{x}, \sigma_0 + i\omega)] = \dot{\mathbb{G}}(\mathbf{x}, 0). \qquad (13)$$

Proof. To estimate the limit (13) we introduce the function

$$\mathbb{F}_{\sigma_0}(\mathbf{x}, s) = -\int_{-s}^{\infty} e^{-\sigma_0\tau}\dot{\mathbb{G}}(\mathbf{x}, \tau)\, d\tau.$$

The hypothesis on $\dot{\mathbb{G}}$ assures that $\dot{\mathbb{F}}_{\sigma_0}$ is Fourier transformable and

$$\lim_{\omega\to\infty} \omega(\widehat{\dot{\mathbb{F}}_{\sigma_0}})_s(\mathbf{x}, \omega) = \dot{\mathbb{F}}_{\sigma_0}(\mathbf{x}, 0) = \dot{\mathbb{G}}(\mathbf{x}, 0).$$

3. – The main result

We consider the problem (2)–(4) and require that the initial history satisfies the following conditions:

i) $\mathbf{u}_0(\cdot, -s) \in H^1_0(\Omega)$, $\dot{\mathbf{u}}_0(\cdot, -s) \in L^2(\Omega)$ for $s \in \mathbb{R}^+$;

ii) the function

$$\mathbf{f}(\mathbf{x},t) = \nabla\cdot\int_t^\infty \dot{\mathbb{G}}(\mathbf{x},s)\nabla\mathbf{u}_0(\mathbf{x},t-s)\,ds$$

belongs to $H^1(\mathbb{R}^+, L^2(\Omega))$ and

$$\int_{\mathbb{R}^+} e^{\lambda t}\|\mathbf{f}(t)\|^2_{L^2}\,dt < \infty. \tag{14}$$

Under these hypotheses, we rewrite the problem (1)–(4) as follows

$$\ddot{\mathbf{u}}(\mathbf{x},t) = \nabla\cdot\left[\mathbb{G}_0(\mathbf{x})\nabla\cdot\mathbf{u}(\mathbf{x},t) + \int_0^t \dot{\mathbb{G}}(\mathbf{x},s)\nabla\cdot\mathbf{u}^t(\mathbf{x},s)\,ds\right] + \mathbf{f}(\mathbf{x},t), \tag{15}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0}\,, \qquad \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{0}\,, \qquad \mathbf{x}\in\Omega, \tag{16}$$

with Dirichlet boundary condition (3). Moreover we require[1]

$$\mathbf{f}(\mathbf{x},0) = \mathbf{0}\,, \qquad \mathbf{x}\in\Omega.$$

Now let consider the Laplace transform of problem (15)–(16):

$$p^2\tilde{\mathbf{u}}(\mathbf{x},p) = \nabla\cdot\left\{\left[\mathbb{G}_0(\mathbf{x}) + \tilde{\dot{\mathbb{G}}}(\mathbf{x},p)\right]\nabla\tilde{\mathbf{u}}(\mathbf{x},p)\right\} + \tilde{\mathbf{f}}(\mathbf{x},p), \tag{17}$$

$$\tilde{\mathbf{u}}(\mathbf{x},p)_{|\partial\Omega} = \mathbf{0}. \tag{18}$$

DEFINITION 1 Let $p\in\mathbb{C}$ with $\mathcal{R}\{p\} > -\lambda$. A function $\tilde{\mathbf{u}}(\cdot,p)\in H^1_0(\Omega)$ is called a weak solution for the elliptic problem (17)–(18), with source $\tilde{\mathbf{f}}(\cdot,p)\in L^2(\Omega)$ if the following relation

$$\int_\Omega \left\{p^2\tilde{\mathbf{u}}(\mathbf{x},p)\cdot\phi^*(\mathbf{x}) + \left[\mathbb{G}_0(\mathbf{x}) + \tilde{\dot{\mathbb{G}}}(\mathbf{x},p)\right]\nabla\tilde{\mathbf{u}}(\mathbf{x},p)\cdot\nabla\phi^*(\mathbf{x})\right\}\,d\mathbf{x}$$
$$= \int_\Omega \tilde{\mathbf{f}}(\mathbf{x},p)\cdot\phi^*(\mathbf{x})\,d\mathbf{x}$$

holds for any $\phi\in H^1_0(\Omega)$, where the symbol $*$ denotes the complex conjugate.

[1] The conditions on the initial values are not restrictive. In fact, if $\mathbf{u}$ is a solution of (15) with initial data $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$, $\dot{\mathbf{u}}(\mathbf{x},0) = \dot{\mathbf{u}}_0(\mathbf{x},0)$ and source $\mathbf{f}$ with $\mathbf{f}(\mathbf{x},0) = \mathbf{f}_0(\mathbf{x})$, then the function $\mathbf{v} = \mathbf{u} + \mathbf{w}$, with $\mathbf{w}\in C^\infty(\mathbb{R}^+, H^1_0(\Omega))$,

$$\mathbf{w}(\mathbf{x},0) = -\mathbf{u}_0(\mathbf{x})\,, \quad \dot{\mathbf{w}}(\mathbf{x},0) = -\dot{\mathbf{u}}_0(\mathbf{x})\,, \quad \ddot{\mathbf{w}}(\mathbf{x},0) = \mathbf{f}_0(\mathbf{x}) + \nabla\cdot[\mathbb{G}_0(\mathbf{x})\nabla\cdot\mathbf{u}_0(\mathbf{x})]$$

and $\mathbf{w}(\cdot,t) = \mathbf{0}$ for every $t > t_0$, satisfies a problem formally equal to (15) –(16) with source

$$\mathbf{g}(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t) - \ddot{\mathbf{w}}(\mathbf{x},t) - \nabla\cdot\left[\mathbb{G}_0(\mathbf{x})\nabla\mathbf{w}(\mathbf{x},t) + \int_0^t \dot{\mathbb{G}}(\mathbf{x},s)\nabla\mathbf{w}^t(\mathbf{x},s)\,ds\right]$$

and

$$\mathbf{v}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t)\,, \quad \mathbf{g}(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t) + \nabla\cdot\int_{t-t_0}^t \dot{\mathbb{G}}(\mathbf{x},s)\nabla\mathbf{w}^t(\mathbf{x},s)\,ds\,, \quad \forall t > t_0.$$

Moreover, the relation

$$\int_{t_0}^{+\infty} e^{\alpha s}\int_{t-t_0}^t \dot{\mathbb{G}}(\mathbf{x},s)\,ds\,dt = \int_0^{t_0} e^{-\alpha t}\int_{t_0}^{+\infty} e^{\alpha s}\,\dot{\mathbb{G}}(\mathbf{x},s)\,ds\,dt$$

assures that $\mathbf{g}$ has the same exponential behavior of $\mathbf{f}$.

LEMMA 2 *Let $p \in \mathbb{C}$ with $\mathcal{R}\{p\} > -\lambda$ and $\tilde{\mathbf{u}}(\cdot,p) \in H_0^1(\Omega)$ a weak solution of (17)–(18) with source $\tilde{\mathbf{f}}(\cdot,p) \in L^2(\Omega)$. If the relaxation tensor and the past history satisfy respectively the constitutive hypotheses $\mathcal{P}_1$– $\mathcal{P}_4$ and the requirements i)– ii), then there exists $\delta > 0$ such that the operator*

$$L(p) = -p^2 + \nabla\cdot\left\{\left[\mathbb{G}_0(\mathbf{x}) + \tilde{\dot{\mathbb{G}}}(\mathbf{x},p)\right]\nabla\right\}$$

is well defined and uniformly elliptic with respect to p for $\mathcal{R}\{p\} > -\delta$ and

$$\|\tilde{\mathbf{u}}(p)\|^2_{H_0^1} \le \kappa\left[\|\tilde{\mathbf{f}}(p)\|^2_{L^2} + \|p\tilde{\mathbf{f}}(p)\|^2_{L^2}\right] \tag{19}$$

where κ is a positive constant.

Proof. We introduce the sesquilinear form

$$a(\mathbf{u},\mathbf{v};p) = p^2 < \mathbf{u},\mathbf{v} > + < \left[\mathbb{G}_0 + \tilde{\dot{\mathbb{G}}}(p)\right]\nabla\mathbf{u},\nabla\mathbf{v} >, \tag{20}$$

where

$$< \mathbf{u},\mathbf{v} >= \int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathbf{v}^*(\mathbf{x})\,d\mathbf{x}$$

and consider the real and imaginary part of (20)

$$\mathcal{R}\{a(\mathbf{u},\mathbf{u};p)\} = (\sigma^2-\omega^2)\|\mathbf{u}\|^2_{L^2} + < \left[\mathbb{G}_0 + \tilde{\dot{\mathbb{G}}}_c(p)\right]\nabla\mathbf{u},\nabla\mathbf{u} > \tag{21}$$

$$\mathcal{I}\{a(\mathbf{u},\mathbf{u};p)\} = 2\sigma\omega\|\mathbf{u}\|^2_{L^2} - < \tilde{\dot{\mathbb{G}}}_s(p)\nabla\mathbf{u},\nabla\mathbf{u} > . \tag{22}$$

We split the complex half-plane $\mathcal{R}\{p\} > -\lambda$ in regions and prove that it is possible to find $\delta < \lambda$ such that inequality (19) holds for any p, with $\mathcal{R}\{p\} > -\delta$.

First we consider a neighbor of $p = 0$. Since

$$a(\mathbf{u},\mathbf{u};0) =< \mathbb{G}_\infty\nabla\mathbf{u},\nabla\mathbf{u} >\ge g_\infty\|\mathbf{u}\|^2_{H_0^1},$$

the continuity of a with respect p assures that there exists $\delta_0' > 0$ such that

$$a(\mathbf{u},\mathbf{u};p) \ge \frac{1}{2}\,g_\infty\|\mathbf{u}\|^2_{H_0^1}, \quad \text{for } |p| < 2\delta_0'.$$

In this way, if $\tilde{\mathbf{u}}(p)$ is a solution of (17)–(18), then

$$\|\tilde{\mathbf{u}}(p)\|^2_{H_0^1} \le \left[\frac{2c}{g_\infty}\right]^2\|\tilde{\mathbf{f}}(p)\|^2_{L^2}, \tag{23}$$

where c denotes the Poincaré constant.

In a similar manner, by using (21), (11) and the continuity of a, we obtain

$$\|\tilde{\mathbf{u}}(p)\|^2_{H_0^1} \le \left[\frac{2c}{\alpha(\sigma)}\right]^2\|\tilde{\mathbf{f}}(p)\|^2_{L^2} \tag{24}$$

for $p = \sigma + i\omega$, with $\sigma > \delta_0$ and $|\omega| < 2m$.

Then (23) and (24) assure that for ω near to zero (precisely $|\omega| < 2m$) and $\sigma > \max\{-\delta_0', -\lambda\} = -\delta_0$

$$\|\tilde{\mathbf{u}}(\sigma + i\omega)\|_{H_0^1}^2 \leq \kappa(\delta_0, m)\|\tilde{\mathbf{f}}(\sigma + i\omega)\|_{L^2}^2 \tag{25}$$

with

$$\kappa(\delta_0, m) = \max\left\{\sup_{\sigma>0}\left[\frac{2c}{\alpha(\sigma)}\right]^2, \left[\frac{2c}{g_\infty}\right]^2\right\}.$$

For $\sigma = 0$ and $m < |\omega| < 2M$, the thermodynamic restriction (7), (22) and continuity of the Laplace transform yield

$$\|\tilde{\mathbf{u}}(0 + i\omega)\|_{H_0^1}^2 \leq \sup_{m<|\omega|<2M}\left[\frac{c}{g_s(\omega)}\right]^2 \|\omega\tilde{\mathbf{f}}(0 + i\omega)\|_{L^2}^2. \tag{26}$$

This assures that there exists δ_1 with $0 < \delta_1 < \delta_0$ such that

$$\|\tilde{\mathbf{u}}(\sigma + i\omega)\|_{H_0^1}^2 \leq \kappa(m, M)\|\omega\tilde{\mathbf{f}}(\sigma + i\omega)\|_{L^2}^2 \tag{27}$$

for $0 < \sigma < \delta_1$, where

$$\kappa(m, M) = \sup_{m<|\omega|<2M}\left[\frac{2c}{g_s(\omega)}\right]^2.$$

On the other hand, for $\sigma < 0$ (22) yields

$$\omega\mathcal{I}\{a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}; \sigma + i\omega)\} \geq \left[2\sigma M^2 c^2 + \frac{1}{2}g_s(\omega)\right]\|\tilde{\mathbf{u}}(\sigma + i\omega)\|_{H_0^1}^2.$$

Let $\delta_2 = \max\{-\frac{1}{8M^2c^2}\inf_{m<|\omega|<2M} g_s(\omega), -\lambda\}$. Then for $\delta_2 < \sigma < 0$ we obtain

$$\|\tilde{\mathbf{u}}(p)\|_{H_0^1}^2 \leq 4\kappa(m, M)\|p\tilde{\mathbf{f}}(p)\|_{L^2}^2. \tag{28}$$

In the region $|\sigma| < \delta_2$ and $|\omega| > M$, we consider the following linear combination of (21) and (22)

$$\begin{aligned} &< \left[\sigma\left(\mathbb{G}_0 + \tilde{\dot{\mathbb{G}}}_c(p)\right) + \frac{\sigma^2 - \omega^2}{2\omega}\,\tilde{\dot{\mathbb{G}}}_s(p)\right]\nabla\tilde{\mathbf{u}}(p), \nabla\tilde{\mathbf{u}}(p) > \\ &= \sigma\mathcal{R}\{< \tilde{\mathbf{f}}(p), \tilde{\mathbf{u}}(p) >\} - \frac{\sigma^2 - \omega^2}{2\omega}\,\mathcal{I}\{< \tilde{\mathbf{f}}(p), \tilde{\mathbf{u}}(p) >\}. \end{aligned} \tag{29}$$

As a consequence of (13) we have

$$\begin{aligned} \lim_{\omega\to\infty} &\left[\sigma\left(\mathbb{G}_0 + \tilde{\dot{\mathbb{G}}}_c(\sigma + i\omega)\right) + \frac{\sigma^2 - \omega^2}{2\omega}\,\tilde{\dot{\mathbb{G}}}_s(\sigma + i\omega)\right] \\ &= \sigma\mathbb{G}_0 - \frac{1}{2}\lim_{\omega\to\infty}\omega\tilde{\dot{\mathbb{G}}}_s(\sigma, \omega) \\ &= \frac{1}{2}(2\sigma\mathbb{G}_0 - \dot{\mathbb{G}}(0)). \end{aligned} \tag{30}$$

The hypotheses on $\dot{\mathbb{G}}(0)$ assure that for $\sigma > \max\{-\frac{g_1}{2G_0}, -\lambda\} = -2\delta_3$ the tensor $2\sigma\mathbb{G}_0 - \dot{\mathbb{G}}(0)$ is positive definite and the continuity of $\widetilde{\mathbb{G}}$ with respect to the parameter p guarantees that there exists M_{δ_3} such that, for $|\sigma| < \delta_3$ and $|\omega| > M_{\delta_3}$,

$$(31) \qquad < \left[\sigma\left(\mathbb{G}_0 + \widetilde{\mathbb{G}}_c(p)\right) + \frac{\sigma^2 - \omega^2}{2\omega}\widetilde{\mathbb{G}}_s(p)\right]\nabla\mathbf{u}, \nabla\mathbf{u} >> \kappa(M_{\delta_3}, \delta_3)\|\mathbf{u}\|^2_{H^1_0}.$$

From (29) and (31) we get

$$(32) \qquad \|\tilde{\mathbf{u}}(p)\|^2_{H^1_0} \leq \left[\frac{2c}{\kappa(M_{\delta_3}, \delta_3)}\right]^2 \|p\tilde{\mathbf{f}}(p)\|^2_{L^2}.$$

Relations (23), (24),(27) and (32) assure that inequality (19) holds in a neighbour of the real and imaginary axes.

Finally, in the region $|\omega| > m$ and $\sigma > \delta = \min_1 |\delta_i|$ $(i = 1,2,3)$, (19) follows from (22) and (12) when $|p|$ is bounded, while, when $|p|$ go to infinity (22) yields

$$(33) \qquad \|\omega\tilde{\mathbf{u}}(p)\|^2_{L^2} \leq \frac{1}{4\delta^2}\|\tilde{\mathbf{f}}(p)\|^2_{L^2},$$

so that (21) and (33) give

$$(34) \qquad < \left[\mathbb{G}_0 + \widetilde{\mathbb{G}}_c(p)\right]\nabla\tilde{\mathbf{u}}(p), \nabla\tilde{\mathbf{u}}(p) >\leq \frac{1}{4\delta_3^2}\|\tilde{\mathbf{f}}(p)\|^2_{L^2} + \|\tilde{\mathbf{f}}(p)\|\|\nabla\tilde{\mathbf{u}}(p)\|.$$

Inequality (19) follows from (34) recalling (5) and that $\lim_{|p|\to\infty}\widetilde{\mathbb{G}}_c(p) = 0$.

4. – Exponential decay result

Lemma 2, together with classic results on elliptic systems depending on a parameter (see [6] Lemma 23.2), allow us to state the following

THEOREM 1 *Under the hypothesis of Lemma 2 for any $p \in \mathbb{C}$ with $\mathcal{R}\{p\} > -\delta$ the problem* (17)–(18) *has one and only one solution $\tilde{\mathbf{u}}(\cdot, p) \in H^1_0(\Omega)$.*

Moreover the function

$$\|\tilde{\mathbf{u}}(p)\|^2_{H^1_0} + \|p\tilde{\mathbf{u}}(p)\|^2_{L^2}$$

is analytic for $\mathcal{R}\{p\} > -\delta$ and goes to zero when p goes to infinity.

As a consequence of Theorem 1 and recalling the relations between the Laplace transform and the Fourier transform of a causal function (see [4]), we can conclude that $\tilde{\mathbf{u}}(\mathbf{x}, p)$ is the Fourier transform of the causal function $e^{\sigma t}\mathbf{u}(\mathbf{x}, t)$ for any $\sigma > -\delta$, with $\mathbf{u}$ solution of the problem (15)–(16). Moreover, from the estimate (19) on the Laplace transform, we have

$$\|\tilde{\mathbf{u}}(p)\|^2_{H^1_0} + \|p\tilde{\mathbf{u}}(p)\|^2_{L^2} \leq M\left[\|\tilde{\mathbf{f}}(p)\|^2_{L^2} + \|p\tilde{\mathbf{f}}(p)\|^2_{L^2}\right]$$

and the Parseval relation assures that

$$\int_{\mathbb{R}^+} e^{\sigma t}\left[\|\mathbf{u}(t)\|^2_{H^1_0} + \|\dot{\mathbf{u}}(t)\|^2_{L^2}\right] dt \leq M\int_{\mathbb{R}^+} e^{\sigma t}\left[\|\mathbf{f}(t)\|^2_{L^2} + \|\dot{\mathbf{f}}(t)\|^2_{L^2}\right] dt < \infty$$

for $\sigma > -\delta$.

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Recovering a memory kernel in an integrodifferential Stefan problem

L. Lorenzi *

1. – Introduction

In this note we deal with a one-dimensional one-phase Stefan problem for materials with memory related to the curvilinear strip $\Omega^s = \{(t,x) \in [0,T] \times \mathbb{R} : x < s(t)\}$. Our problem consists of determining a pair of functions $s : [0,T] \to \mathbb{R}$ and $u : \Omega^s \to \mathbb{R}$ such that

$$\begin{aligned} & D_t\Big(u(t,x) - \int_0^t k(t-\tau)u(\tau,x)d\tau\Big) \\ (1) \qquad & = D_{xx}u(t,x) + f(t,x), \quad t \in [0,T],\ x \leq s(t), \end{aligned}$$

$$(2) \qquad u(0,x) = u_0(x), \quad x \leq s(0),$$

$$(3) \qquad u(t,s(t)) = u_1(t), \quad t \in [0,T],$$

$$(4) \qquad s'(t) = -D_x u(t,s(t)) + u_2(t), \quad t \in [0,T].$$

In order that the time convolution term appearing in the differential equation may be meaningful we assume that s is a non-increasing function in $[0,T]$. So we deal with free boundaries moving to the left.

As is well known, (4) accounts for the so-called Stefan condition needed to determine the free boundary $x = s(t)$. When also the memory kernel k is itself unknown, we have to prescribe an additional condition of the form

$$(5) \qquad g(t) = \Psi[u(t,\cdot)] := \int_{-\infty}^{s(t)} \psi(y)u(t,y)dy,$$

generalizing the energy condition considered in [3].

Equation (1) is a particular case of the more general equation

$$\begin{aligned} & D_t\Big(u(t,x) - \int_0^t k(t-\tau)u(\tau,x)d\tau\Big) \\ (6) \qquad & = D_{xx}u(t,x) + \int_0^t \varphi(t-\tau)D_{xx}u(\tau,x)d\tau + f(t,x), \end{aligned}$$

that occurs when dealing with the heat conduction in a rigid one-dimensional body with memory. As is well known, the terms

$$e(t,x) = u(t,x) - \int_0^t k(t-\tau)u(\tau,x)d\tau,$$

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and

$$q(t,x) = D_{xx}u(t,x) + \int_0^t \varphi(t-\tau)D_{xx}u(\tau,x)d\tau,$$

account, respectively, for the internal energy and the heat flux, while f stands for the heat supply.
Since the identification problem (1)-(5) seems to be new and rather complicated in the general formulation (6), (2)-(5), we restrict ourselves to the simpler problem (1)-(5).

We recall that Stefan problems and, more generally, phase transition problems have been the topics of many papers in the last five decades. We quote [4], [5] and all the papers cited in the book [1, Chapter 12], which covers the years between the 70's and the beginning of the 80's, as well as the monographs [2], [10] and the references therein cited.
More recently, in [6], [7], [8] the authors deal with phase transitions problems arising in heat conduction for materials with memory. In particular in [8] they are concerned with weak solutions (u, χ) to the equation, similar to (6),

$$D_t\big(u(t,x) + (\psi \star \chi)(t,x)\big) = \Delta(k \star u)(t,x) + f(t,x),$$

related to bounded domains $\Omega \subset \mathbb{R}^N$ in the case of two memory kernels k and ψ which may depend also on the space variables. Here "$\star$" denotes time convolution.

As far as identification problems for one-phase Stefan problem are concerned, we quote the papers [14], [11], [12], [9]. In [14] the author considers a one-dimensional, one-phase Stefan problem related to the bounded (for any fixed t) space domain $(0, s(t))$. Assuming that the problem depends also on six constants $k_1, \ldots, k_6$, the author shows how to determine, by an additional measurement on the face $x = 0$, both the solution u and a pair of constants (k_i, k_j) $(i, j = 1, \ldots, 6,\ i \neq j)$ whenever the remaining four constants are supposed to be known.
In the more recent papers [11], [12] the authors deal with the problem of recovering an unknown time-dependent diffusion coefficient in a one-dimensional, one-phase Stefan problem both theoretically, using a least-square method (see [12]) and numerically (see [11]).
Finally in [9] the authors are concerned with the problem of identifying a moving solid/liquid interface of a one-dimensional melting solid from measurements of both the temperature and the flux on the solid part of the interface.

In this note we limit ourselves to stating that problem (1)-(5) admits a unique (local in time) classical solution (u, s, k) with Hölder regularity, and we point out the main techniques used to get such a result. We refer the reader to [13] for a complete proof of the stated result.

Let us introduce the functional spaces we deal with in this note.

DEFINITION 1 For any $T > 0$ and any $\alpha \in (0,1)$, we denote by $C^{\alpha/2,\alpha}(\Omega_T)$ $(\Omega_T := [0,T] \times (-\infty, 0])$ the function space

$$C^{\alpha/2,\alpha}(\Omega_T) \quad = \quad \{f : \Omega_T \to \mathbb{R} : f(\cdot, x) \in C^{\alpha/2}([0,T]) \;\; \forall x \leq 0,$$

$$\sup_{x\leq 0}\|f(\cdot,x)\|_{C^{\alpha/2}([0,T])}<+\infty,\ f(t,\cdot)\in C^{\alpha}((-\infty,0])\ \ \forall t\in[0,T],$$
$$\sup_{t\in[0,T]}[f(t,\cdot)]_{C^{\alpha}((-\infty,0])}<+\infty\}.$$

We endow $C^{\alpha/2,\alpha}(\Omega_T)$ with the norm

$$\|f\|_{\alpha/2,\alpha,T}=\sup_{t\in[0,T]}\|f(t,\cdot)\|_{C^{\alpha}((-\infty,0])}+\sup_{x\leq 0}[f(\cdot,x)]_{C^{\alpha/2}([0,T])}.$$

Then we define by $C^{1,2}(\Omega_T)$ the space of all the functions $f:\Omega_T\to\mathbb{R}$ which are once continuously differentiable with respect to time and twice continuously differentiable with respect to the space variable in Ω_T. We endow $C^{1,2}(\Omega_T)$ with the norm

$$\|f\|_{1,2,T}=\sum_{2i+j\leq 2}\|D_t^iD_x^jf\|_{C(\Omega_T)}.$$

Finally, for any $\alpha\in(0,1)$, we denote by $C^{1+\alpha/2,2+\alpha}(\Omega_T)$ the space of all the functions $u:\Omega_T\to\mathbb{R}$ such that u, D_tu, D_xu and $D_{xx}u$ belong to $C^{\alpha/2,\alpha}(\Omega_T)$. We endow $C^{1+\alpha/2,2+\alpha}(\Omega_T)$ with the norm

$$\|u\|_{1+\alpha/2,2+\alpha,T}=\sum_{2i+j\leq 2}\|D_t^iD_x^ju\|_{\alpha/2,\alpha,T}.$$

We can now state the main result of this note.

THEOREM 1 *Suppose that* $f\in C^{1+\alpha/2,2+\alpha}(\Omega_T)$, $D_{tx}f\in C^{\alpha/2,\alpha}(\Omega_T)$, $u_0\in C^{4+\alpha}((-\infty,0])$, $\psi\in C^3((-\infty,0])\cap W^{3,1}((-\infty,0))$, $u_1,g\in C^{2+\alpha/2}([0,T])$, $u_2\in C^{1+\alpha/2}([0,T])$. *Further assume that*

H1) $u_1(0)\neq 0$,

H2) the equation $u_0(y)=u_1(0)$ *admits a unique solution* $s_0<0$,

H3) $0\neq\chi:=\int_0^{s_0}\psi(y)u_0(y)dy$,

H4) $m_0:=\sup_{t\in[0,T]}u_2(t)-u_0'(s_0)<0$,

and the compatibility conditions

$$g(0)=\int_{-\infty}^{s_0}\psi(y)u_0(y)dy,$$

$$\begin{aligned}g'(0)&=u_1(0)\psi(s_0)(u_2(0)-u_0'(s_0))+k_0g(0)\\&\quad+\int_{-\infty}^{s_0}\psi(y)[u_0''(y)+f(0,y)]dy,\end{aligned}$$

$$\begin{aligned}g''(0)&=(u_2(0)-u_0'(s_0))\psi(s_0)[u_0''(s_0)+u_1'(0)+u_1(0)u_0''(s_0)+f(0,s_0)]\\&\quad+u_1(0)\psi(s_0)[u_2'(0)-u_0'''(s_0)-D_xf(0,s_0)]+h_0g(0)\\&\quad+k_0[g'(0)-u_0'(s_0)\psi(s_0)u_1(0)]+u_1(0)\psi'(s_0)(u_2(0)-u_0'(s_0))^2\\&\quad+\int_{-\infty}^{s_0}\psi(y)[u_0''''(y)+k_0u_0''(y)+D_{xx}f(0,y)+D_tf(0,y)]dy\end{aligned}$$

hold, where

$$k_0 = (u_0(s_0))^{-1}[u_1'(0) - u_0''(s_0) - u_0'(s_0)(u_2(0) - u_0'(s_0)) - f(0,s_0)], \tag{7}$$

$$\begin{aligned} h_0 \;=\; \chi^{-1}\Big\{ & u_1''(0) - (u_2(0) - u_0'(s_0))[2u_0'''(s_0) + 2D_x f(0,s_0) + u_0''(s_0)u_2(0) \\ & \qquad - 2u_0'(s_0)u_0''(s_0) + 2k_0u_0'(s_0)] \\ & - D_{xx}f(0,s_0) - u_0''''(s_0) - u_0'(s_0)[u_2'(0) - u_0'''(s_0) - D_x f(0,s_0)] \\ & - D_t f(0,s_0) - k_0[2u_0''(s_0) + f(0,s_0) + k_0u_0(s_0) - (u_0'(s_0))^2]\Big\}. \end{aligned}$$

Then there exist $\overline{M}_0$, $\overline{M}_1 \in \mathbb{R}_+$ such that for any $M_j \geq \overline{M}_j$ $(j = 0,1)$ we can find out $T_0 > 0$ and a unique triplet (u,s,k) solution to problem (1)-(5) such that $s \in C^{2+\alpha/2}([0,T_0])$ is a non-increasing function with $s(0) = s_0$, $k \in C^{1+\alpha/2}([0,T_0])$ with $k(0) = k_0$ and $\|k'\|_{C^{\alpha/2}([0,T_0])} \leq M_1$, and such that the function $\tilde{u}(t,x) \to u(t,x-s(t))$ belongs to $C^{1,2}(\Omega_{T_0})$ with $D_t\tilde{u} \in C^{1+\alpha/2,2+\alpha}(\Omega_{T_0})$ and $\|D_t\tilde{u}\|_{1+\alpha/2,2+\alpha,T_0} \leq M_0$.

2. – Fixing the boundary

Before fixing the domain, we observe that if k is a differentiable function, then equation (1) can be rewritten in the equivalent form

$$\begin{aligned} D_t u(t,x) \;&=\; D_{xx}u(t,x) + k(0)u(t,x) \\ &\quad + \int_0^t k'(t-\tau)u(\tau,x)d\tau + f(t,x), \quad t \in [0,T],\ x \leq s(t). \end{aligned} \tag{8}$$

Assuming that u, its first-order time derivative and first- and second-order space derivatives are continuous at $(0,s(0))$ we can show that $k(0)$ is uniquely determined from the initial data, thanks to *H1-H2*, and $k(0) = k_0$ (see (7)). Hence we can replace equation (1) with the equation

$$\begin{aligned} D_t u(t,x) \;&=\; D_{xx}u(t,x) + k_0u(t,x) \\ &\quad + \int_0^t h(t-\tau)u(\tau,x)d\tau + f(t,x), \quad t \in [0,T],\ x \leq s(t), \end{aligned} \tag{9}$$

and we can deal with problem (9), (2)-(5) showing that it admits a unique (locally in time) solution (u,h,s). Coming back to our original problem (1)-(5), it is then easy to show that it has a unique solution (u,k,s) where

$$k(t) = k_0 + \int_0^t h(\tau)d\tau.$$

So, let us consider problem (9), (2)-(5). As usually when dealing with a free boundary problem, we fix the boundary setting

$$\tilde{u}(t,x) = u(t,x+s(t)).$$

If (u, s) is a solution to problem (9), (2)-(4), with $s(0) = s_0$, then the pair $(\widetilde{u}, s)$ is easily seen to solve the problem:

$$
\begin{aligned}
D_t\widetilde{u}(t,x) &= D_{xx}\widetilde{u}(t,x) + k_0\widetilde{u}(t,x) + \int_0^t h(t-\tau)\widetilde{u}(\tau, x+s(t)-s(\tau))d\tau \\
&\quad + s'(t)D_x\widetilde{u}(t,x) + f(t, x+s(t)), \quad t \in [0,T],\ x \le 0, \qquad (10)
\end{aligned}
$$

$$\widetilde{u}(0,x) = u_0(x+s_0), \quad x \le 0, \qquad (11)$$

$$\widetilde{u}(t,0) = u_1(t), \quad t \in [0,T], \qquad (12)$$

$$s'(t) = -D_x\widetilde{u}(t,0) + u_2(t), \quad t \in [0,T]. \qquad (13)$$

Applying the operator Ψ, defined in (5), to both sides in (9) and observing that

$$
\begin{aligned}
\Psi[D_{xx}u(t,\cdot)] &= \psi(s(t))(u_2(t) - s'(t)) - \int_{-\infty}^{s(t)} \psi'(y)D_x\widetilde{u}(t, y-s(t))dy, \\
\Psi[D_t u(t,\cdot)] &= g'(t) - u_1(t)\psi(s(t))s'(t),
\end{aligned}
$$

for any $\psi \in C^1((-\infty, 0]) \cap W^{1,1}((-\infty, 0))$ and any $t \in [0,T]$, we easily get

$$
\begin{aligned}
g'(t) &= \psi(s(t))(u_2(t) - s'(t)) + u_1(t)\psi(s(t))s'(t) + k_0 g(t) \\
&\quad - \int_{-\infty}^0 \psi'(y+s(t))D_x\widetilde{u}(t,y)dy + \int_{-\infty}^0 \psi(y+s(t))f(t, y+s(t))dy \\
&\quad + \int_0^t h(t-\tau)d\tau \int_{-\infty}^0 \psi(y+s(t))\widetilde{u}(\tau, y+s(t)-s(\tau))dy, \qquad t \in [0,T]. \qquad (14)
\end{aligned}
$$

Since our goal is to transform problem (10)-(13) into a fixed-point system, we set $v = D_t\widetilde{u}$ and differentiate problem (10)-(13) with respect to time, getting the following problem for the pair (s, v):

$$D_t v(t,x) = D_{xx}v(t,x) + k_0 v(t,x) + \mathcal{F}(h, s, v)(t,x), \quad t \in [0,T],\ x < 0, \qquad (15)$$

$$
\begin{aligned}
v(0,x) &= u_0''(x+s_0) + u_0'(x+s_0)(u_2(0) - u_0'(s_0)) \\
&\quad + k_0 u_0(x+s_0) + f(0, x+s_0), \quad x \le 0, \qquad (16)
\end{aligned}
$$

$$v(t,0) = u_1'(t), \quad t \in [0,T], \qquad (17)$$

$$s'(t) = u_2(t) - u_0'(s_0) - \int_0^t D_x v(\tau, 0)d\tau, \qquad (18)$$

where

$$
\begin{aligned}
\mathcal{F}(h,s,v)(t,x) &= s'(t)D_x f(t, x+s(t)) + s''(t)u_0'(x+s_0) + s'(t)D_x v(t,x) \\
&\quad + h(t)u_0(x+s(t)) + D_t f(t, x+s(t)) \\
&\quad + s''(t)\int_0^t D_x v(\tau, x)d\tau + \int_0^t h(t-\tau)v(\tau, x+s(t)-s(\tau))d\tau \\
&\quad + \int_0^t h(t-\tau)(s'(t) - s'(\tau))d\tau \int_0^\tau D_x v(\sigma, x+s(t)-s(\tau))d\sigma \\
&\quad + \int_0^t h(t-\tau)(s'(t) - s'(\tau))u_0'(x+s(t)-s(\tau)+s_0)d\tau. \qquad (19)
\end{aligned}
$$

Differentiating equation (14) with respect to t we get

$$\begin{aligned}
g''(t) &= \psi(s(t))(u_2'(t) - s''(t)) + u_1'(t)\psi(s(t))s'(t) + u_1(t)\psi'(s(t))(s'(t))^2 \\
&\quad + u_1(t)\psi(s(t))s''(t) + f(t,s(t))\psi(s(t))s'(t) + k_0 g'(t) \\
&\quad + \int_{-\infty}^0 \psi(y+s(t))D_t f(t,y+s(t))dy - \int_{-\infty}^0 \psi'(y+s(t))D_x v(t,y)dy \\
&\quad + s'(t)\int_{-\infty}^0 \psi'(y+s(t))dy \int_0^t D_{xx}v(\tau,y)d\tau \\
&\quad + s'(t)\int_{-\infty}^0 \psi'(y+s(t))u_0''(y+s_0)dy + h(t)\int_{-\infty}^0 \psi(y+s(t))u_0(y+s(t))dy \\
&\quad + \psi(s(t))s'(t)\int_0^t h(t-\tau)d\tau \int_0^\tau v(\sigma, s(t)-s(\tau))d\sigma \\
&\quad + \psi(s(t))s'(t)\int_0^t h(t-\tau)u_0(s(t)-s(\tau)+s_0)d\tau \\
&\quad + \int_0^t h(t-\tau)d\tau \int_{-\infty}^0 \psi(y+s(t))v(\tau, y+s(t)-s(\tau))dy \\
&\quad - \int_0^t h(t-\tau)s'(\tau)d\tau \int_{-\infty}^0 \psi(y+s(t))dy \int_0^\tau D_x v(\sigma, y+s(t)-s(\tau))d\sigma \\
&\quad - \int_0^t h(t-\tau)s'(\tau)d\tau \int_{-\infty}^0 \psi(y+s(t))u_0'(y+s(t)-s(\tau)+s_0)dy, \qquad (20)
\end{aligned}$$

for any $t \in [0,T]$.

Under the assumptions of Theorem 1, the problems (10)-(13), (14) (say P1) and (15)-(18), (20) (say P2) are equivalent in the following sense: let (u,s,h) be a solution to problem (P1) such that u, $D_t u \in C^{1,2}(\Omega_T)$, $h \in C([0,T])$ and $s \in C^2([0,T])$, s being a non-increasing function with $s(0) = s_0$, then the triplet $(h, s, D_t u)$ solves problem (P2). Conversely, if (h,s,v) solves problem (P2) with $v \in C^{1,2}(\Omega_T)$, $h \in C([0,T])$ and $s \in C^2([0,T])$, s being a non-increasing function in $[0,T]$ with $s(0) = s_0$, then the triplet (h,s,u), where

$$u(t,x) = \int_0^t v(\tau,x)d\tau + u_0(x+s_0), \qquad \forall (t,x) \in \Omega_T,$$

is a solution to problem (P1) with $u \in C^{1,2}(\Omega_T)$ and $D_t u \in C^{1,2}(\Omega_T)$. For a detailed proof see [13].

In order to transform problem (15)-(18), (20) into a fixed-point problem, we recall that (see [2, Chapter 3]) for any $\lambda \in \mathbb{R}$, any $v_0 \in C^{2+\alpha}((-\infty,0])$, $v_1 \in C^{1+\alpha/2}([0,T])$, $f \in C^{\alpha/2,\alpha}(\Omega_T)$ satisfying the compatibility conditions

$$v_0(0) = v_1(0), \qquad v_1'(0) = v_0''(0) + \lambda v_0(0) + f(0,0),$$

the problem

$$D_t v(t,x) = D_{xx}v(t,x) + \lambda v(t,x) + f(t,x), \quad t \in [0,T], \ x \le 0, \qquad (21)$$

$$v(0,x) = v_0(x), \quad x \le 0, \qquad (22)$$

$$(23) \qquad v(t,0) = v_1(t), \quad t \in [0,T],$$

admits a unique solution $v \in C^{1+\alpha/2,2+\alpha}(\Omega_T)$ and it satisfies

$$\|v\|_{1+\alpha/2,2+\alpha,T} \le C(\|v_0\|_{C^{2+\alpha}((-\infty,0])} + \|v_1\|_{C^{1+\alpha/2}([0,T])} + \|f\|_{\alpha/2,\alpha,T}),$$

for some positive constant C, continuously depending on λ and T, but independent of the data. It follows that any solution $v \in C^{1+\alpha/2,2+\alpha}(\Omega_T)$ to problem (15)-(17) is a solution to the fixed-point equation $v = \mathcal{M}(h,s,v)$, where $\mathcal{M}(h,s,v)$ is the solution to problem (21)-(23) with f, v_0, v_1 being replaced, respectively, by $\mathcal{F}(h,s,v)$ (see (19)), the right-hand side of (16), and u_1'.
From (20) we get the other two fixed-point equations we need. To be more precise (see also [13]), h turns out to be a solution to the equation $h = \mathcal{N}(h,s,v)$, where

$$\begin{aligned}
&\mathcal{N}(h,s,v)(t) \\
= \;& \chi^{-1}\Big[g''(t) - \psi(s(t))(u_2'(t) - s''(t)) - u_1'(t)\psi(s(t))s'(t) - u_1(t)\psi'(s(t))(s'(t))^2 \\
&- u_1(t)\psi(s(t))s''(t) - f(t,s(t))\psi(s(t))s'(t) - k_0 g'(t) \\
&- \int_{-\infty}^0 \psi(y+s(t))D_t f(t,y+s(t))dy + \int_{-\infty}^0 \psi'(y+s(t))D_x v(t,y)dy \\
&- s'(t)\int_{-\infty}^0 \psi'(y+s(t))dy \int_0^t D_{xx}v(\tau,y)d\tau \\
&- s'(t)\int_{-\infty}^0 \psi'(y+s(t))u_0''(y+s_0)dy \\
&- h(t)\int_{-\infty}^0 [\psi(y+s(t))u_0(y+s(t)) - \psi(y+s_0)u_0(y+s_0)]dy \\
&- \psi(s(t))s'(t)\int_0^t h(t-\tau)d\tau \int_0^\tau v(\sigma, s(t)-s(\tau))d\sigma \\
&- \psi(s(t))s'(t)\int_0^t h(t-\tau)u_0(s(t)-s(\tau)+s_0)d\tau \\
&- \int_0^t h(t-\tau)d\tau \int_{-\infty}^0 \psi(y+s(t))v(\tau, y+s(t)-s(\tau))dy \\
&+ \int_0^t h(t-\tau)s'(\tau)d\tau \int_{-\infty}^0 \psi(y+s(t))dy \int_0^\tau D_x v(\sigma, y+s(t)-s(\tau))d\sigma \\
&+ \int_0^t h(t-\tau)s'(\tau)d\tau \int_{-\infty}^0 \psi(y+s(t))u_0'(y+s(t)-s(\tau)+s_0)dy\Big],
\end{aligned}$$

for any $t \in [0,T]$, and s is a solution to the functional equation

$$s(t) \;=\; \mathcal{S}(v)(t) := \int_0^t u_2(\tau)d\tau - u_0'(s_0)t - \int_0^t d\tau \int_0^\tau D_x v(\sigma,0)d\sigma + s_0, \quad \forall t \in [0,T].$$

Summing up any triplet (h,s,v), solution to problem (15)-(20) with $s(0) = s_0$, is a fixed-point of the system

$$(24) \qquad \begin{cases} (a) \quad h = \mathcal{N}(h,s,v), \\ (b) \quad s = \mathcal{S}(v), \\ (c) \quad v = \mathcal{M}(h,s,v). \end{cases}$$

Replacing the expression of s given by (24b) into (24a) and (24c), we reduce the number of unknowns, obtaining the equivalent problem (for the pair (h, v))

$$(25) \qquad \begin{cases} (a) \quad h = \widetilde{\mathcal{N}}(h, v), \\ \\ (b) \quad v = \widetilde{\mathcal{M}}(h, v), \end{cases}$$

where

$$(26) \qquad \widetilde{\mathcal{M}}(h, v) = \mathcal{M}(h, \mathcal{S}(v), v), \qquad \widetilde{\mathcal{N}}(h, v) = \mathcal{N}(h, \mathcal{S}(v), v).$$

If problem (25) is uniquely solvable with solution (h, v) also problem (24) is, and its (unique) solution is given by the triplet (h, v, s) where $s = \mathcal{S}(v)$.

Unfortunately the operator $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ is not a contraction not even for small T. To overcome this difficulty, we transform problem (25) into an equivalent one by replacing the operator $\widetilde{\mathcal{M}}$ with the operator $\overline{\mathcal{M}}$ that with any v and any h associates the solution to problem (21)-(23) with f, v_0, v_1 being now replaced, respectively, by $\overline{\mathcal{F}}(h, v)$, the right-hand side of (16), u_1', and $\overline{\mathcal{F}}(h, v)$ is obtained from $\mathcal{F}$ by replacing s with its expression given by (24b), and the term $h(t)u_0(x + \mathcal{S}(v))$ by the more regular term $\widetilde{\mathcal{N}}(h, v)u_0(x + \mathcal{S}(v))$. To the so obtained fixed-point equation we can successfully apply the Banach fixed-point theorem, and finally we can prove Theorem 1. For a more detailed proof we refer the reader to [13].

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The fundamental solutions of the time-fractional diffusion equation

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1. – Introduction

Time-fractional diffusion equations, obtained from the standard diffusion equation by replacing the first order time-derivative by a fractional derivative (of order $0 < \beta \le 2$, in Riemann-Liouville or Caputo sense), have been treated by a number of authors, see, *e.g.* Engler [11], Fujita [15], Gorenflo, Luchko and Mainardi [18, 19], Hanyga [21], Mainardi [23, 24, 25, 26], Metzler and Klafter [34], Prüss [38], Saichev and Zaslavsky [39], Schneider and Wyss [41], Uchaikin and Zolotarev [43]. For other treatments of the time-fractional diffusion equations we refer the reader to the references cited therein. In this paper we intend to provide more insights for the fundamental solutions of the general time-fractional diffusion equation, based on the recent results by Mainardi, Luchko and Pagnini [28].

By *time-fractional diffusion equation* we mean the evolution equation

$$\frac{\partial^\beta}{\partial t^\beta}\, u(x,t) = \frac{\partial^2}{\partial x^2}\, u(x,t)\,, \quad 0 < \beta \le 2\,, \quad x \in I\!R\,, \quad t \in I\!R_0^+\,, \tag{1}$$

where the time-fractional derivative is intended in the Caputo sense, see Appendix A. When β is not integer ($\beta \neq 1\,,\,2$) the L.H.S of (1) reads:

$$\frac{\partial^\beta}{\partial t^\beta}\, u(x,t) := \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_0^t \left[\frac{\partial}{\partial \tau}\, u(x,\tau)\right] \frac{d\tau}{(t-\tau)^\beta}\,, & \text{if } 0 < \beta < 1\,, \\ \dfrac{1}{\Gamma(2-\beta)} \displaystyle\int_0^t \left[\frac{\partial^2}{\partial \tau^2}\, u(x,\tau)\right] \frac{d\tau}{(t-\tau)^{\beta-1}}\,, & \text{if } 1 < \beta < 2\,. \end{cases} \tag{2}$$

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When $\beta = 1, 2$ we recover well-known evolution equations, namely, for $\beta = 1$, the diffusion equation:

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)\,, \quad x \in I\!R\,, \quad t \in I\!R_0^+\,; \tag{3}$$

for $\beta = 2$, the D'Alembert wave equation:

$$\frac{\partial^2}{\partial t^2} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)\,, \quad x \in I\!R\,, \quad t \in I\!R_0^+\,. \tag{4}$$

For $1 < \beta < 2$ the fractional equation (1) is expected to interpolate (3) and (4), thus in this case it can be referred to as the time-fractional diffusion-wave equation.

Suitable processes of integration allow us to eliminate the time-fractional derivative in the L.H.S of (1); recalling the definition of the Caputo derivative we easily obtain the following integro-differential equations:
if $0 < \beta < 1$,

$$u(x,t) = u(x,0^+) + \frac{1}{\Gamma(\beta)} \int_0^t \left[\frac{\partial^2}{\partial x^2} u(x,\tau)\right] (t-\tau)^{\beta-1}\, d\tau\,; \tag{5}$$

if $1 < \beta < 2$,

$$u(x,t) = u(x,0^+) + t\, u_t(x,0^+) + \frac{1}{\Gamma(\beta)} \int_0^t \left[\frac{\partial^2}{\partial x^2} u(x,\tau)\right] (t-\tau)^{\beta-1}\, d\tau\,. \tag{6}$$

In order to formulate and solve the Cauchy problem for (1) we have to select explicit initial conditions concerning $u(x,0^+)$ if $0 < \beta \le 1$ and $u(x,0^+)$, $u_t(x,0^+)$ if $1 < \beta \le 2$. If $\phi(x)$ and $\psi(x)$ denote sufficiently well-behaved real functions [1] defined on $I\!R$, the Cauchy problem consists in finding the solution of (1) subjected to the additional conditions:

$$u(x,0^+) = \phi(x)\,, \quad x \in I\!R\,, \quad \text{if} \quad 0 < \beta \le 1\,; \tag{7a}$$

$$\begin{cases} u(x,0^+) = \phi(x)\,, \\ u_t(x,0^+) = \psi(x)\,, \end{cases} \quad x \in I\!R\,, \quad \text{if} \quad 1 < \beta \le 2\,. \tag{7b}$$

We note that if we set $\psi(x) \equiv 0$ *in (7b) we ensure the continuous dependence of the corresponding solution with respect to the parameter* β *in the transition from* $\beta = 1^-$ *to* $\beta = 1^+$*, as it turns out by comparing the representations (5) and (6).*

The paper is divided as follows. In Section 2 we treat the Cauchy problem for the equation (1) by making use of the Fourier and Laplace transforms with respect to the space and time variables. We state the concept of fundamental solutions (the so-called Green functions*) for which we derive the general scaling properties in*

[1] In what follows we shall meet only functions that are defined and continuous in $x \in I\!R$ and/or $t \in (0,T)$, $\forall T > 0$ except, possibly, at isolated points where these functions can be infinite. Following Marichev [31] we restrict our attention to the classes of such functions for which the Riemann improper integrals in x and in t absolutely converge on $I\!R$ and $(0,T)$, respectively. We denote these classes as $L^c(I\!R)$, $L^c(0,T)$.

terms of a similarity variable. In Section 3 we provide a general representation for the (reduced) Green functions in terms of Mellin-Barnes integrals in the complex plane, which allow us to obtain them in computational form. We note that these functions are peculiar "higher transcendental" functions of the Wright-type, that, in their turn, are special cases of the more general Fox H functions. Finally, Section 4 is devoted to concluding discussions, and a summary of the results in which we present plots of the (reduced) Green functions for a number of cases. After the Appendix A devoted to the Caputo fractional derivative, we report in Appendix B some historical notes on the Italian mathematician S. Pincherle, who can be considered the pioneer of the Mellin-Barnes integrals.

2. – The Green functions: scaling and similarity properties

The Cauchy problems stated in the Introduction can be conveniently treated by making use of the most common integral transforms, *i.e.* the Fourier transform (in space) and the Laplace transform (in time) whose notations are briefly recalled in [2]. Indeed, the combined Fourier-Laplace transforms of the solutions of the two Cauchy problems:

$$\text{(a)}\quad \{(1)+(7a)\}\quad \text{if}\quad 0<\beta\le 1\,,\qquad \text{(b)}\quad \{(1)+(7b)\}\quad \text{if}\quad 1<\beta\le 2\,,$$

turn out to satisfy the following algebraic equations

$$(8a)\qquad -\kappa^2\,\widehat{\widetilde{u}}(\kappa,s) = s^\beta\,\widehat{\widetilde{u}}(\kappa,s) - s^{\beta-1}\,\widehat{\phi}(\kappa)\,,\quad 0<\beta\le 1\,,$$

$$(8b)\qquad -\kappa^2\,\widehat{\widetilde{u}}(\kappa,s) = s^\beta\,\widehat{\widetilde{u}}(\kappa,s) - s^{\beta-1}\,\widehat{\phi}(\kappa) - s^{\beta-2}\,\widehat{\psi}(\kappa)\,,\quad 1<\beta\le 2\,,$$

---

[2]Let

$$\widehat{f}(\kappa) = \mathcal{F}\{f(x);\kappa\} = \int_{-\infty}^{+\infty} \mathrm{e}^{\,+i\kappa x}\,f(x)\,dx\,,\quad \kappa\in I\!R\,,$$

be the Fourier transform of a function $f(x)\in L^c(I\!R)$, and let

$$f(x) = \mathcal{F}^{-1}\left\{\widehat{f}(\kappa);x\right\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\,-i\kappa x}\,\widehat{f}(\kappa)\,d\kappa\,,\quad x\in I\!R\,,$$

be the inverse Fourier transform.
Let

$$\widetilde{f}(s) = \mathcal{L}\{f(t);s\} = \int_0^\infty \mathrm{e}^{\,-st}\,f(t)\,dt\,,\quad \Re(s) > a_f\,,$$

be the Laplace transform of a function $f(t)\in L^c(0,T)$, $\forall T>0$ and let

$$f(t) = \mathcal{L}^{-1}\left\{\widetilde{f}(s);t\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{e}^{\,st}\,\widetilde{f}(s)\,ds\,,\quad \Re(s)=\gamma>a_f\,,$$

with $t>0$, be the inverse Laplace transform. Above a_f denotes the abscissa of convergence: for its existence a sufficient condition is that the original function is of exponential type.
We remind that if the original functions are piecewise differentiable, then the two inversion formulas hold true where the functions are continuous and the corresponding integrals must be understood in the sense of the Cauchy principal value.

from which we obtain

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + \kappa^2}\, \widehat{\phi}(\kappa)\,, \quad 0 < \beta \le 1\,, \tag{9a}$$

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + \kappa^2}\, \widehat{\phi}(\kappa) + \frac{s^{\beta-2}}{s^\beta + \kappa^2}\, \widehat{\psi}(\kappa)\,, \quad 1 < \beta \le 2\,. \tag{9b}$$

By fundamental solutions (or Green functions) of the above Cauchy problems we mean the (generalized) solutions corresponding to the initial conditions

$$G_\beta^{(1)}(x, 0^+) = \delta(x)\,, \quad \text{if} \quad 0 < \beta \le 1\,; \tag{10a}$$

$$\begin{cases} G_\beta^{(1)}(x, 0^+) = \delta(x)\,, \\ \dfrac{\partial}{\partial t}\, G_\beta^{(1)}(x, 0^+) = 0\,, \end{cases} \quad \begin{cases} G_\beta^{(2)}(x, 0^+) = 0\,, \\ \dfrac{\partial}{\partial t}\, G_\beta^{(2)}(x, 0^+) = \delta(x)\,, \end{cases} \quad \text{if} \quad 1 < \beta \le 2\,. \tag{10b}$$

Here $\delta(x)$ is the delta-Dirac generalized function whose (generalized) Fourier transform is known to be identically 1. Thus, the combined space-Fourier and time-Laplace transforms of these Green functions turn out to be

$$\widehat{\widetilde{G_\beta^{(1)}}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + \kappa^2}\,, \quad 0 < \beta \le 2\,, \tag{11a}$$

$$\widehat{\widetilde{G_\beta^{(2)}}}(\kappa, s) = \frac{s^{\beta-2}}{s^\beta + \kappa^2}\,, \quad 1 < \beta \le 2\,. \tag{11b}$$

We note that the function $G_\beta^{(2)}(x, t)$ along with its combined Fourier-Laplace transform is well defined also for $0 < \beta \le 1$ even if it loses its meaning of being a fundamental solution of (1).

Then, by recalling the Fourier convolution property in the inversion of the Fourier-Laplace transforms of (9a)-(9b), we note that the Green functions allow us to represent the solutions of the above two Cauchy problems through the relevant integral formulas

$$u(x, t) = \int_{-\infty}^{+\infty} G_\beta^{(1)}(\xi, t)\, \phi(x - \xi)\, d\xi\,, \quad 0 < \beta \le 1\,; \tag{12a}$$

$$u(x, t) = \int_{-\infty}^{+\infty} [G_\beta^{(1)}(\xi, t)\, \phi(x - \xi) + G_\beta^{(2)}(\xi, t)\, \psi(x - \xi)]\, d\xi\,, \ 1 < \beta \le 2\,. \tag{12b}$$

By using the known scaling rules for the Fourier and Laplace transforms,

$$f(ax) \overset{\mathcal{F}}{\leftrightarrow} a^{-1}\, \widehat{f}(\kappa/a)\,, \quad a > 0\,, \qquad f(bt) \overset{\mathcal{L}}{\leftrightarrow} b^{-1}\, \widetilde{f}(s/b)\,, \quad b > 0\,, \tag{13}$$

we infer directly from (13) (thus without inverting the two transforms) the following *scaling properties* of the Green functions,

$$\text{(14)} \qquad \begin{cases} G_\beta^{(1)}(ax\,,\,bt) = b^{-\nu} G_\beta^{(1)}(ax/b^\nu\,,\,t)\,, \\ G_\beta^{(2)}(ax\,,\,bt) = b^{-\nu+1} G_\beta^{(2)}(ax/b^\nu\,,\,t)\,, \end{cases} \qquad \nu = \beta/2\,.$$

Consequently, introducing the *similarity variable* $x/t^{\beta/2}$, we can write

$$\text{(15)} \qquad \begin{cases} G_\beta^{(1)}(x,t) = t^{-\beta/2}\, K_\beta^{(1)}(x/t^{\beta/2})\,, \\ G_\beta^{(2)}(x,t) = t^{-\beta/2+1}\, K_\beta^{(2)}(x/t^{\beta/2})\,, \end{cases}$$

where the one-variable functions $K_\beta^{(1)}(x)$, $K_\beta^{(2)}(x)$ are referred to as the *reduced Green functions.* We note that both the Green functions are symmetric with respect to x and

$$\text{(16)} \qquad K_\beta^{(j)}(x) = G_\beta^{(j)}(x,1) = K_\beta^{(j)}(-x)\,, \quad j = 1,2\,.$$

In view of (15) and (12), the knowledge of the *reduced Green functions* is sufficient to provide the complete solutions of the Cauchy problems.

3. – Mellin-Barnes integral representation of the Green functions

To determine the two Green functions in the space-time domain we can follow two alternative strategies related to the different order in carrying out the inversion of the combined Fourier-Laplace transforms in (11)-(12). Indeed we can

(S1) : invert the Fourier transforms getting $\widetilde{G_\beta^{(1)}}(x,s)$, $\widetilde{G_\beta^{(2)}}(x,s)$, and then invert these Laplace transforms,

(S2) : invert the Laplace transforms getting $\widehat{G_\beta^{(1)}}(\kappa,t)$, $\widehat{G_\beta^{(2)}}(\kappa,t)$, and then invert these Fourier transforms.

<u>Strategy (S1)</u>

Recalling the Fourier transform pair,

$$\text{(17)} \qquad \frac{a_j}{b+\kappa^2} \stackrel{\mathcal{F}}{\leftrightarrow} \frac{a_j}{2b^{1/2}}\,\mathrm{e}^{-|x|b^{1/2}}\,, \quad b > 0\,,$$

and setting $a_j = s^{\beta-j}$, $b = s^\beta$ we get

$$\text{(18)} \qquad \widetilde{G_\beta^{(j)}}(x,s) = \frac{s^{\beta/2-j}}{2}\,\mathrm{e}^{-|x|s^{\beta/2}}\,, \quad j = 1,2\,.$$

<u>Strategy (S2)</u>

Recalling the Laplace transform pair, see *e.g.* [12], [20], [36]

$$\text{(19)} \qquad \frac{s^{\beta-j}}{s^\beta + c} \stackrel{\mathcal{L}}{\leftrightarrow} t^{j-1}\,E_{\beta,j}(-ct^\beta)\,, \quad c > 0\,,$$

where $E_{\beta,j}$ denotes the two-parameter Mittag-Leffler function[3] and setting $c = \kappa^2$ we get

$$\widehat{\widetilde{G_\beta^{(j)}}}(\kappa,t) = t^{j-1}\, E_{\beta,j}(-\kappa^2 t^\beta)\,, \quad j = 1,2\,. \tag{20}$$

The strategy (S1) has been followed by Mainardi [23, 24, 25, 26] to obtain the first Green function as

$$G_\beta^{(1)}(x,t) = \frac{1}{2} t^{-\beta/2}\, M_{\beta/2}\left(|x|/t^{\beta/2}\right)\,, \quad -\infty < x < +\infty\,, \quad t \geq 0\,, \tag{21}$$

where $M_{\beta/2}$ denotes the so-called M function of order $\beta/2$, see also [36], which is a noteworthy case of the Wright function[4].

[3]The Mittag-Leffler function $E_{\beta,\mu}$ with $\beta, \mu > 0$ is an entire transcendental function of order $\rho = 1/\beta$, defined in the complex plane by the power series

$$E_{\beta,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta\, n + \mu)}\,, \quad \beta, \mu > 0\,, \quad z \in \mathbb{C}\,.$$

Originally, at the beginning of 1900, Mittag-Leffler introduced and investigated (in five notes from 1902 to 1905) the function

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}\,, \quad \alpha > 0\,, \quad z \in \mathbb{C}\,,$$

as an instructive example of an entire function that generalises the exponential. For detailed information on the Mittag-Leffler-type functions the reader may consult *e.g.* [10],[12], [20], [22], [27], [36], [40].

[4]The function $M_\nu(z)$ is defined for any order $\nu \in (0,1)$ and $\forall z \in \mathbb{C}$ by

$$M_\nu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\, \Gamma[-\nu n + (1-\nu)]}\,, \quad 0 < \nu < 1\,, \quad z \in \mathbb{C}\,.$$

It turns out that $M_\nu(z)$ is an entire function of order $\rho = 1/(1-\nu)$, which provides a generalization of the Gaussian and of the Airy function. In fact we obtain

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-z^2/4\right)\,, \quad M_{1/3}(z) = 3^{2/3}\, \mathrm{Ai}\left(z/3^{1/3}\right)\,.$$

The M function is a special case of the Wright function defined by the series representation, valid in the whole complex plane,

$$\Phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\, \Gamma(\lambda n + \mu)}\,, \quad \lambda > -1\,, \quad \mu \in \mathbb{C}\,, \quad z \in \mathbb{C}\,.$$

Indeed, we recognize

$$M_\nu(z) = \Phi_{-\nu,1-\nu}(-z)\,, \quad 0 < \nu < 1\,.$$

Originally, Wright introduced and investigated this function with the restriction $\lambda \geq 0$ in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions. Only later, in 1940, he considered the case $-1 < \lambda < 0$. For detailed information on the Wright-type functions the interested reader may consult, *e.g.* [12] (where, presumably for a misprint, λ is restricted to be non negative), [18, 19], [22].

As far as the second Green function is concerned, we note from (18) that $\widetilde{G_\beta^{(2)}}(x,s) = \widetilde{G_\beta^{(1)}}(x,s)/s$, so

$$G_\beta^{(2)}(x,t) = \int_0^t G_\beta^{(1)}(x,\tau)\, d\tau\,. \tag{22}$$

Closed form solutions are found in the special case $\beta = 1$ (diffusion equation) and in the limiting case $\beta = 2$ (D'Alembert wave equation). We easily recognize for $\beta = 1$:

$$G_1^{(1)}(x,t) = \frac{t^{-1/2}}{2\sqrt{\pi}}\, \mathrm{e}^{-x^2/(4t)}\,, \quad G_1^{(2)}(x,t) = \frac{t^{1/2}}{\sqrt{\pi}}\, \mathrm{e}^{-x^2/(4t)} - \frac{x}{2}\,\mathrm{erfc}\left(\frac{x}{2t^{1/2}}\right), \tag{23}$$

where erfc denotes the complementary error function, and, for $\beta = 2$:

$$G_2^{(1)}(x,t) = \frac{\delta(x+t) + \delta(x-t)}{2}\,, \quad G_2^{(2)}(x,t) = \frac{\theta(x+t) - \theta(x-t)}{2}\,, \tag{24}$$

where θ denotes the unit-step Heaviside function.

The strategy (S2) has been followed by Gorenflo, Iskenderov & Luchko [17] and by Mainardi, Luchko & Pagnini [28] to obtain the first Green function of space-time fractional diffusion equations. These authors have inverted the relevant Fourier transforms by using the Mellin transform[5].

[5] Let

$$\mathcal{M}\{f(r);s\} = f^*(s) = \int_0^{+\infty} f(r)\, r^{s-1}\, dr, \quad \gamma_1 < \Re(s) < \gamma_2$$

be the Mellin transform of a sufficiently well-behaved function $f(r)$, and let

$$\mathcal{M}^{-1}\{f^*(s);r\} = f(r) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s)\, r^{-s}\, ds$$

be the inverse Mellin transform, where $r > 0$, $\gamma = \Re(s)$, $\gamma_1 < \gamma < \gamma_2$. We refer to specialised treatises and/or handbooks, see *e.g.* [13], [31], [37], for more details and tables on the Mellin transform. Here, for our convenience we recall the main rules that are useful to adapt the formulae from the handbooks and, meantime, are relevant in the following. Denoting by $\overset{\mathcal{M}}{\leftrightarrow}$ the juxtaposition of a function $f(r)$ with its Mellin transform $f^*(s)$, the main rules are:

$$f(ar) \overset{\mathcal{M}}{\leftrightarrow} a^{-s} f^*(s)\,,\ a > 0\,; \quad r^a f(r) \overset{\mathcal{M}}{\leftrightarrow} f^*(s+a)\,; \quad f(r^p) \overset{\mathcal{M}}{\leftrightarrow} \frac{1}{|p|} f^*(s/p)\,, \quad p \neq 0\,;$$

$$h(r) = \int_0^\infty \frac{1}{\rho} f(\rho)\, g(r/\rho)\, d\rho \overset{\mathcal{M}}{\leftrightarrow} h^*(s) = f^*(s)\, g^*(s)\,.$$

The Mellin convolution formula is useful in treating integrals of Fourier type for $x = |x| > 0$ as $I_c(x) = \frac{1}{\pi}\int_0^\infty f(\kappa)\, \cos(\kappa x)\, d\kappa$, when the Mellin transform $f^*(s)$ of $f(\kappa)$ is known.

Referring to [28] for details, we get

$$I_c(x) = \frac{1}{\pi x}\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s)\,\Gamma(1-s)\,\sin\left(\frac{\pi s}{2}\right) x^s\, ds\,,\ x > 0\,,\ 0 < \gamma < 1\,.$$

Here we follow the same approach based on Mellin transform. For this purpose we note that the Mittag-Leffler function admits a Mellin transform type representation, see *e.g.* [31], which will be used to get the Fourier anti-transform of (20) (with $t = 1$), namely

$$E_{\beta,j}(-r^2) \overset{\mathcal{M}}{\leftrightarrow} \frac{1}{2}\frac{\Gamma(s/2)\,\Gamma(1-s/2)}{\Gamma(j-\beta s/2)}\,, \quad 0<\beta\le 2\,, \quad 0<\Re(s)<2\,. \tag{25}$$

For the determination of the reduced Green functions $K_\beta^{(j)}(x) = G_\beta^{(j)}(x,1)$ we can restrict our attention to $x > 0$, and thus write in view of (20) and (25)

$$K_\beta^{(j)}(x) = \frac{1}{\pi}\int_0^\infty \cos(\kappa x)\, E_{\beta,j}\left(-\kappa^2\right)\, d\kappa\,. \tag{26}$$

So, using the final result in footnote [(6)] and the reflection formula for the Gamma function, we obtain

$$K_\beta^{(j)}(x) = \frac{1}{2x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-s)}{\Gamma(j-\beta s/2)}\, x^{\,s}\, ds\,, \quad 0<\gamma<1\,. \tag{27}$$

The integral at the RHS of (27), in that it contains only gamma functions in the fraction multiplying x^s, is a particular *Mellin-Barnes integral* according to a usual terminology. In this respect the interested reader can find in [12] the discussion on the general conditions of convergence for the typical Mellin-Barnes integral, based on the asymptotic representation (Stirling formula) of the gamma function. The names refer to the two authors, who in the first 1900's developed the theory of these integrals using them for a complete integration of the hypergeometric differential equation. However, as pointed out in [12], these integrals were first used by the Italian mathematician S. Pincherle in 1888, see Appendix B.

Readers acquainted with the "higher transcendental" H functions (introduced by Fox [14] in 1961) can recognize in the R.H.S of (27) the representation of a certain function of this class see *e.g.* [22], [31], [32], [37], [40], [42]. Unfortunately, as far as we know, computing routines for this general class of special functions are not yet available. Here, following the approach adopted by Mainardi, Luchko & Pagnini [28], we intend to compute the (reduced) Green functions in any space domain by matching a convergent power series (suitable for small $|x|$) with an asymptotic representation (suitable for large $|x|$).

In order to obtain the convergent power series we transform the original contour in (27) to the loop $L_{+\infty}$ encircling all the poles $s_n = 1+n$, $n \in I\!N_0$ of the function $\Gamma(1-s)$ and apply the residue theorem. We obtain

$$K_\beta^{(j)}(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-x)^n}{n!\,\Gamma[-\beta n/2 + (j-\beta/2)]}\,, \quad j = 1,2\,. \tag{28}$$

The asymptotic representation can be obtained by using the arguments by Braaksma [3] (see also [28]), and turns out to be

$$K_\beta^{(j)}(x) \sim A_j\, x^{\,a_j} \exp\left(-b\, x^{\,c}\right)\,, \quad x \to +\infty\,, \tag{29}$$

where

$$(30) \qquad A_j = \left\{2\pi(2-\beta)\, 2^{[\beta-4(j-1)]/(2-\beta)}\, \beta^{[2(2j-1)-2\beta]/(2-\beta)}\right\}^{-1/2},$$

$$(31) \qquad a_j = \frac{2\beta-2(2j-1)}{2(2-\beta)}, \quad b = (2-\beta)\, 2^{-2/(2-\beta)} \beta^{\beta/(2-\beta)}, \quad c = \frac{2}{2-\beta}.$$

We now can complement Mainardi's result (21) providing also the second Green function in the space-time domain in terms of a Wright-type function. Indeed, using (15) and (28) we can write

$$(32) \qquad G_\beta^{(2)}(x,t) = \frac{1}{2} t^{1-\beta/2}\, P_{\beta/2}\left(|x|/t^{\beta/2}\right), \quad -\infty < x < +\infty, \quad t \geq 0,$$

where $P_{\beta/2}$ denotes a suitable Wright-type function briefly discussed in [6].

4. – Concluding discussion and plots

We conclude with a discussion about some general features occurring in the Cauchy problem of our time fractional diffusion equation (1)-(2). A first general feature concerns the scaling property of the two Green functions which allows us to express them in terms of functions of a single variable, the *reduced Green functions* $K_\beta^{(j)}(x)$, $j = 1, 2$, see (15). In this paper we have focused our attention on deriving a computational form for $K_\beta^{(j)}(x)$ in all of $\mathbb{R}$. In this respect the representation of $K_\beta^{(j)}(x)$ through the Mellin-Barnes integral, see (27), was found useful. More precisely, to compute the functions $K_\beta^{(j)}(x)$ we used the series expansions (28) and asymptotic representations (29)-(31), which were derived from (27).

Hereafter we shall exhibit some plots of the reduced Green functions $K_\beta^{(j)}(x)$ for some "characteristic" values of the parameter β. All the plots were drawn by using the MATLAB system for the values of the independent variable x in the range $|x| \leq 5$. To give the reader a better impression about the behaviour of the tails, the logarithmic scale was adopted. Both the *reduced Green functions*, being non-negative and normalized are of the greatest interest in view of their interpretation as probability densities. However, only the first Green function $G_\beta^{(1)}(x,t)$ keeps the normalization when it evolves in time, see Mainardi and Pagnini [30].

[6]The function $P_\nu(z)$ is defined for any order $\nu \in (0,1)$ and $\forall z \in \mathbb{C}$ by

$$P_\nu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n + (2-\nu)]}, \quad 0 < \nu < 1, \quad z \in \mathbb{C}.$$

It turns out that $P_\nu(z)$ is an entire function of order $\rho = 1/(1-\nu)$, and is a special case of the Wright function being

$$P_\nu(z) = \Phi_{-\nu,2-\nu}(-z), \quad 0 < \nu < 1.$$

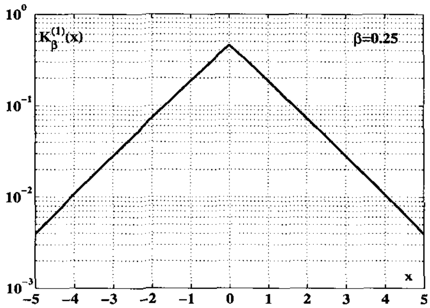

Fig. 1

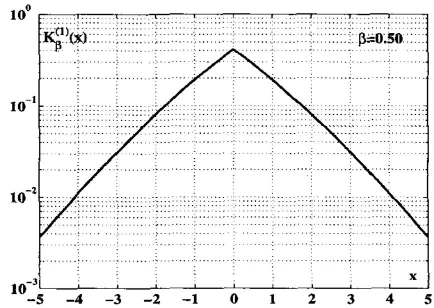

Fig. 2

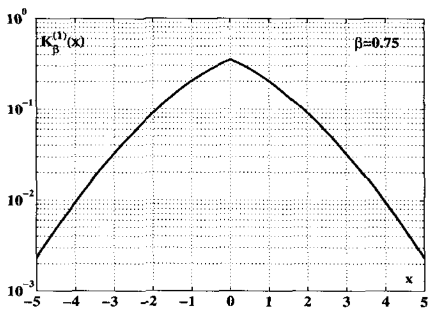

Fig. 3

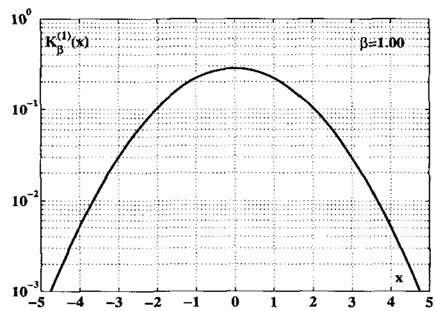

Fig. 4

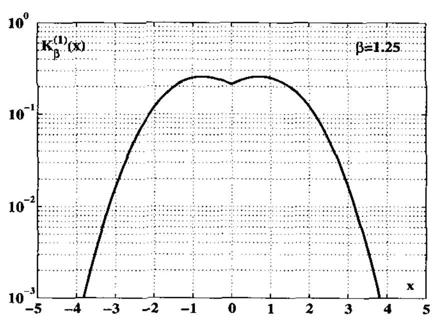

Fig. 5

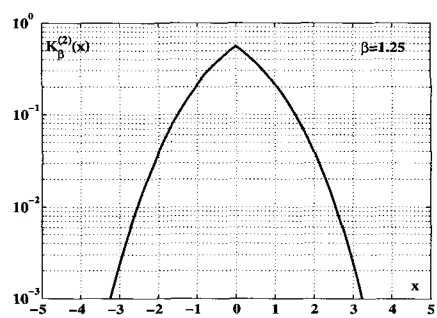

Fig. 6

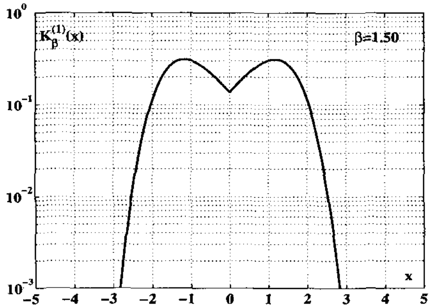

Fig. 7

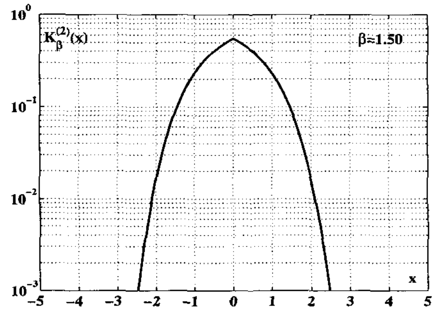

Fig. 8

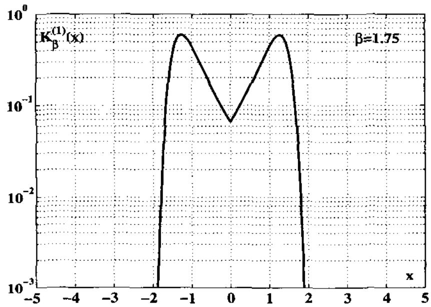

Fig. 9

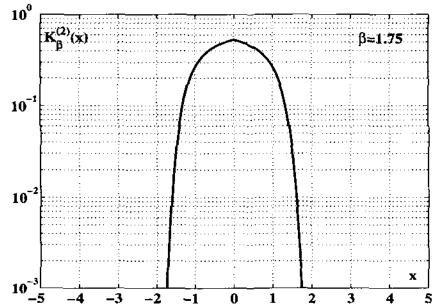

Fig. 10

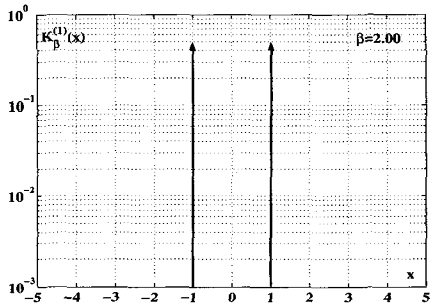

Fig. 11

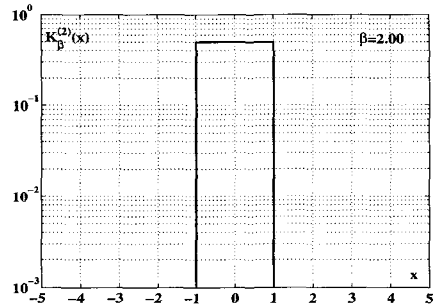

Fig. 12

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Appendix A: The Caputo time-fractional derivative

Now we present an introduction to the *Caputo* fractional derivative starting from its representation in the Laplace transform domain and contrasting it to the standard *Riemann-Liouville* fractional derivative.

Let

$$(A.1a) \qquad \widetilde{f}(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt\,, \quad \Re(s) > a_f\,,$$

be the Laplace transform of a function $f(t) \in L^c(0,T)$, $\forall T > 0$ and let

$$(A.1b) \qquad f(t) = \mathcal{L}^{-1}\left\{\widetilde{f}(s); t\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{e}^{st}\, \widetilde{f}(s)\, ds\,, \quad \Re(s) = \gamma > a_f\,,$$

with $t > 0$, be the inverse Laplace transform.

For a sufficiently well-behaved function $f(t)$ we define the *Caputo* time-fractional derivative of order β $(m-1 < \beta \le m\,,\ m \in I\!N)$ through its Laplace transform

$$(A.2) \qquad \mathcal{L}\left\{{}_tD_*^\beta\, f(t); s\right\} = s^\beta\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-1-k}\, f^{(k)}(0^+)\,, \quad m-1 < \beta \le m\,.$$

This leads us to define, see *e.g.* [6], [20],

$$(A.3) \qquad {}_tD_*^\beta\, f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)\, d\tau}{(t-\tau)^{\beta+1-m}}\,, & m-1 < \beta < m\,, \\ \dfrac{d^m}{dt^m} f(t)\,, & \beta = m\,. \end{cases}$$

The operator defined by (A.2)-(A.3) has been referred to as the *Caputo* fractional derivative since it was introduced by Caputo in the late 1960's for modelling the energy dissipation in some anelastic materials with memory, see [5, 6]. A former review of the theoretical aspects of this derivative with applications in viscoelasticity was given in 1971 by Caputo and Mainardi [9], with special emphasis to the long-memory effects.

The reader should observe that the *Caputo* fractional derivative differs from the usual *Riemann-Liouville* fractional derivative which, defined as the left inverse of the Riemann-Liouville fractional integral, is here denoted as ${}_tD^\beta\, f(t)$. We have, see *e.g.* [40],

$$(A.4) \qquad {}_tD^\beta\, f(t) := \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t \frac{f(\tau)\, d\tau}{(t-\tau)^{\beta+1-m}}\right], & m-1 \le \beta < m\,, \\ \dfrac{d^m}{dt^m} f(t)\,, & \beta = m\,. \end{cases}$$

When the order is not integer, Gorenflo and Mainardi have shown the following relationships between the two fractional derivatives (when both of them exist), see *e.g.* [20],

$$(A.5)\qquad {}_tD_*^\beta\, f(t) \,=\, {}_tD^\beta \left[f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\, \frac{t^k}{k!} \right], \quad m-1<\beta<m\,,$$

or

$$(A.6)\qquad {}_tD_*^\beta\, f(t) \,=\, {}_tD^\beta\, f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\, \frac{t^{k-\beta}}{\Gamma(k-\beta+1)}\,, \quad m-1<\beta<m\,.$$

The *Caputo* fractional derivative, practically ignored in the mathematical treatises[7], represents a sort of regularization in the time origin for the *Riemann-Liouville* fractional derivative. Recently, it has been extensively investigated by Gorenflo and Mainardi [20] and by Podlubny [36] in view of its major utility in treating physical and engineering problems which require standard initial conditions. Several applications have been treated by Caputo himself up to nowadays, see *e.g.* [7, 8] and references therein.

We point out that the *Caputo* fractional derivative satisfies the relevant property of being zero when applied to a constant, and, in general, to any power function of non-negative integer degree less than m, if its order β is such that $m-1<\beta<m$. Furthermore, since

$$(A.7)\qquad {}_tD^\beta\, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta)}\, t^{\gamma-\beta}\,, \quad \beta>0\,, \quad \gamma>-1\,, \quad t>0\,,$$

we note that

$$(A.8)\qquad {}_tD^\beta\, f(t) \,=\, {}_tD^\beta\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{\beta-j}\,,$$

whereas, using also (A.5) or (A.6),

$$(A.9)\qquad {}_tD_*^\beta\, f(t) \,=\, {}_tD_*^\beta\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{m-j}\,.$$

In these formulae the coefficients c_j are arbitrary constants. We also note the different behaviour of ${}_tD_*^\beta$ at the end points of the interval $(m-1, m)$,

$$(A.10)\qquad \lim_{\beta\to(m-1)^+} {}_tD_*^\beta\, f(t) = f^{(m-1)}(t) - f^{(m-1)}(0^+)\,, \quad \lim_{\beta\to m^-} {}_tD_*^\beta\, f(t) = f^{(m)}(t)\,.$$

The last limit can be formally obtained by recalling the formal representation of the m-th derivative of the Dirac function, $\delta^{(m)}(t) = t^{-m-1}/\Gamma(-m)$, $t\ge 0$, see [16]. As a consequence of (A.10), with respect to the order, the *Caputo* derivative is an operator left-continuous at any positive integer.

[7] According to Samko, Kilbas and Marichev[40] and Butzer and Westphal[4] the "regularized" fractional derivative was considered by Liouville himself (but then disregarded).

Appendix B: Pincherle and the Mellin-Barnes integrals

In Vol. 1, p. 49 of *Higher Transcendental Functions* of the Bateman Project [12] we read "Of all integrals which contain gamma functions in their integrands the most important ones are the so-called Mellin-Barnes integrals. Such integrals were first introduced by S. Pincherle, in 1888 [35]; their theory has been developed in 1910 by H. Mellin (where there are references to earlier work) [33] and they were used for a complete integration of the hypergeometric differential equation by E.W. Barnes [2]."

For a revisited analysis of the pioneering work of Pincherle we refer the interested reader to our recent paper [29]. Here we limit ourselves to give some biographical notes and to report the original quotations by Barnes and Mellin. This may help to recall the attention of mathematicians towards Pincherle.

Salvatore Pincherle (1853 - 1936) was Professor of Mathematics at the University of Bologna from 1880 to 1928. He retired from the University just after the International Congress of Mathematicians that he had organized in Bologna, following the invitation received at the previous Congress held in Toronto in 1924. He wrote several treatises and lecture notes on Algebra, Geometry, Real and Complex Analysis. His main book related to his scientific activity is entitled "Le Operazioni Distributive e loro Applicazioni all'Analisi"; it was written in collaboration with his assistant, Dr. Ugo Amaldi, and was published in 1901 by Zanichelli, Bologna.

Pincherle can be considered one of the most prominent founders of Functional Analysis, as pointed out by J. Hadamard in his review lecture "Le développement et le rôle scientifique du Calcul fonctionnel", given at the Congress of Bologna (1928).

A description of Pincherle's works, requested to the author by Mittag-Leffler, the Editor of Acta Mathematica, appeared (in French) in Vol. 46, pp. 341-362 (1925) of this prestigious journal under the title "Notice sur les travaux de S. Pincherle". Furthermore, a collection of selected papers (38 from 247 notes plus 24 treatises) was edited by Unione Matematica Italiana (UMI) on the occasion of the centenary of his birth, and published by Cremonese, Roma 1954. Note that S. Pincherle was the first President of UMI, from 1922 to 1936.

Here we point out that the 1888 pioneering work of S. Pincherle on *Generalized Hypergeometric Functions* led him to introduce the later so named Mellin-Barnes integral to represent the solution of a hypergeometric differential equation investigated by Goursat in 1883. Pincherle's priority was explicitly recognized by Mellin and Barnes themselves, as reported below.

In 1907 Barnes, see p. 63 in [1], wrote: "The idea of employing contour integrals involving gamma functions of the variable in the subject of integration appears to be due to Pincherle, whose suggestive paper was the starting point of the investigations of Mellin (1895) though the type of contour and its use can be traced back to Riemann."

In 1910 Mellin, see p. 326 in [33], devoted a section (§10: Beweis eines Satzes von Pincherle = Proof of Theorems of Pincherle) to revisiting the original work of Pincherle; in particular, he wrote "Ehe wir zum Beweise dieses Satzes schreithen, welcher einen speziellen Fall eines noch allgemeinerem Satzes von Herrn Pincherle bildet, wollen wir die Linien L näher angeben, über welche die Integration vorzugsweise erstreckt wird."

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Some results of pointwise stability for solutions to the Navier-Stokes system

P. Maremonti *

1. – Introduction

Let us assume that $\mathcal{F}$ is a viscous incompressible fluid whose motion m_0, in a region Ω of three-dimensional space $\mathbb{R}^3$, is governed by the Navier-Stokes equations. Denote by $(\mathbf{v}, \tilde{\pi})$ the kinetic and pressure fields of the motion m_0, where m_0 can be supposed steady or not. The study of continuous dependence and of the stability of fluid motion is essentially based on the so called energy methods. More precisely, let $(\mathbf{u}, \pi)$ denote the perturbation to the kinetic and pressure field of the motion m_0 of the fluid $\mathcal{F}$, Serrin in [11] proved that the perturbations, which have finite energy (say $|\mathbf{u}_0|_2 < \infty$) to the initial instant, consent to give, for suitable Reynolds number, a variational formulation of the stability, that ensures the stability in energy of the basic motion. The result of paper [11] for the stability in energy can be resumed in the following way: it is known that the perturbation $(\mathbf{u}, \pi)$ of the motion m_0 is a solution to the system:

$$(1) \qquad \begin{aligned} &\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u} + \nabla \pi = \nu \Delta \mathbf{u}, \\ &\nabla \cdot \mathbf{u} = 0, \ \text{in } \Omega \times (0, \mathrm{T}), \\ &\mathbf{u}(x, 0) = \mathbf{u}_0(x) \text{ in } \Omega, \ \mathbf{u}(\mathrm{x}, \mathrm{t}) = 0 \ \forall (\mathrm{x}, \mathrm{t}) \in \partial\Omega \times (0, \mathrm{T}), \\ &\text{if } \Omega \text{ is not bounded,} \forall \mathrm{t} \geq 0, \mathbf{u}(\mathrm{x}, \mathrm{t}) \to 0 \text{ for } |\mathrm{x}| \to \infty, \end{aligned}$$

$\mathbf{u}_t = \frac{\partial \mathbf{u}(x,t)}{\partial t}$, $\mathbf{a} \cdot \nabla \mathbf{b} = a_k \frac{\partial b_i}{\partial x_k} \mathbf{e}_i$; formally a solution of system (1) satisfies the relation:

$$(2) \quad \frac{1}{2} \frac{d}{dt} |\mathbf{u}(t)|_2^2 + \nu |\nabla \mathbf{u}(t)|_2^2 = -\int_{\partial\Omega} \mathbf{u}(x,t) \cdot \nabla \mathbf{v}(x,t) \cdot \mathbf{u}(x,t) dx, \forall t \in (0,T).$$

In [11] the following functional is considered

$$F(\mathbf{u}) = \left| \frac{\int_{\partial\Omega} (\mathbf{u}(x,t) \cdot \nabla \mathbf{v}(x,t) \cdot \mathbf{u}(x,t) dx}{|\nabla \mathbf{u}(t)|^2} \right|, \forall t \geq 0;$$

if $\sup_{\mathbf{u}} F(\mathbf{u}) = \nu^*(\mathbf{t}) \leq \nu$ for any $t \geq 0$, then integrated version of relation (2) becomes the energy inequality:

$$|\mathbf{u}(t)|_2^2 + 2 \int_0^t (\nu - \nu^*) |\nabla \mathbf{u}(\tau)|_2^2 d\tau \leq |\mathbf{u}_0|_2^2, \forall t \geq 0.$$

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Therefore from the energy inequality the definition of stability is well posed and if $\nu^*(t) < \nu$, uniformly in t, with $\nu* < \nu$, the asymptotic stability also, that is

$$|\mathbf{u}(t)|_2^2 \leq |\mathbf{u}_0|_2^2, \forall t \geq 0;$$

$$\lim_{t\to\infty} |\mathbf{u}(t)|_2 = 0.$$

In [11] the energy stability is formulated in the hypothesis that the region of motion Ω is bounded, however it can be also formulated in more general context and in unbounded regions Ω. For us the cases of an exterior domain or of the half-space $\mathbb{R}^3_+$ (provided that the unperturbed motion satisfies suitable hypotheses for large spatial distances) are of particular interest. Subsequently to the paper [11], in [10] there is an extension to the magnetohydrodynamics in the case of Ω bounded and it is proved that the variational formulation for the functional $F(\mathbf{u})$, in order to deduce the existence of the maximum, is well posed in the function space $H^1(\Omega)$, which is *a.e.* in t the function space of the perturbations which are weak solutions to system (1), thus perturbations corresponding to any $\mathbf{u}_0 \in L^2(\Omega)$. For the case of Ω unbounded a variational formulation of the energy stability is given in [2], and in [6] results of asymptotic stability are given. In this connection in [7], sharp orders of decay of $|\mathbf{u}(t)|_2$ are also deduced for perturbations to the rest state. In the case of half-space the energy method is studied in [3].

The approach to the study of the continuous dependence and of the stability which is considered in the this note is quite different.[1] We limit ourself to the case of $\mathbb{R}^3_+$. In this case the study of the pointwise stability regards two different formulations (see definition 1 and 3). As far as we known, results of stability in the sense of definition 1 and 3, if we exclude the results of [12] and [4], are not known either for solutions to Stokes system or for solutions to the Navier-Stokes system. In [12] the problem of the maximum modulus for the initial boundary value problem of Stokes system is considered (the result is essentially stated). Instead in [4] the study of stability in the sense of the definition 3 is considered for Cauchy problem of Navier-Stokes system . Finally, results of continuous dependence for solutions to the Cauchy problem associated to the Navier-Stokes system can be found in the papers [1,4,5].

2. – Definition of stability in $C(\overline{\Omega})$ and of Pointwise Stability

In the following definitions, the field $\mathbf{u}(x,t)$ can be interpreted as a perturbation to the kinetic field of the motion m_0 of a fluid, and moreover this can be regarded indifferently as a solution to Stokes or Navier-Stokes system.

DEFINITION **1 (Stability in $C(\overline{\Omega})$)** A motion m_0 of a fluid $\mathcal{F}$ is said stable in the norm of $C(\overline{\Omega})$ if

$$\forall\varepsilon > 0, \exists\delta(\varepsilon) > 0 \text{ such that } \forall\mathbf{u}_0(x) \text{ with } |\mathbf{u}_0(x)| < \delta, \forall x \in \overline{\Omega},$$
$$\Rightarrow |\mathbf{u}(x,t)| < \varepsilon, \forall (x,t) \in \overline{\Omega} \times (0,T).$$

[1]In this note we present some results which will be published in [8].

Definition 2 **(Asymptotic Stability in $C(\overline{\Omega})$)** A motion m_0 of a fluid $\mathcal{F}$ is said asymptotically stable if it is stable in $C(\overline{\Omega})$ and if

$$\begin{aligned}&\exists\gamma \in (0,\infty) \text{ such that } \forall \mathbf{u}_0(x) \text{ with } |\mathbf{u}_0(x)| < \gamma, \forall x \in \overline{\Omega},\\ &\qquad \Rightarrow \forall x \in \overline{\Omega}, \lim_{t\to\infty} |\mathbf{u}(x,t)| = 0.\end{aligned}$$

Definition 3 **(Pointwise Stability)** A motion m_0 of a fluid $\mathcal{F}$ is said pointwise stable if there exists a constant $A > 0$ such that

$$\begin{aligned}&\forall \mathbf{u}_0(x) \text{ with } |\mathbf{u}_0(x)| \le U_0/(|x|+1)^\alpha, \text{ for some } \alpha \in [0,3),\\ &\Rightarrow \forall\beta \in (0,\alpha], |\mathbf{u}(x,t)| \le AU_0/(|x|+1)^{\alpha-\beta}(t+1)^{\frac{\beta}{2}}, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times [0,\infty),\\ &\quad \text{and for } \beta = 0,\ |\mathbf{u}(x,t)| \le AU_0/(|x|+1)^\alpha, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (0,T),\\ &\qquad\qquad \text{and } \forall x \in \mathbb{R}^3_+, \lim_{t\to\infty} \mathbf{u}(x,t)(|x|+1)^\alpha = 0.\end{aligned} \tag{3}$$

Remark 1 Definition 3 of stability, which is of particular interest in the case of unbounded regions, gives the behavior of the solution in any point of the region of motion of the fluid. It ensures that, modulo a constant factor, the behavior of the perturbation, for any t, is not different, in any point of the space, from the one of the perturbation to the initial instant. Moreover, it gives asymptotic stability; indeed, it also gives an explicit order of decay of the kinetic field for large t. In this connection, analogously with the results of the L^p theory (*cf. e.* [7,9,13]), the order of decay with respect to t is connected with the spatial one following a suitable dimensional balance typical of the heat equation. Such a result was obtained for the first time in [4] for the solutions to the Cauchy problem of Navier-Stokes system. As far as we known, the results, in the sense of definition 1-3, concerning the initial boundary value problem either Stokes or Navier-Stokes system are new, in the particular case of half-space R^3_+ also. Among these results, the particular case of the maximum modulus theorem (with the exception of paper [12] already quoted) is also new.

3. – The results obtained in the case of the half-space

The following results concerns the Stokes and Navier-Stokes system (1) with $\mathbf{v} = 0$ (in other words perturbations to the rest state):

$$\begin{aligned}&\mathbf{w}_t + \nabla\pi = \nu\Delta\mathbf{w},\\ &\nabla\cdot\mathbf{w} = 0, \qquad \text{in } \mathbb{R}^3_+ \times (0,T),\\ &\mathbf{w}(x,0) = \mathbf{w}_0(x) \text{ in } \mathbb{R}^3_+,\ \mathbf{w}(x,t) = 0,\ \forall (x,t) \in x_3 = 0 \times (0,T),\\ &\mathbf{w}(x,t) \to 0 \text{ per } |x| \to \infty, \forall t \ge 0.\end{aligned}$$

$$\begin{aligned}&\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla\pi = \nu\Delta\mathbf{v},\\ &\nabla\cdot\mathbf{v} = 0, \qquad \text{in } \mathbb{R}^3_+ \times (0,T),\\ &\mathbf{v}(x,0) = \mathbf{v}_0(x) \text{ in } \mathbb{R}^3_+,\ \mathbf{v}(x,t) = 0,\ \forall (x,t) \in x_3 = 0 \times (0,T),\\ &\mathbf{v}(x,t) \to 0 \text{ per } |x| \to \infty, \forall t \ge 0.\end{aligned}$$

For the Stokes system we have

THEOREM 1 *Assume* $\mathbf{w}_0(x) \in C(\overline{\mathbb{R}}^3_+), \nabla \cdot \mathbf{w}_0(x) = 0$, *with* $\mathbf{w}_0(x)_{|x_3=0} = 0$ *and* $\mathbf{w}_0(x) \to 0$ *for* $|x| \to \infty$. *Then there exists a unique classical solution* $(\mathbf{w}, \pi)$ *of Stokes system defined for* $t \in (0, \infty)$, *such that*

$$\mathbf{w}(x,t) \in C([0,T) \times \overline{\mathbb{R}}^3_+) \cap C^1(\eta, T \times \overline{\mathbb{R}}^3_+),$$
$$\nabla \mathbf{w}(x,t), \pi(x,t) \in C(\eta, T; C^1(\overline{\mathbb{R}}^3_+)), \forall \eta > 0.$$

Moreover,

$$|\mathbf{w}(x,t)| \leq C \max_{\overline{\mathbb{R}}^3_+} |\mathbf{w}_0(x)|, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (0,\infty),$$
$$\forall x \in \overline{\mathbb{R}}^3_+, \lim_{t \to \infty} |\mathbf{w}(x,t)| = 0.$$

The following theorem concerns the pointwise stability

THEOREM 2 *Assume* $\mathbf{w}_0(x) \in C(\overline{\mathbb{R}}^3_+), \nabla \cdot \mathbf{w}_0(x) = 0$, *with* $\mathbf{w}_0(x)_{|x_3=0} = 0$ *and* $|\mathbf{w}_0(x)| \leq W_0/(|x|+1)^\alpha$, *for some* $\alpha \in [0,3)$. *Then there exists a unique classical solution* $(\mathbf{w}, \pi)$ *of Stokes system defined for* $t \in (0, \infty)$, *such that*

$$\mathbf{w}(x,t) \in C([0,T) \times \overline{\mathbb{R}}^3_+) \cap C^1(\eta, T \times \overline{\mathbb{R}}^3_+),$$
$$\nabla \mathbf{w}(x,t), \pi(x,t) \in C(\eta, T; C^1(\overline{\mathbb{R}}^3_+)), \forall \eta > 0.$$

Moreover,

$$|\mathbf{w}(x,t)| \leq A \max_{\overline{\mathbb{R}}^3_+} |\mathbf{w}_0(x)|, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (0,\infty),$$
$$|\mathbf{w}(x,t)| \leq \frac{AW_0}{(|x|+1)^{\alpha-\beta}(t+1)^{\frac{\beta}{2}}}, \beta \in [0,\alpha], \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times [1,\infty],$$
$$\textit{and for } \beta = 0 \ \lim_{t \to \infty} |\mathbf{w}(x,t)|(|x|+1)^\alpha = 0, \forall x \in \overline{\mathbb{R}}^3_+,$$

with A *independent of* $\mathbf{w}_0$.

As far as Navier-Stokes system is concerned, we have

THEOREM 3 *Assume* $\mathbf{v}_0(x) \in C(\overline{\mathbb{R}}^3_+), \nabla \cdot \mathbf{v}_0(x) = 0, \mathbf{v}_0(x)_{|x_3=0} = 0$ *and* $\mathbf{v}_0(x) \to 0$ *for* $|x| \to \infty$. *Then there exists a unique solution* $(\mathbf{v}(x,t), \pi(x,t))$ *of Navier-Stokes system in some interval* $[0,T)$ *with* $T = T(\max_{\overline{\mathbb{R}}^3_+} |\mathbf{v}_0(x)|)$, *such that*

$$\mathbf{v}(x,t) \in C([0,T) \times \overline{\mathbb{R}}^3_+) \cap C^1(\eta, T \times \overline{\mathbb{R}}^3_+),$$
$$\nabla \mathbf{v}(x,t), \pi(x,t) \in C(\eta, T; C^1(\overline{\mathbb{R}}^3_+)), \forall \eta > 0.$$

Moreover,

$$|\mathbf{v}(x,t)| \leq c(t) \max_{\overline{\mathbb{R}}^3_+} |\mathbf{v}_0(x)|, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (0,\infty),$$

(a priori $c(t) \to \infty$ *for* $t \to T$.*)*

The next result gives a sufficient condition for global (in time) solution to Navier-Stokes system:

THEOREM 4 *Assume* $\mathbf{v}_0(x) \in C(\overline{\mathbb{R}}^3_+), \nabla \cdot \mathbf{v}_0(x) = 0$, *with* $\mathbf{v}_0(x)_{|x_3=0} = 0$ *and* $|\mathbf{v}_0(x)| \leq V_0/(|x|+1)$, *with* V_0 *"sufficiently small". Then there exists a unique classical solution* $(\mathbf{v}, \pi)$ *to Navier-Stokes system, defined for* $t \in (0, \infty)$, *such that*

$$\mathbf{v}(x,t) \in C([0,T) \times \overline{\mathbb{R}}^3_+) \cap C^1(\eta, T \times \overline{\mathbb{R}}^3_+),$$
$$\nabla \mathbf{v}(x,t), \pi(x,t) \in C(\eta, T; C^1(\overline{\mathbb{R}}^3_+)), \forall \eta > 0.$$

Moreover,

$$|\mathbf{v}(x,t)| \leq C \max_{\overline{\mathbb{R}}^3_+} |\mathbf{v}_0(x)|, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (0,\infty),$$
$$|\mathbf{v}(x,t)| \leq C V_0/(|x|+1), \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (0,\infty),$$
$$|\mathbf{v}(x,t)| \leq C V_0/t^{-1/2}, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (1,\infty).$$

The following proposition contains an interesting result on the behavior of the solution in a neighborhood of $t = 0$.

PROPOSITION 1 *The solutions obtained in Theorem 1 satisfy the following properties*

$$|\nabla \mathbf{w}(x,t)| \leq C \max_{\overline{\mathbb{R}}^3_+} |\mathbf{w}_0(x)|/t^{1/2}, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (0,\infty),$$
$$|\mathbf{w}_t(x,t)| + |\nabla \pi(x,t)| + |D^2 \mathbf{w}(x,t)| \leq C \max_{\overline{\mathbb{R}}^3_+} |\mathbf{w}_0(x)|/t, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (0,\infty),$$

with C *independent of* $\mathbf{w}_0$.

REMARK 2 Theorems 3 and 4 can be obtained for solutions to system (1) also, where the unperturbed motion $\mathbf{v}$ can be assumed steady or not. In the case of Theorem 4 it is necessary to require that $\mathbf{v}$ satisfies suitable hypotheses of *smallness* and behavior for large spatial distances as

$$|\mathbf{v}(x,t)| \leq V_0/(1+|x|)^\gamma, \gamma \geq 1, \forall (x,t) \in \overline{\mathbb{R}}^3_+ \times (0,T),$$

with V_0 sufficiently small with respect to ν.

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Asymptotic behavior for a model of transverse vibration of a bar with linear memory

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1. – Introduction

We want to study the asymptotic behavior of transverse vibrations of a bar slightly modifying a model proposed in 1950 by Woinowsky-Krieger [1], which we recall briefly. Let l be the length of the bar in the reference configuration supposed subjected to a given tension S_0. Let furthermore ρ and B be two positive constants measuring the density of the bar and its rigidity. In the deformed configuration, the equation governing the motion $t \mapsto u(x,t)$ of the bar is

$$\rho\frac{\partial^2 u}{\partial t^2} + B\frac{\partial^4 u}{\partial x^4} - (S_0 + S_1)\frac{\partial^2 u}{\partial x^2} = 0.$$

In order to evaluate the extra tension S_1 due to the deflection, note that the length of the bar in the deformed configuration is

$$l' = \int_0^l \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx$$

so that, supposing small deformations, a series expansion bears

$$l' = \int_0^l \left[1 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right] dx = l + \int_0^l \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx := l + \Delta l.$$

We make the assumption that $S_1 = EA\Delta l/l$, where E is the Young modulus and A is the area of the section of the bar.

Normalizing the constants, we get the equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} - \left(\beta + \int_0^l \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

to which the boundary conditions

$$u(0,t) = u(l,t) = 0 = u'(0,t) = u'(l,t)$$

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and the initial conditions

$$u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x)$$

are associated.

In 1973 Ball [2] studied the problem

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} - \left(\beta + \int_0^l \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi - \sigma \int_0^l \frac{\partial u}{\partial \xi}\frac{\partial^2 u}{\partial \xi \partial t} d\xi\right)\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^5 u}{\partial x^4 \partial t} + \delta\frac{\partial u}{\partial t} = 0$$

where $\alpha, \gamma, \sigma > 0$, $\beta, \delta \in \mathbb{R}$ and with analogous boundary conditions, showing that under some assumptions on the constants the solutions of the problem tend to zero or remain bounded in the solution space.

2. – Position of the problem

We are thus interested in the addition of a linear memory term in equation (1), namely

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} + \int_0^{+\infty} \alpha'(s)\frac{\partial^4 u(x,t-s)}{\partial x^4} ds + \gamma\frac{\partial^5 u}{\partial x^4 \partial t} - \left(\beta + \int_0^l \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u}{\partial x^2} = 0$$
(2)

with the above initial-boundary value conditions.

In order to study the asymptotic behavior of solutions to (2) it is useful to introduce the auxiliary variable

$$w(t,s) = u(t) - u(t-s), \qquad (s \geq 0)$$

so that, setting $\alpha_\infty = \lim_{s\to+\infty}\alpha(s)$ and $\mu(s) = -\alpha'(s)$, an integration by parts shows immediately that

$$\int_0^{+\infty} \alpha'(s)\frac{\partial^4 u(x,t-s)}{\partial x^4} ds = (\alpha_\infty - \alpha(0))\frac{\partial^4 u}{\partial x^4} + \int_0^\infty \mu(s)\frac{\partial^4 w(x,s)}{\partial x^4} ds.$$

Therefore equation (2) becomes a system

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \overline{\alpha}\dfrac{\partial^4 u}{\partial x^4} + \displaystyle\int_0^{+\infty} \mu(s)\frac{\partial^4 w(s)}{\partial x^4} ds + \gamma\frac{\partial^5 u}{\partial x^4 \partial t} - \left(\beta + \int_0^l \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u}{\partial x^2} = 0 \\ \dfrac{\partial w}{\partial t}(t,s) = \dfrac{\partial u}{\partial t}(t) - \dfrac{\partial w}{\partial s}(t,s) \end{cases}$$
(3)

(where $\overline{\alpha} = \alpha_\infty - \alpha(0) + \alpha \geq 0$) with the same boundary conditions for w as for u (in view of the regularity), while the initial condition for w becomes

$$w(0,s) = w_0(s) = u_0 - u(-s)$$

which is supposed to be given.

We need now suitable assumptions on the kernel μ, that is

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu(s) \geq 0, \quad \mu'(s) \leq 0 \ \forall s \geq 0 \tag{4}$$

$$\mu'(s) + \delta\mu(s) \leq 0 \quad \forall s \geq 0 \ \text{ and for some } \ \delta > 0 \tag{5}$$

With the hypotheses given above (at this point it is enough, moreover, to suppose (4)) using standard techniques, it is possible to show that the above problem has a unique solution $u \in L^\infty(0,T;H_0^2)$ such that $\partial u/\partial t \in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^2)$ and $w \in L^2(0,T;L^2_\mu(\mathbb{R}^+;H_0^2))$, where $L^2_\mu(\mathbb{R}^+;H_0^2)$ is just the Hilbert space of functions with values in H_0^2 with weighted scalar product

$$< \phi, \psi >_\mu = \int_0^\infty \mu(s)(\phi,\psi)_{H_0^2}\, ds.$$

Furthermore, if $u_0 \in H_0^2 \cap H^4 := K$, $u_1 \in H_0^2$ and $w_0 \in L^2_\mu(K)$, then the solution is regular in the sense that $u \in L^\infty(0,T;K)$, $\partial u/\partial t \in L^\infty(0,T;H_0^2) \cap L^2(0,T;K)$, $\partial^2 u/\partial t^2 \in L^2(0,T;L^2)$ and finally $w \in L^2(0,T;L^2_\mu(\mathbb{R}^+;K))$.

With these properties, we can define a semigroup

$$S(t) : (u_0, u_1, w_0) \mapsto (u(t), \partial u/\partial t(t), w(t))$$

acting on the space $\mathcal{H} := H_0^2 \times L^2 \times L^2_\mu$.

3. – Results

DEFINITION 1 A semigroup in a Banach space X is said to have an *absorbing set* $\mathcal{A}$ iff for all $x_0 \in X$

$$||x_0||_X \leq R \quad \Rightarrow \quad \exists T \geq 0 : \forall t \geq T : S(t)x_0 \in \mathcal{A}.$$

Intuitively, $\mathcal{A}$ is a set in the phase space into which every trajectory eventually enters.

The main result of this contribution is the following, which generalizes some results of [3].

THEOREM 1 *Under hypotheses* (4)–(5), *the semigroup associated to problem* (3) *possesses an absorbing set in the space* $\mathcal{H}$.

The proof relies on uniform estimates of the solution and its derivatives. More precisely, if $\varepsilon > 0$ and if we set

$$E_\varepsilon(t) = ||u_t + \varepsilon u||^2 + (\alpha_\infty - \delta\varepsilon)||u_{xx}||^2 + \varepsilon^2||u||^2 + ||w||^2_\mu + \frac{1}{2}(\beta + ||u_x||^2)^2$$

($||\cdot||$ is the norm on L^2 and $||\cdot||_\mu$ the norm on L^2_μ), then it is possible to show that for ε sufficiently small, E_ε is a norm on $\mathcal{H}$ and it holds

$$E_\varepsilon(t) \leq E_\varepsilon(0)e^{-\varepsilon t} + \frac{\beta^2}{\varepsilon}.$$

It is therefore clear that for every $\eta > 0$, the sets $\{E < \beta^2/\varepsilon + \eta\}$ are absorbing.

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Balance equations in two-fluid models of helium II

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1. – Introduction

Superfluidity is the property of flowing without viscosity in narrow capillaries or gaps. This property is of quantum character and occurs at temperatures close to absolute zero. However, only helium remains a fluid down to absolute zero and all other fluids solidify well before quantum effects become noticeable. That is why superfluidity is associated with helium.

Superfluidity was discovered by Kapitza in 1938. A theory of liquid helium II was set up by Landau in 1941. He assumed that the energy levels of liquid helium II, not of separate helium atoms, consisted of two sets of overlapping energy states. One state represents the levels of phonons (sound quanta), the other represents the levels of rotons (vortex motions). In this way Landau constructed a quantum hydrodynamics that was satisfactory for the properties of helium II.

F. London envisaged the formation of liquid helium II from liquid helium I at the λ-point as a peculiar type of quantum condensation known as Bose-Einstein condensation. This phenomenon is framed in momentum space where the condensed particles have zero-point energy and momentum. Next Feynman succeeded in deriving, on the basis of quantum statistics, the energy level and energy gap picture of Landau while at the same time retaining the Bose-Einstein condensation idea of London.

Both Landau's and London's points of view were redescribed by Tisza within a phenomenological theory of liquid helium known as the two-fluid model. According to Tisza's papers, liquid helium II is to be viewed as a mixture of two liquids,
one composed of normal atoms with normal viscosity,
one composed of superfluid atoms with zero-point energy and zero entropy and capable of moving through the normal atoms without viscosity.

Though Tisza's picture regards helium II as a mixture of two liquids, it was emphasized that the mixture of two liquids or fluids is no more than a convenient description of the phenomena which occur in a quantum fluid. Indeed, a quantum fluid, such as helium II, can undergo two motions at once each of which involves its own effective mass. One of these motions is normal, namely it has the properties of an ordinary viscous fluid, the other is the motion of a superfluid. The two motions

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occur without the transfer of momentum from one to other. In other words, we might speak of superfluid flow and normal flow without associating them with the components of a mixture. It is essential, though, that the mathematical model accounts for the phenomena observed through experiments.

In this paper we start from the equations of motion derived by F. London through a variational principle [1]. The derivation is briefly reviewed also because London's equations have been reconsidered in [2] and [3] through a quantum approach which traces back to Landau [4]. Next London's equations are framed in a consistent thermodynamic scheme for mixtures of fluids while allowing for heat conduction and viscosity of the normal component. Some consequences are examined for the thermomechanical effects and compared with known results. Also, the propagation modes and speeds are derived for the so-called first, second and fourth sounds.

2. – Variational derivation of the equations of motion

Helium II is regarded as a mixture of two fluids, the normal fluid and the superfluid, occupying a region Ω of $\mathbb{R}^3$. Throughout the subscripts n and s denote the quantities pertaining to the normal fluid and the superfluid. Hence, $\mathbf{v}_n$ and $\mathbf{v}_s$ are the time-dependent velocity fields on, $\Omega \times \mathbb{R}$, while ρ_n and ρ_s are the mass densities. We denote by $\rho = \rho_n + \rho_s$ the mass density of the mixture and let $x = \rho_n/\rho \in [0,1]$ be the fraction of normal fluid. Also we let $t \in \mathbb{R}$ be the time and ∂_t the partial time differentiation.

In view of the mixture model, the continuity equations for the two fluids are written in the form

$$\partial_t \rho_s + \nabla \cdot (\rho_s \mathbf{v}_s) = \chi, \qquad \partial_t \rho_n + \nabla \cdot (\rho_n \mathbf{v}_n) = -\chi$$

as for chemically-reacting constituents. The non-conservation of mass for the constituents is not new. It traces back, e.g., to F. London [1] where the right-hand sides are called source density or (sink density) of the normal fluid and of the superfluid[1] The non-reacting mixture is recovered by letting $\chi = 0$. In any case, the continuity equation for the mixture can be written as

$$\partial_t \rho + \nabla \cdot (\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s) = 0.$$

A backward prime denotes the pertinent material or Lagrangian derivative, e.g., $\dot{\mathbf{v}}_s = \partial_t \mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s$, $\dot{\mathbf{v}}_n = \partial_t \mathbf{v}_n + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n$.

It is perhaps the essential feature of the superfluid that the specific entropy η_s is taken to vanish,

$$\eta_s = 0.$$

Accordingly, the superfluid motion involves no entropy transfer and then no heat transfer. The entropy density η of the mixture is then given by

$$\rho\eta = \rho_n \eta_n.$$

[1] Recently Kosinski and Cimmelli [5] have considered liquid helium as a reacting mixture with internal state variables.

The entropy η is taken to satisfy

$$\partial_t(\rho\eta) + \nabla \cdot (\rho\eta\mathbf{v}_n) = 0,$$

which means that dissipative processes are not allowed in the mixture (helium II).

Let $e(\rho_n, \rho_s, \eta)$ be the internal energy per unit mass and ϕ a given field of potential energy for external body forces. The lagrangian density $\mathcal{L}$ is taken as the kinetic energy minus the potential energy, namely

$$\mathcal{L} = \rho_n \mathbf{v}_n^2/2 + \rho_s \mathbf{v}_s^2/2 - \rho e(\rho_n, \rho_s, \eta) - \rho\phi].$$

The variational principle is then taken as the assertion that

$$\int_{t_0}^{t_1} \int_\Omega \{\mathcal{L} - \alpha[\partial_t \rho + \nabla \cdot (\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)] - \beta[\partial_t(\rho\eta) + \nabla \cdot (\rho\eta\mathbf{v}_n)]\} dv\, dt$$

is stationary for all functions $\rho_n, \rho_s, \eta, \mathbf{v}_n, \mathbf{v}_s$ which take fixed values at the extreme times $t = t_0, t_1$ and at the boundary $\partial\Omega$. The variation with respect to $\rho_n, \rho_s, \eta, \mathbf{v}_n, \mathbf{v}_s$ provide the associated five Euler-Lagrange equations in the form

(1) $$\mathbf{v}_n^2/2 - e - \rho\partial_{\rho_n} e - \phi + \partial_t \alpha + \mathbf{v}_n \cdot \nabla\alpha + \eta(\partial_t \beta + \mathbf{v}_n \cdot \nabla\beta) = 0,$$

(2) $$\mathbf{v}_s^2/2 - e - \rho\partial_{\rho_s} e - \phi + \partial_t \alpha + \mathbf{v}_s \cdot \nabla\alpha + \eta(\partial_t \beta + \mathbf{v}_n \cdot \nabla\beta) = 0,$$

(3) $$\rho(-\partial_\eta e + \partial_t \beta + \mathbf{v}_n \cdot \nabla\beta) = 0,$$

(4) $$\rho_n \mathbf{v}_n + \rho_n \nabla\alpha + \rho\eta\nabla\beta = 0,$$

(5) $$\rho_s(\mathbf{v}_s + \nabla\alpha) = 0.$$

Equation (5) means that $\mathbf{v}_s = -\nabla\alpha$ and hence

$$\nabla \times \mathbf{v}_s = 0,$$

the velocity field of the superfluid is irrotational. Meanwhile if we let

$$T := \partial_\eta e$$

we see from (3) that

$$\partial_t \beta + \mathbf{v}_n \cdot \nabla\beta = T.$$

It is natural to regard T as the temperature of the mixture. Equation (4) gives

$$\nabla\beta = \frac{\rho_n}{\rho\eta}(\mathbf{v}_s - \mathbf{v}_n).$$

Consider (2) and replace $\nabla\alpha$ with $-\mathbf{v}_s$ and $\partial_t \beta + \mathbf{v}_n \cdot \beta$ with T. Apply then the gradient to obtain

$$-\partial_t \mathbf{v}_s - \nabla\mathbf{v}_s^2/2 = \nabla(e + \rho\partial_{\rho_s} e + \phi) - \nabla(\eta T).$$

Incidentally, taking the curl gives

$$\partial_t \nabla \times \mathbf{v}_s = 0$$

and hence the irrotationality of $\mathbf{v}_s$ is consistent with the equation of motion.

Because $\mathbf{v}_s$ is irrotational we have

$$\partial_t \mathbf{v}_s + \nabla \mathbf{v}_s^2/2 = \partial_t \mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = \dot{\mathbf{v}}_s$$

and the equation of motion for the superfluid becomes

$$\dot{\mathbf{v}}_s = \nabla(e + \rho\partial_{\rho_s} e + \phi) - \nabla(\eta T).$$

Now, let $f(\rho_n, \rho_s) := (e + \rho\partial_{\rho_s} e)|_\eta$, namely f is the function $e + \rho\partial_{\rho_s} e$ at constant η. We have

$$\nabla(e + \rho\partial_{\rho_s} e) = \nabla f + (\rho\partial_{\rho_s} T + T)\nabla\eta.$$

In conclusion we have

$$\rho_s \dot{\mathbf{v}}_s = \rho_s \nabla f + \rho_s \rho(\partial_{\rho_s} T)\nabla\eta + \rho_s \eta \nabla T - \rho_s \nabla\phi.$$

To obtain the equation of motion for the normal fluid we observe that, by (1) and (2),

$$-\mathbf{v}_s \cdot \mathbf{v}_n + \mathbf{v}_s^2/2 + \mathbf{v}_n^2/2 + \rho(\partial_{\rho_s} e - \partial_{\rho_n} e) = 0.$$

Now

$$\partial_t \mathbf{v}_n = \partial_t \mathbf{v}_s - \partial_t(\eta_n \nabla\beta) = -\nabla \mathbf{v}_s^2/2 - \nabla(e + \rho\partial_{\rho_s} e + \phi) + \nabla(\eta T) - \nabla\beta\partial_t \eta_n - \eta_n \nabla\partial_t \beta.$$

Adding $(\mathbf{v}_n \cdot \nabla)\mathbf{v}_n$ and suitable manipulations yield

$$\begin{aligned} \rho_n \dot{\mathbf{v}}_n \;=\;& -\rho_n \nabla f - \rho_s \eta \nabla T - \rho_n \rho(\partial_{\rho_s} T)\nabla\eta \\ & + \rho_n[(\mathbf{v}_n \cdot \nabla)\mathbf{v}_n - (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s] + \rho\eta\nabla(\mathbf{v}_n \cdot \nabla\beta) - \nabla\beta[\chi\eta_n - \rho_n(\mathbf{v}_n \cdot \nabla)\eta_n]. \end{aligned}$$

The contribution to $\rho_n \dot{\mathbf{v}}_n$ by the density of mass production χ is

$$\nabla\beta\eta_n\chi = \chi(\mathbf{v}_n - \mathbf{v}_s)$$

as in [1], p. 132. It may be reasonable that $\partial_{\rho_s} T = 0$. Hence, apart from the contributions of χ and of the nonlinear terms in the velocities, it follows that the force on the normal fluid per unit volume is given by $-\rho_n \nabla f - \rho_s \eta \nabla T$.

3. – Remarks about the balance equations

It is worth reviewing the thermodynamical argument followed by Landau [4] to determine the force on the superfluid (cf. [3]). Let

$$dU = TdS - pdV + \mu dM$$

be the differential of the internal energy due to the variations dS, dV and dM of the entropy, volume, and mass. Here μ is the chemical potential, namely the Gibbs free energy per unit mass. Assume that the mass of the region under consideration is changed by keeping the volume constant ($dV = 0$) and adding superfluid ($dS = 0$).

Hence $dU/dM = \mu$. Now consider a mass ΔM of superfluid at $\mathbf{x}_1$ subject to the force $\mathbf{F}(\mathbf{x}_1)\Delta M$. As the mass ΔM is displaced from $\mathbf{x}_1$ to $\mathbf{x}_2$ we have

$$\Delta U = [\mu(\mathbf{x}_2) - \mu(\mathbf{x}_1)]\Delta M = \nabla\mu(\mathbf{x}_1)\cdot(\mathbf{x}_2 - \mathbf{x}_1)\Delta M$$

as $\mathbf{x}_2 \to \mathbf{x}_1$ and $\Delta M \to 0$. The force $\mathbf{F}$, at $\mathbf{x}_1$, is such that

$$\mathbf{F}\cdot(\mathbf{x}_2 - \mathbf{x}_1)\Delta M = -\Delta U = \nabla\mu(\mathbf{x}_1)\cdot(\mathbf{x}_2 - \mathbf{x}_1)\Delta M$$

as $\mathbf{x}_2 \to \mathbf{x}_1$ and $\Delta M \to 0$. Hence we have

$$\mathbf{F} = -\nabla\mu.$$

Moreover, from $\mu = e + p/\rho - T\eta$ and

$$de = -(p/\rho^2)d\rho + Td\eta$$

we have

$$d\mu = (1/\rho)dp - \eta dT$$

which means that

$$1/\rho = \partial_p\mu(p,T), \qquad \eta = -\partial_T\mu(p,T).$$

Consequently we can write the equation[2]

$$\dot{\mathbf{v}}_s = -(1/\rho)\nabla p + \eta\nabla T.$$

Really, Landau regards μ as

$$\mu = \mu_0(p,T) - \frac{\mathbf{P}^2}{2M_nM_s}$$

where M_n, M are the masses of normal fluid and total fluid while $\mathbf{P} = M_n(\mathbf{v}_n - \mathbf{v}_s)$. Hence he takes μ in the form

$$\mu = \mu_0 - \frac{\rho_n}{2\rho}(\mathbf{v}_n - \mathbf{v}_s)^2.$$

We now deal with the case in which both components are present. The velocities are taken to be small and no force of interaction between the two fluids is considered. The entropy of He II is carried by the normal fluid and so the corresponding entropy current density equals $\rho\eta\mathbf{v}_n$. Sometimes[3] the equations of motions are considered for incompressible flow in which ρ_s, ρ_n and η are all constant while $\nabla\cdot\mathbf{v}_s$ and $\nabla\cdot\mathbf{v}_n$ vanish. Here we let the two components be compressible fluids.

Viscosity properties are associated only to the normal fluid and are modelled by allowing for a stress tensor in the form

$$\boldsymbol{\sigma}'_n = 2\nu_n\mathbf{D}_n + \lambda_n(\nabla\cdot\mathbf{v}_n)\mathbf{1}$$

[2] As it stands, the equation is not trivially compatible with $\nabla\times\mathbf{v}_s = 0$.
[3] Cf., e.g. [2].

in addition a pressure stress tensor. The symbol $\mathbf{D}_n$ represents the symmetric part of the velocity gradient, ν_n and λ_n are the viscosity coefficients. The motion of the whole fluid is taken to obey the Navier-Stokes equation

$$\rho_s \dot{\mathbf{v}}_s + \rho_n \dot{\mathbf{v}}_n = -\nabla p + \nu_n \Delta \mathbf{v}_n + (\nu_n + \lambda_n)\nabla(\nabla \cdot \mathbf{v}_n).$$

Once we account for the equation of motion for the superfluid,

$$\rho_s \dot{\mathbf{v}}_s = -(\rho_s/\rho)\nabla p + \rho_s \eta \nabla T,$$

we are left with

$$\rho_n \dot{\mathbf{v}}_n = -(\rho_n/\rho)\nabla p - \rho_s \eta \nabla T + \nu_n \Delta \mathbf{v}_n + (\nu_n + \lambda_n)\nabla(\nabla \cdot \mathbf{v}_n).$$

Hence the motion of the normal fluid is governed by an equation of the Navier-Stokes type but contains an additional term $-\rho_s \eta \nabla T$. This unusual term occurs because $\rho_s \eta \nabla T$ occurs in the equation for the superfluid and because it is not allowed in the equation for the fluid as a whole.

The balance equations are now examined in the context of mixtures of fluids. To this end we review the essential topics.

4. – Entropy inequality for a two-fluid model

According to the results of [1] and [2], the equations of motion for the superfluid and the normal fluid are[4]

$$\rho_s \dot{\mathbf{v}}_s = -(1-x)\nabla p + (1-x)\rho\eta\nabla T + x(1-x)\nabla|\mathbf{v}_n - \mathbf{v}_s|^2/2 + \rho_s \mathbf{f}_s, \tag{6}$$

$$\rho_n \dot{\mathbf{v}}_n = -x\nabla p - (1-x)\rho\eta\nabla T - x(1-x)\nabla|\mathbf{v}_n - \mathbf{v}_s|^2/2 + \rho_n \mathbf{f}_n, \tag{7}$$

and essentially coincide with those proposed by Landau [4]. Here we have inserted the body forces $\rho_s \mathbf{f}_s$ and $\rho_n \mathbf{f}_n$ with a view to the next thermodynamic analysis. If the normal fluid is regarded as viscous then an additional force term $\nu_n \nabla \cdot \mathbf{D}_n + \lambda_n \nabla(\nabla \cdot \mathbf{v}_n)$ has to be inserted in the equation for $\dot{\mathbf{v}}_n$. These equations are compatible with the form for the fluid components of a mixture provided we can write the pressure contributions as $-\nabla p_s$ and $-\nabla p_n$ for suitable partial pressures p_s, p_n. One way to do this is to observe that

$$-(1-x)\nabla p + (1-x)\rho\eta\nabla T = -\nabla(1-x)p + (1-x)\rho\eta\nabla T - p\nabla x$$

and

$$-x\nabla p + -(1-x)\rho\eta\nabla T = -\nabla x p - (1-x)\rho\eta\nabla T + p\nabla x.$$

Hence we can make the identifications

$$p_s = (1-x)p, \quad p_n = xp, \quad \mathbf{m}_s = -\mathbf{m}_n = (1-x)\rho\eta\nabla T - p\nabla x + (1-x)x\nabla|\mathbf{v}_n - \mathbf{v}_s|^2/2.$$

We can then exploit the balance of energy and entropy as follows. Let $\mathbf{q}_s = 0$ and

$$\boldsymbol{\sigma}_s = -p_s \mathbf{1}, \qquad \boldsymbol{\sigma}_n = -p_n \mathbf{1} + 2\nu_n \mathbf{D}_n + \lambda_n(\nabla \cdot \mathbf{v}_n)\mathbf{1}.$$

[4]We omit the body force term $-\nabla\phi$ because it is inessential in our considerations.

The balances of energy become

$$\rho_s\dot{\epsilon}_s = -p_s\nabla\cdot\mathbf{v}_s + l_s + \rho_s(\mathbf{f}_s\cdot\mathbf{v}_s + r_s) - \mathbf{m}_s\cdot\mathbf{v}_s - \chi(\epsilon_s - \mathbf{v}_s^2),$$

$$\begin{aligned}\rho_n\dot{\epsilon}_n = -p_n\nabla\cdot\mathbf{v}_n - \mathbf{m}_n\cdot\mathbf{v}_n + l_n + \rho_n(\mathbf{f}_n\cdot\mathbf{v}_n + r_n) + \chi(\epsilon_n - \mathbf{v}_n^2/2)\\ +2\nu_n\mathbf{D}_n\cdot\mathbf{D}_n + \lambda_n(\nabla\cdot\mathbf{v}_n)^2 - \nabla\cdot\mathbf{q}_n\end{aligned}$$

where $l_n + l_s = 0$. By replacing $\nabla\cdot\mathbf{v}_n$ and $\nabla\cdot\mathbf{v}_s$ through the continuity equations we obtain the entropy inequality in the form

$$\begin{aligned}(8)\ -\rho_s\dot{\psi}_s - \rho_n(\dot{\psi}_n + \eta_n\dot{T}_n) + p_s\dot{\rho}_s/\rho_s + p_n\dot{\rho}_n/\rho_n + 2\nu_n\mathbf{D}_n\cdot\mathbf{D}_n + \lambda_n(\nabla\cdot\mathbf{v}_n)^2\\ +p(\mathbf{v}_s - \mathbf{v}_n)\cdot\nabla x + x(1-x)(\mathbf{v}_n - \mathbf{v}_s)\cdot\nabla|\mathbf{v}_n - \mathbf{v}_s|^2/2\\ -T^{-1}\nabla T\cdot[\mathbf{q}_n + \gamma(\mathbf{v}_s - \mathbf{v}_n)] + \chi(\mu_n - \mu_s) \geq 0\end{aligned}$$

where

$$\gamma = T(1-x)\rho\eta, \qquad \mu_n = \psi_n + p_n/\rho_n - \mathbf{v}_n^2/2$$

and analogously for μ_s. Since $p_n/\rho_n = p_s/\rho_s = p/\rho$ it follows that

$$\mu_n - \mu_s = \psi_n - \psi_s - \mathbf{v}_n^2/2 + \mathbf{v}_s^2/2.$$

5. – Thermodynamic restrictions

To obtain thermodynamic restrictions, as a consequence of the entropy inequality (8), we need specifying possible constraints. Both Bose-Einstein condensation (cf. [1], p. 40) and Andronikashvili's experiment (cf. [2], p. 7) indicate that, as $T < T_\lambda$, ρ_s/ρ and ρ_n/ρ are known monotone functions of T. Let

$$\rho_s/\rho = 1 - x = h(T).$$

Experiments show that $h' := dh/dT \leq 0$. This constraint results in a restriction on the mass production χ. Time differentiation of $\rho_s/(\rho_s + \rho_n) = h(T)$ yields

$$(1+h)\partial_t\rho_s + h\partial_t\rho_n = h'\partial_t T.$$

Comparison with the continuity equations in the form

$$\partial_t\rho_s + \nabla\cdot(\rho_s\mathbf{v}_s) = \chi, \qquad \partial_t\rho_n + \nabla\cdot(\rho_n\mathbf{v}_n) = -\chi$$

allows us to express χ in the form

$$\chi = h'\partial_t T - (1+h)\nabla\cdot(\rho_s\mathbf{v}_s) - h\nabla\cdot(\rho_n\mathbf{v}_n).$$

This means that, given any pair of velocity functions $\mathbf{v}_s, \mathbf{v}_n$ and temperature T, the mass production χ and the derivatives $\partial_t\rho_s$, $\partial_t\rho_n$ are determined.

The occurrence of $\mathbf{f}_s, \mathbf{f}_n, r_s, r_n$, though practically inessential, allows us to say that the balance equations for momentum and energy place no restriction on the pertinent fields.

Sufficient conditions for the inequality (8) to hold are that

$$\eta_n = -\partial_T \psi_n, \quad p_n = \rho_n^2 \partial_{\rho_n} \psi_n, \quad p_s = \rho_s^2 \partial_{\rho_s} \psi_s$$

and

$$2\nu_n \mathbf{D}_n \cdot \mathbf{D}_n + \lambda_n (\nabla \cdot \mathbf{v}_n)^2 + x(1-x)(\mathbf{v}_n - \mathbf{v}_s) \cdot \nabla |\mathbf{v}_n - \mathbf{v}_s|^2/2$$
$$-T^{-1}\nabla T \cdot [\mathbf{q}_n + \zeta(\mathbf{v}_s - \mathbf{v}_n)] + \chi(\mu_n - \mu_s) \geq 0$$

where $\zeta := (\rho_s \eta + p h')T$. To investigate the possible negative definiteness of the term

$$\nabla T \cdot [\mathbf{q}_n + \zeta(\mathbf{v}_s - \mathbf{v}_n)]$$

we observe that the vector

$$\boldsymbol{\phi} = \sum_\alpha \rho_\alpha T \eta_\alpha (\mathbf{v}_\alpha - \mathbf{v})$$

can be viewed as a flux due to diffusion. Since $\eta_s = 0$ then

$$\boldsymbol{\phi} = \rho_n T \eta_n (\mathbf{v}_n - \mathbf{v}).$$

Because

$$\mathbf{v}_n - \mathbf{v} = \mathbf{v}_n - \frac{\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s}{\rho_n + \rho_s} = \frac{\rho_s}{\rho}(\mathbf{v}_n - \mathbf{v}_s)$$

we have

$$\boldsymbol{\phi} = \rho_s \eta T (\mathbf{v}_n - \mathbf{v}_s).$$

Consequently $\boldsymbol{\phi}/T$ is the convective entropy flux.

Concerning the constitutive equation for $\mathbf{q}_n$ there are a few simple cases. First,

$$0 = \tilde{\mathbf{q}} := \mathbf{q}_n - \zeta(\mathbf{v}_n - \mathbf{v}_s)$$

in which case the contribution to the entropy inequality is zero. Second,

$$\tilde{\mathbf{q}} = -\kappa \nabla T, \qquad \kappa > 0,$$

as for a Fourier-like case. Third,

$$-\nabla T = \tilde{\mathbf{q}}/\kappa + (\tau_0/\kappa)\partial_t \tilde{\mathbf{q}}$$

in which case the contribution to the entropy inequality is $\tilde{\mathbf{q}}^2/\kappa + (\tau_0/\kappa)\partial_t \tilde{\mathbf{q}}^2$. It is worth mentioning that, e.g. in [4] and [6],

$$\mathbf{q}_n = \rho T \eta \mathbf{v}_n$$

which equals $T\rho_s \eta(\mathbf{v}_n - \mathbf{v}_s)$ if $\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n = 0$ namely when the net mass flux is zero.

6. – Description of the main effects

Since the ∇T terms are non-standard in the equations of motion, it is of interest to check that the two-fluid model is consistent with the experiments involving thermal gradients.

Fountain effect. As observed first by Allen and Jones in 1938, the level of liquid in a vessel, communicating through a narrow capillary to a surrounding bath of liquid, raises above that of the surrounding bath when heat is supplied to the liquid in the vessel. Let T be the temperature in the vessel and T_0 that in the surrounding. To fix ideas let $T > T_0$. The superfluid in the vessel is subject to the equation of motion

$$\rho_s \dot{\mathbf{v}}_s = -(\rho_s/\rho)\nabla p + \rho_s \eta \nabla T.$$

It is likely that at the beginning $\rho_s \eta \nabla T$ prevails. Anyway the superfluid flow stops provided that

$$-\nabla p + \rho \eta \nabla T = 0.$$

This means that, along the capillary,

$$\frac{dp}{dT} = \rho \eta,$$

which is called the *fountain effect equation* (H. London 1939) [7]; cf. [1], §12. Incidentally, this equation provides a way for evaluating entropy from data on dp/dT.

Mechano-caloric effect. Two reservoirs, A and B, are connected by a narrow capillary and are enclosed in a constant-temperature bath at the temperature T. An excess pressure is applied to A, $p_A > p_B$. The pressure gradient so established makes the superfluid in A to pass through the capillary into B. Of course the superfluid carries no entropy. The liquid already in B possesses entropy $\rho\eta$ per unit volume; to bring the newly arrived superfluid into equilibrium an amount of heat Q equal to $\rho\eta T$ times the change of volum must be supplied to vessel B. Conversely, the same amount of heat must be extracted from A so that the liquid there remains at the same temperature. Thus the transfer of superfluid at constant temperature with the normal fluid stationary is accompanied by a flow of heat. This effect was checked experimentally by Brewer and Edwards (1958) [8].

Connected reservoirs. Two resevoirs A and B are connected by a channel in which the normal fluid can flow as well as the superfluid. If the temperature in B is raised then the normal fluid is subject to the force

$$-(\rho_n/\rho)\nabla p - \rho_s \eta \nabla T.$$

In stationary conditions $\nabla p = \rho \eta \nabla T$ and hence the force equals

$$-\rho \eta \nabla T.$$

Along the channel, the force, per unit length is $-\rho \eta dT/dx$. According to Poiseuille's law, the volume flow rate in A is

$$\partial_t V_n = \frac{\beta}{\nu_n} \rho \eta \frac{dT}{dx}$$

where β is a geometrical factor. Meanwhile the superfluid, in A, is subject to the opposite force which forces it to flow from A to B. Since the superfluid carries no entropy, the net entropy change in A is given by

$$\frac{1}{T}\partial_t Q = \frac{\beta}{\nu_n}(\rho\eta)^2\frac{dT}{dx}.$$

This equation traces back to F. London and Zilsel (1948) [6].

If $-\rho_s\eta\nabla T$ does not occur in the equation of motion for the normal fluid then the force is $-(\rho_n/\rho)\nabla p$ and $dp/dT = \rho\eta$ at equilibrium. Hence the force would be

$$-(\rho_n/\rho)\nabla p = -\rho_n\eta\nabla T$$

rather than $-\rho\eta\nabla T$.

7. – The propagation of sound

We examine the propagation of sound in helium II by neglecting viscosity. The linear approximation is used throughout. In particular we let

$$\rho_s\partial_t\mathbf{v}_s = -(\rho_s/\rho)\nabla p_s + \rho_s\eta\nabla T, \qquad \rho_n\partial_t\mathbf{v}_n = -(\rho_n/\rho)\nabla p - \rho_s\eta\nabla T$$

and write the continuity equation in the form

$$\partial_t\rho = -\nabla\cdot(\rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n).$$

Summing the equations of motion we have

$$\rho_n\partial_t\mathbf{v}_n + \rho_s\partial_t\mathbf{v}_s = -\nabla p$$

or $\partial_t\mathbf{j} = -\nabla p$. Comparison gives

$$\partial_t^2\rho = \Delta p.$$

If p is regarded as a function of ρ and η then

$$\partial_t^2\rho = \frac{\partial p}{\partial\rho}\Delta\rho + \frac{\partial p}{\partial\eta}\Delta\eta.$$

Multiplying the equations of motion by ρ_n and ρ_s, respectively, and taking the difference we have

$$\rho_n\rho_s\partial_t(\mathbf{v}_n - \mathbf{v}_s) = -(\rho_n + \rho_s)\rho_s\eta\nabla T.$$

We now consider the (linearized) balance of energy as

$$\rho\partial_t\epsilon = -\nabla\cdot\mathbf{q}_n.$$

Also $\mathbf{q}_n = \tilde{\mathbf{q}} + \zeta(\mathbf{v}_n - \mathbf{v}_s)$ and

$$\tilde{\mathbf{q}} + \tau\partial_t\tilde{\mathbf{q}} = -\kappa\nabla T.$$

If $\epsilon = \epsilon(\rho, T)$ we have

$$\partial_t \epsilon = \frac{\partial \epsilon}{\partial \rho} \partial_t \rho + \frac{\partial \epsilon}{\partial T} \partial_t T$$

and hence the higher-order derivatives result in

$$\tau \rho (\frac{\partial \epsilon}{\partial \rho} \partial_t^2 \rho + \frac{\partial \epsilon}{\partial T} \partial_t^2 T) = (\kappa + \tau \zeta \eta T \frac{\rho}{\rho_n}) \Delta T.$$

If also we let η be a function of ρ and T then we have a system of two equations in two unknowns. Let c denote the unknown speed, so that, e.g., $\partial_t^2 \rho = c^2 \Delta \rho$ and look for approximate solutions.

First sound. It corresponds to vanishing amplitudes for η and ∇T and hence $\mathbf{v}_n = \mathbf{v}_s$. In such a case a speed c_1 occurs such that

$$c_1^2 = \frac{\partial p}{\partial \rho}(\rho, \eta),$$

Second sound. Because the ratio ρ_n / ρ_s depends (strongly) on the temperature, a rapid local temperature fluctuation is likely to originate a rapid local variation of ρ_n / ρ_s without altering the sum $\rho = \rho_n + \rho_s$. An ordinary sound wave is the propagation of fluctuations of ρ (first sound). Fluctuations of ρ_n / ρ_s without change of ρ were predicted by Tisza and Landau and are called second sound. Let $C_m = \partial \epsilon / \partial T$ be the specific heat (per unit mass). Taking only the terms in the perturbation of T we have

$$c_2^2 = \frac{1}{C_m} \Big[\frac{\kappa}{\tau \rho} + \frac{\rho_s \eta + p h'}{\rho_n} T \eta \Big].$$

If $\kappa = 0, h' = 0$ then the second sound speed c_2 is given by

$$c_2^2 = \frac{\eta^2 T}{C_m} \frac{\rho_s}{\rho_n}.$$

This simple expression for the speed of second sound was applied by Klerk, Hudson and Pellam to determine the temperature variation of ρ_n / ρ and hence the function h.

Fourth sound. In narrow capillaries it is possible that the wavelength of the fluctuations becomes comparable to or greater than the diameter of the pipe. In such a case the normal fluid is stationary and the sound propagation is due to the superfluid. These fluctuations are called fourth sound after Atkins. To find the speed of propagation we can argue as follows.

Since $\mathbf{v}_n = 0$ we have

$$\partial_t \rho + \rho_s \nabla \cdot \mathbf{v}_s = 0.$$

Comparison with the equation of motion for the superfluid gives

$$\partial_t^2 \rho = (\rho_s / \rho) \Delta p - \rho_s \eta \Delta T.$$

The approximation $\partial_t(\rho \eta) = 0$ and neglect of thermal expansion $(\partial_T \rho \simeq 0, \partial_p \eta \simeq 0)$ yields

$$\Delta T = -(\eta T / \rho c_1^2 C_m) \Delta p.$$

Substitution and account of $\partial_t^2 \rho = \partial_t^2 p / c_1^2$, $\Delta p = \partial_t^2 p / c_4^2$ yield

$$c_4^2 = \frac{\rho_s}{\rho} c_1^2 + \frac{\rho_n}{\rho} c_2^2.$$

8. – Comments

This paper provides a continuum account of a two-fluid model of helium II. Also, the condition is considered that the mass densities are given functions of the temperature. This constraint has consequences on the thermodynamic restrictions and on the speed of waves, specifically of second sound. It is at least of interest to investigate if the connection between mass densities and temperature may more conveniently be modelled in another way. This aspect is the subject of a future research.

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Thermoelastic plate with thermal interior control

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1. – Introduction

In this note we consider a thermoelastic plate in the light of [6]. The resulting model consists of an elastic motion equation and a heat equation, which are coupled in such a way that the energy transfer between them is taken into account. Transverse shear effects are neglected, and the plate is hinged on its edge. Accounting for thermal effects, we assume that the heat flux law involves only the temperature gradient by the Fourier law. The main result of this paper is to provide the null controllability for the model when the interior control (heat source) acts in the thermal equation.

The variable u represents the *vertical deflection* of a plate occupying a two dimensional domain Ω with a sufficiently smooth boundary. The variable θ describes the *variation of temperature* of the plate with respect to its reference temperature. Let ω be any open subset of Ω, $T > 0$ and set $Q := (0,T) \times \Omega$, $\Sigma := (0,T) \times \partial\Omega$. In absence of exterior forces, and with hinged mechanical and Dirichlet thermal boundary conditions, the system to look at is given by

$$\left\{ \begin{array}{ll} u_{tt} + \Delta^2 u + \Delta\theta = 0 & \text{in } Q \\ \theta_t - \Delta\theta - \Delta u_t = f & \text{in } Q \\ u = 0, \ \Delta u = 0, \ \theta = 0 & \text{on } \Sigma \\ u(0) = u^0 \, , \ u_t(0) = u^1 \, , \ \theta(0) = \theta^0 & \text{on } \Omega. \end{array} \right. \tag{1}$$

where the *control function* $f \in L^2((0,T) \times \omega)$. The subscript $\cdot_t$ denotes time derivative, and u^0, u^1, θ^0 are initial data in a suitable space.

Two results are obtained (see [5]). Firstly, we study the case when $\omega \equiv \Omega$, and we find the null controllability at any time $T > 0$. Then, we prove the same result in the case $\omega \subset\subset \Omega$ and the closure of ω does not intersect the boundary of Ω.

It will be said that (see, for instance, [7, 13]) that a system is *exactly controllable* at given time $T > 0$ if it can be driven from any state to any state belonging to the same space of states where the system evolves. A system is *null controllable* at time

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$T > 0$ if an arbitrary state can be transferred to 0 in time T, or, equivalently any state can be joined to any trajectory (e.g. attainability of the trajectories). The null controllability does not yield the exact controllability of the system (see, for instance, the heat equation with distributed control in the domain Ω [13]).

2. – Preliminaries

We introduce the Hilbert space $H := (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$ equipped with the inner product

$$\langle z_1, z_2 \rangle_H = \int_\Omega (\Delta u_1 \cdot \Delta u_2 + v_1 \cdot v_2 + \theta_1 \cdot \theta_2)\, dx\,, \quad \text{where } z_i = \begin{bmatrix} u_i \\ v_i \\ \theta_i \end{bmatrix}, \; i = 1, 2.$$

The induced norm is denoted by $\|\cdot\|_H$. Putting $v = u_t$ and $z(t) = \begin{bmatrix} u(t) \\ v(t) \\ \theta(t) \end{bmatrix}$, $z^0 = \begin{bmatrix} u^0 \\ v^0 \\ \theta^0 \end{bmatrix}$, problem (1) can be rewritten as an abstract linear evolution equation in H of the form

$$\text{(2)} \qquad \begin{cases} z_t = A\,z + B\,f \\ z(0) = z^0 \in H \end{cases}$$

where we set the operator $A : D(A) \to H$ by

$$\text{(3)} \qquad A = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{bmatrix}$$

with domain $D(A) = \{z \in H : \; \Delta u, v, \theta \in H^2(\Omega) \cap H_0^1(\Omega)\}$, and the control operator $B : L^2(\omega) \to H$ by

$$\text{(4)} \qquad B\,f = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}.$$

Given $T > 0$, the problem of the null controllability of system (2) consists in to prove that, for any $z^0 \in H$, there exists a control $f \in L^2((0,T) \times \omega)$ such that the solution $z(t; z^0, f)$ of (2) satisfies $z(T; z^0, f) = 0$. This property is equivalent to (see for instance [13], Theorem 2.6, p. 213): there exists a positive constant C_T such that

$$\text{(5)} \qquad \|e^{A^*T} y^0\|_H^2 \le C_T \int_0^T \|B^*\, e^{A^*t} y^0\|_{L^2(\omega)}^2 dt\,, \quad \forall y^0 \in H\,.$$

We compute

$$A^* = \begin{bmatrix} 0 & -I & 0 \\ \Delta^2 & 0 & \Delta \\ 0 & -\Delta & \Delta \end{bmatrix}$$

with domain $D(A^*) = D(A)$, and $B^* = \begin{bmatrix} 0 & 0 & I \end{bmatrix}$. The adjoint system with respect to (1) is

$$(6) \qquad \begin{cases} \varphi_{tt} + \Delta^2\varphi + \Delta w = 0 & \text{in } Q \\ w_t - \Delta w - \Delta\varphi_t = 0 & \text{in } Q \\ \varphi = 0, \ \Delta\varphi = 0, \ w = 0 & \text{on } \Sigma \\ \varphi(0) = \varphi^0\,, \ \varphi_t(0) = \varphi^1\,, \ w(0) = w^0 & \text{on } \Omega. \end{cases}$$

Its solution can be written as

$$(7) \qquad \begin{bmatrix} \varphi(t) \\ \varphi_t(t) \\ w(t) \end{bmatrix} = e^{A^*t} \begin{bmatrix} \varphi^0 \\ \varphi^1 \\ w^0 \end{bmatrix}$$

and

$$B^*\, e^{A^*t} \begin{bmatrix} \varphi^0 \\ \varphi^1 \\ w^0 \end{bmatrix} = w(t)\,.$$

Then, condition (5) is equivalent to require that there exists a positive constant C_T such that

$$(8) \qquad \|\Delta\varphi(T)\|^2_{L^2(\Omega)} + \|\varphi_t(T)\|^2_{L^2(\Omega)} + \|w(T)\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|w(t)\|^2_{L^2(\omega)} dt$$

for any solution (7) of system (6).

3. – Results

Our main result is

THEOREM 1 *Problem* (1) *is null controllable at any time* $T > 0$ *on the space* H *within the class of* $L^2((0,T) \times \omega)$*-controls, when*

(a) $\omega \equiv \Omega$;

(b) $\omega \subset\subset \Omega$ *and the closure of* ω *does not intersect the boundary of* Ω.

REMARK 1 (a) *In the case of* Theorem 1-(a), *an analogous result was obtained by Lasiecka and Triggiani in* [8]. *Our procedure* [2] *is supported by introducing a quadratic function depending on the time. Multipliers method is applied to construct this function* [1, 3, 4].

(b) *In the case of* Theorem 1-(b), *by applying an iterative method and the observability estimates on the eigenfunctions of the Laplacian operator due to Lebeau and Robbiano in* [10] *(see also* [11]*), we show that system* (1) *is null controllable at any time* $T > 0$. *In our proof, the analyticity property of semigroup associated to the thermoelastic system (recall* $\gamma = 0$, *see Lasiecka and Triggiani* [9]*), and the commutative property of the operators, which comes from the hinged boundary conditions, are crucial.*

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A non-stationary model in superconductivity

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1. – Introduction

In this paper we analyze the Ginzburg-Landau model for the superconductivity. The first part is devoted to the stationary problem, while in the second part we consider the time-dependent case. In both cases, we formulate the system of equations by means of observable variables, such as the density of the superconducting electrons $f = |\psi|$, the magnetic field $\mathbf{H}$ or the superconducting current $\mathbf{J}_s$, instead of the complex order parameter ψ and the vector and scalar potential $\mathbf{A}$ and ϕ and for the time-dependent problem we obtain some uniqueness results.

The approach we have followed allows a better macroscopic interpretation of the phenomenon and therefore gives the possibility to study the compatibility between the model and the principles of Thermodynamics. From this analysis it comes out that the time-dependent system of equations, introduced by Gor'kov and Eliashberg, needs to be studied in a deeper way. In fact, in this model the electric field $\mathbf{E}$ is supposed to be such that $\dfrac{\partial \mathbf{E}}{\partial t}$ is negligible, so that the Ampere equation is written as

$$\nabla \times \mathbf{H} = \mathbf{J} = \mathbf{J}_n + \mathbf{J}_s, \tag{1}$$

where $\mathbf{J}$ and $\mathbf{J}_n$ are respectively the total and the normal currents. As a consequence of (1) and of the continuity equation, we have that

$$\dot{\rho} = -\nabla \cdot \mathbf{J} = 0,$$

therefore the charge density ρ results constant in time, that is

$$\rho(\mathbf{x}, t) = \rho_0(\mathbf{x}),$$

ρ_0 being the initial charge density. Moreover, since $\epsilon \nabla \cdot \mathbf{E} = \rho_0$, we get

$$\nabla \cdot \mathbf{J}_s = \frac{\sigma}{\epsilon} \rho_0.$$

This new relation implies a slight modification to the system proposed by Gor'kov and Eliashberg, which we present in the last section of this paper.

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2. – Steady case

Two main phenomena characterize the superconductivity: the absence of electrical resistence in stationary conditions and the so called Meissner effect, which consists in the expulsion of the magnetic field from the superconductor whenever the temperature is below a critical temperature T_c, characteristic of the considered metal.

These two phenomena are well described by the two-fluids model of London ([8, 7]), which makes use of the Maxwell equations and assumes the electric current $\mathbf{J}$ as the sum of a normal component $\mathbf{J}_n$ and a superconducting component $\mathbf{J}_s$, i.e.

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s.$$

Since $\mathbf{J}_n$ represents the flux of the current due to the normal electrons, from the Ohm law we have

$$\mathbf{J}_n = \sigma \mathbf{E},$$

where σ is the conductivity. The constitutive law for $\mathbf{J}_s$ is given by the London equation ([7])

$$\nabla \times \Lambda \mathbf{J}_s = -\mu \mathbf{H},$$

Λ being a scalar coefficient defined by the fraction

$$\Lambda = \frac{m^*}{e^{*2} n_s}$$

where m^*, e^*, n_s are respectively the mass, the charge and the relative density of the superconducting electrons. Whenever $T > T_c$, i.e. when we are in presence of the normal state, then $n_s = 0$, while $n_s = 1$ if $T = 0$.

This model allows to study the superconductivity when the temperature T is constant and far from the critical temperature T_c, i.e. $T < T_c$. In this case the relative density n_s can be assumed to be constant and the evolution problem in a sufficiently regular domain Ω is governed by the differential system

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \Lambda \mathbf{J}_s &= -\mu \mathbf{H} \\ \mathbf{J} &= \mathbf{J}_s + \sigma \mathbf{E} \end{aligned}$$

together with the boundary conditions

$$\mathbf{E} \cdot \mathbf{n}_{|\partial\Omega} = 0, \qquad \mathbf{H} \times \mathbf{n}_{|\partial\Omega} = \mathbf{H}_{ex} \times \mathbf{n}.$$

REMARK 1 The last boundary condition means that $\mathbf{H} \times \mathbf{n}$ is continuous across the boundary. Hence, $\mathbf{H}_{ex} \in \mathcal{R}(\Omega) = \{\mathbf{A} \in L^2(\Omega) : \nabla \times \mathbf{A} \in L^2(\Omega)\}$ is the applied magnetic field outside the superconducting sample and $\mathbf{H}_{ex} \times \mathbf{n}$ stands for its trace.

When the temperature T is close to the critical temperature T_c or when the magnetic field is close to the critical value $\mathbf{H}_c$ even if $T < T_c$, we are in presence of an intermediate phase between the normal and the superconducting state. This phase is described by the time independent Ginzburg-Landau equations ([5])

$$-i\frac{\hbar e^*}{2m^*}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{{e^*}^2}{m^*}|\psi|^2\mathbf{A} = \frac{1}{\mu}\nabla\times\nabla\times\mathbf{A}, \tag{2}$$

$$\frac{1}{2m^*}(i\hbar\nabla + e^*\mathbf{A})^2\psi - a\psi + b|\psi|^2\psi = 0, \tag{3}$$

where $\hbar$ is the Planck's constant, the complex function $\psi = f\exp(i\theta)$ is such that its squared modulus f^2 represents the relative density n_s of the superconducting electrons of mass m^* and charge e^*, and $\mathbf{A}$ stands for the vector potential of the magnetic induction $\mathbf{B}$, for which

$$\mathbf{B} = \mu\mathbf{H} = \nabla\times\mathbf{A}, \qquad \mathbf{A}\cdot\mathbf{n}_{|\partial\Omega} = 0. \tag{4}$$

In the Ginzburg-Landau theory equations (2), (3) are obtained as stationary points of the Gibbs' free energy

$$E(\Omega) = \int_\Omega \left[\frac{1}{2m^*}|i\hbar\nabla\psi + e^*\mathbf{A}\psi|^2 + \frac{1}{2\mu}|\nabla\times\mathbf{A}|^2 - a|\psi|^2 + \frac{b}{2}|\psi|^4\right] dx - \int_{\partial\Omega}\mathbf{A}\times\mathbf{H}_{ex}\cdot\mathbf{n}\, da. \tag{5}$$

System (2)-(3) must be combined with Maxwell equations which, in the stationary case, can be written as

$$\nabla\times\mathbf{E} = \mathbf{0}\ , \qquad \nabla\cdot\mathbf{E} = 0, \tag{6}$$

$$\nabla\times\mathbf{H} = \mathbf{J}_s + \sigma\mathbf{E}\ , \qquad \nabla\cdot\mathbf{H} = 0, \tag{7}$$

and the following boundary condition, which is typical of the superconductivity,

$$\mathbf{E}\cdot\mathbf{n}_{|\partial\Omega} = 0 \tag{8}$$

must be added.

Equations (6) and (8) give the solution $\mathbf{E} = \mathbf{0}$ on Ω. Therefore system (2)-(3) must be associated to the equations (4), $(7)_1$, which becomes

$$\nabla\times\mathbf{H} = \mathbf{J}_s, \tag{9}$$

and to the boundary conditions

$$(i\hbar\nabla + e^*\mathbf{A})\psi\cdot\mathbf{n}_{|\partial\Omega} = 0, \quad (\nabla\times\mathbf{A})\times\mathbf{n}_{|\partial\Omega} = \mu\mathbf{H}_{ex}\times\mathbf{n}, \tag{10}$$

that can be obtained from the stationarity of the functional (5).

System (2)-(3) can be rewritten in the non-dimensional form

$$\mathbf{J}_s = -\frac{i}{\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2\mathbf{A}, \tag{11}$$

$$0 = \left(\frac{i}{2\kappa}\nabla + \mathbf{A}\right)^2\psi - \psi + |\psi|^2\psi, \tag{12}$$

where $\kappa^2 = \dfrac{bm^{*2}}{\hbar^2 e^{*2}}$. Moreover it should be invariant under gauge transformations of the type

$$\mathbf{A}' = \mathbf{A} + \frac{1}{\kappa}\nabla\chi, \qquad \psi' = \psi\exp(i\kappa\chi). \tag{13}$$

In [3] it has been proved that, starting from the free energy

$$\begin{aligned}\int_\Omega \tilde{e}\, d\,x \;=\; & \frac{1}{2}\int_\Omega \left[\frac{1}{2}f^4 - f^2 - \mu\mathbf{H}^2 + \frac{1}{\kappa^2}|\nabla f|^2 - f^{-2}|\nabla\times\mathbf{H}|^2\right] d\,x \\ & - \int_{\partial\Omega} f_a^{-2}(\nabla\times\mathbf{H})\times\mathbf{H}_{ex}\cdot\mathbf{n}\; d\,a,\end{aligned} \tag{14}$$

where f_a represents the assigned value of $f = |\psi|$ on $\partial\Omega$, it is possible to obtain a system, which is equivalent to (11)-(12) but is written in terms of the real variables f and $\mathbf{H} = \dfrac{1}{\mu}\nabla\times\mathbf{A}$.

The variations of (14) with respect to f and $\mathbf{H}$ lead to the system

$$\nabla\times(f^{-2}\nabla\times\mathbf{H}) \;=\; -\mu\mathbf{H}, \tag{15}$$

$$-\frac{1}{\kappa^2}\nabla^2 f + f^{-3}|\nabla\times\mathbf{H}|^2 - f + f^3 \;=\; 0, \tag{16}$$

and to the boundary conditions

$$\nabla f\cdot\mathbf{n}_{|\partial\Omega} = 0, \quad \mathbf{H}\times\mathbf{n}_{|\partial\Omega} = \mathbf{H}_{ex}\times\mathbf{n}. \tag{17}$$

Thanks to Maxwell equation (9) and by putting $\mathbf{p}_s = f^{-2}\mathbf{J}_s$, problem (15)-(17) can be rewritten as follows:

$$\nabla\times\nabla\times\mathbf{p}_s \;=\; -\mu f^2\mathbf{p}_s, \tag{18}$$

$$-\frac{1}{\kappa^2}\nabla^2 f + f\mathbf{p}_s^2 - f + f^3 \;=\; 0, \tag{19}$$

$$\nabla f\cdot\mathbf{n}_{|\partial\Omega} = 0, \qquad (\nabla\times\mathbf{p}_s)\times\mathbf{n}_{|\partial\Omega} \;=\; \mu\mathbf{H}_{ex}\times\mathbf{n}. \tag{20}$$

As observed in [3, 4], system (18)-(20) is equivalent to the system given by (10)-(12). In fact, equation (18) is exactly equation (11), while equation (19) is the real part of the coefficient of $\exp(i\theta)$ in equation (12).

As for the imaginary part of the coefficient of $\exp(i\theta)$ in (12), it gives the relation

$$\nabla\cdot\left(f^2(\frac{1}{\kappa}\nabla\theta + \mathbf{A})\right) = \nabla\cdot\mathbf{J}_s = 0. \tag{21}$$

This last equation is a direct consequence of the Maxwell equation (9).

3. – Time-dependent problem

The generalization of the Ginzburg-Landau theory to the non-stationary case has been proposed by Schmid ([9]) and then developed by Gor'kov and Éliashberg ([6]) in the context of the BCS theory. This model holds under the hypothesis that the temperature T be close to the critical temperature T_c. The Maxwell equations for this evolutive problem are written as

$$\nabla \times \mathbf{H} = \mathbf{J} \quad , \qquad \nabla \cdot \mathbf{H} = 0, \tag{22}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad , \qquad \epsilon \nabla \cdot \mathbf{E} = \rho, \tag{23}$$

and the two-fluids model is considered for the current $\mathbf{J}$, so that

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s. \tag{24}$$

Actually, equation $(22)_1$ is obtained under the hypothesis of a slowly variating in time electric field $\mathbf{E}$, in such a way that $\dfrac{\partial \mathbf{E}}{\partial t}$ is negligible.

In the theory developed by Gor'kov and Éliashberg ([6, 10]), system (22)-(23) is combined with the equations

$$\gamma \left(\frac{\partial \psi}{\partial t} + i\kappa \Phi \psi \right) = -\left(\frac{i}{\kappa} \nabla + \mathbf{A} \right)^2 \psi + \left(1 - |\psi|^2 \right) \psi, \tag{25}$$

$$\mathbf{J}_s = -\frac{i}{2\kappa} (\psi^* \nabla \psi - \psi \nabla \psi^*) - |\psi|^2 \mathbf{A}, \tag{26}$$

where γ is a positive relaxation coefficient, while the function Φ represents the scalar potential of the electromagnetic field. Therefore $\mathbf{A}$ and Φ are such that

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi. \tag{27}$$

As one can see, equation (26) is the same as (2) and also the associated boundary conditions do not change from the ones in the stationary case, that is (10) still holds. The initial conditions are

$$\psi(\mathbf{x}, 0) = \psi_0(\mathbf{x}), \qquad \mathbf{A}(\mathbf{x}, 0) = \mathbf{A}_0(\mathbf{x}). \tag{28}$$

System (25)-(26) must be invariant for gauge transformations of the type

$$\psi' = \psi \exp(i\kappa\chi), \quad \mathbf{A}' = \mathbf{A} + \nabla \chi, \quad \Phi' = \Phi - \frac{\partial \chi}{\partial t},$$

where χ is an arbitrary and regular function of space and time.

In literature ([10, 11, 2]) different choices of gauge have been proposed. Among all we recall the London gauge

$$\nabla \cdot \mathbf{A} = 0, \qquad \mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

the Lorentz gauge

$$\phi = -\nabla \cdot \mathbf{A},$$

and the zero electric potential gauge $\phi = 0$.

Also in the time-dependent case it is possible to study the problem through the unknowns $f, \mathbf{H}, \mathbf{p}_s, \mathbf{E}$. In fact, as a consequence of (22), (23), (25) and (26), we have

$$\begin{aligned}
\gamma\frac{\partial f}{\partial t} &= \frac{1}{\kappa^2}\nabla^2 f - f\mathbf{p}_s^2 + f - f^3, && (29)\\
\nabla\times\mathbf{p}_s &= -\mu\mathbf{H}, && (30)\\
\nabla\times\mathbf{H} &= f^2\mathbf{p}_s + \sigma\mathbf{E}, && (31)\\
\nabla\times\mathbf{E} &= -\mu\frac{\partial\mathbf{H}}{\partial t}. && (32)
\end{aligned}$$

A comparison between (30) and (32) gives the identity

$$\frac{\partial\mathbf{p}_s}{\partial t} = \mathbf{E} + \nabla\phi_s, \tag{33}$$

where ϕ_s is an arbitrary and regular function that we put equal to $(-\frac{1}{\kappa}\frac{\partial\theta}{\partial t} - \Phi)$.

System (29)-(32) can be therefore reduced to

$$\begin{aligned}
\gamma\frac{\partial f}{\partial t} &= \frac{1}{\kappa^2}\nabla^2 f - f\mathbf{p}_s^2 + f - f^3, && (34)\\
\frac{1}{\mu}\nabla\times\nabla\times\mathbf{p}_s &= -f^2\mathbf{p}_s + \sigma(-\frac{\partial\mathbf{p}_s}{\partial t} + \nabla\phi_s). && (35)
\end{aligned}$$

In order to study the evolution problem we must add a constitutive equation relating ϕ_s, f and $\mathbf{p}_s$. A possible one is given by

$$\frac{1}{\kappa^2}\nabla\cdot(f^2\mathbf{p}_s) = \gamma f^2\phi_s. \tag{36}$$

With such a choice, it can be proved ([1]) that system (34)-(36) is completely equivalent to the system introduced by Gor'kov and Éliashberg.

We are now interested in obtaining some uniqueness results for problem (34)-(36) together with the boundary conditions

$$\nabla f\cdot\mathbf{n}_{|\partial\Omega} = 0, \quad f\mathbf{p}_s\cdot\mathbf{n}_{|\partial\Omega} = 0, \quad (\nabla\times\mathbf{p}_s)\times\mathbf{n}_{|\partial\Omega} = \mu\mathbf{H}_{ex}\times\mathbf{n} \tag{37}$$

and the initial conditions

$$f(\mathbf{x},0) = f_0(\mathbf{x}), \quad \mathbf{p}_s(\mathbf{x},0) = \mathbf{p}_{s0}(\mathbf{x}). \tag{38}$$

To this end, let first consider the decomposition

$$\mathbf{p}_s = \mathbf{A} - \frac{1}{\kappa}\nabla\theta,$$

where $\nabla\cdot\mathbf{A} = -\sigma\Phi$. Then system (34)–(36) can be rewritten in terms of $f, \mathbf{A}$ and θ as

$$\begin{aligned}
\frac{1}{\mu}\nabla\times\nabla\times\mathbf{A} &= -f^2(\mathbf{A} - \frac{1}{\kappa}\nabla\theta) - \sigma\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi\right), && (39)\\
\gamma\frac{\partial f}{\partial t} &= \frac{1}{\kappa^2}\nabla^2 f - f(\mathbf{A} - \frac{1}{\kappa}\nabla\theta)^2 + f - f^3, && (40)\\
\gamma f(\kappa\phi + \frac{\partial\theta}{\partial t}) &= -\frac{1}{\kappa}[2\nabla f\cdot(\mathbf{A} - \frac{1}{\kappa}\nabla\theta) + f\nabla\cdot(\mathbf{A} - \frac{1}{\kappa}\nabla\theta)], && (41)
\end{aligned}$$

or, in an equivalent way, as

$$\frac{1}{\mu}\nabla\times\nabla\times\mathbf{A} = -f^2(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)-\sigma\left(\frac{\partial\mathbf{A}}{\partial t}+\nabla\phi\right), \tag{42}$$

$$\gamma\frac{\partial f}{\partial t}\cos\theta-\gamma\sin\theta f(\frac{\partial\theta}{\partial t}+\kappa\phi) = \left[\frac{1}{\kappa^2}\nabla^2 f - f\mathbf{p}_s^2+f(1-f^2)\right]\cos\theta + \frac{1}{\kappa}[2\nabla f\cdot\mathbf{p}_s+f\nabla\cdot\mathbf{p}_s]\sin\theta, \tag{43}$$

$$\gamma\frac{\partial f}{\partial t}\sin\theta+\gamma\cos\theta f(\frac{\partial\theta}{\partial t}+\kappa\phi) = \left[\frac{1}{\kappa^2}\nabla^2 f - f\mathbf{p}_s^2+f(1-f^2)\right]\sin\theta - \frac{1}{\kappa}[2\nabla f\cdot\mathbf{p}_s+f\nabla\cdot\mathbf{p}_s]\cos\theta, \tag{44}$$

while the boundary and the initial conditions (37)-(38) become

$$\nabla f\cdot\mathbf{n}_{|\partial\Omega}=0,\quad f(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)\cdot\mathbf{n}_{|\partial\Omega}=0,\quad(\nabla\times\mathbf{A})\times\mathbf{n}_{|\partial\Omega}=\mu\mathbf{H}_{ex}\times\mathbf{n}, \tag{45}$$

$$f(\mathbf{x},0)=f_0(\mathbf{x}),\quad(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)(\mathbf{x},0)=\mathbf{p}_{s0}(\mathbf{x}). \tag{46}$$

In order to give a precise formulation of the problem, we introduce the following spaces:

$$\begin{aligned}
H^1_n(\Omega) &= \left\{\mathbf{A}\in H^1(\Omega):\mathbf{A}\cdot\mathbf{n}|_{\partial\Omega}=0\right\},\\
\mathcal{K}(\Omega) &= \left\{(f,\theta): f\cos\theta\in H^1(\Omega), f\sin\theta\in H^1(\Omega)\right\},\\
\mathcal{I}(\Omega) &= \left\{(f,\theta): f\cos\theta\in (H^1(\Omega))', f\sin\theta\in (H^1(\Omega))'\right\},\\
\mathcal{H}(Q) &= \left\{u=((f,\theta),\mathbf{A}):(f,\theta)\in L^2(0,T;\mathcal{K}(\Omega))\cap H^1(0,T;\mathcal{I}(\Omega)),\right.\\
&\qquad \left.\mathbf{A}\in L^2(0,T;\mathbf{H}^1_n(\Omega))\cap H^1(0,T;(H^1_n(\Omega))')\right\},
\end{aligned}$$

where $\Omega\subset\mathbb{R}^2$ is a sufficiently regular domain and $(X)'$ denotes the dual of the space X.

DEFINITION 1 A pair $u=((f,\theta),\mathbf{A})\in\mathcal{H}(Q)$ is a weak solution of problem (42)-(44) with boundary conditions (45) if the relations

$$\begin{aligned}
&\int_\Omega\left[\gamma\frac{\partial(f\cos\theta)}{\partial t}-\gamma\kappa f\phi\sin\theta+f(f^2-1)\cos\theta\right]h\,dx\\
&+\int_\Omega\left[f\cos\theta(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)-\frac{1}{\kappa}\nabla f\sin\theta\right]\cdot\mathbf{A}h\,dx\\
&+\int_\Omega\left[\frac{1}{\kappa^2}\nabla(f\cos\theta)\cdot\nabla h+\frac{1}{\kappa}f\sin\theta\mathbf{A}\cdot\nabla h\right]\,dx=0
\end{aligned} \tag{47}$$

$$\begin{aligned}
&\int_\Omega\left[\gamma\frac{\partial(f\sin\theta)}{\partial t}+\gamma\kappa f\phi\cos\theta+f(f^2-1)\sin\theta\right]\,dx\\
&+\int_\Omega\left[f\sin\theta(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)+\frac{1}{\kappa}\nabla f\cos\theta\right]\cdot\mathbf{A}h\,dx
\end{aligned} \tag{48}$$

$$+\int_\Omega \left[\frac{1}{\kappa^2}\nabla(f\sin\theta)\cdot\nabla h - \frac{1}{\kappa} f\cos\theta \mathbf{A}\cdot\nabla h\right]\,d\,x = 0$$

(49)
$$\int_\Omega \left[\sigma\frac{\partial \mathbf{A}}{\partial t}\cdot\mathbf{a} + \frac{1}{\mu}\nabla\times\mathbf{A}\cdot\nabla\times\mathbf{a} + \nabla\cdot\mathbf{A}\nabla\cdot\mathbf{a}\right]\,d\,x$$
$$+\int_\Omega f^2(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)\cdot\mathbf{a}\,d\,x + \int_{\partial\Omega}\mathbf{H}_{ex}\times\mathbf{a}\cdot\mathbf{n}\,d\,a = 0$$

hold for any $(h,\mathbf{a})\in H^1(\Omega)\times\mathbf{H}^1_n(\Omega)$, together with the initial conditions

(50)
$$f(0)=f_0,\quad \theta(0)=\theta_0,\quad \mathbf{A}(0)=\mathbf{A}_0$$

where $\mathbf{A}_0-\dfrac{1}{\kappa}\nabla\theta_0=\mathbf{p}_{s_0}$.

THEOREM 1 *Let $((f,\theta),\mathbf{A})$ be a weak solution of problem (42)-(44) in the sense of the definition 1. Then for any $T>0$*

(51)
$$\|f(t)\|_{L^2}+\|\mathbf{A}(t)\|_{L^2}\le C,\qquad 0\le t\le T,$$

(52)
$$\int_0^T (\|(f\cos\theta)(t)\|_{H^1}+\|(f\sin\theta)(t)\|_{H^1}+\|\mathbf{A}(t)\|_{H^1})\,d\,t\le C,$$

where C is a constant depending on $\|f_0\|_{L^2(\Omega)}$, $\|\mathbf{A}_0\|_{L^2(\Omega)}$ and $\|\mathbf{H}_{ex}\times\mathbf{n}\|_{H^{-1/2}(\partial\Omega)}$.

PROOF. By choosing $h=f\cos\theta$ in (47), $h=f\sin\theta$ in (48) and then summing the two expressions, we get

$$\int_\Omega\left[\frac{\gamma}{2}\frac{\partial}{\partial t}f^2+f^4+\frac{1}{\kappa^2}|\nabla f|^2+|f(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)|^2\right]\,d\,x=\int_\Omega f^2\,d\,x,$$

from which it follows that for all $0\le t\le T$,

$$\|f(t)\|^2_{L^2(\Omega)}\le C$$

and also

$$\int_0^T\|f(t)\|^4_{L^4(\Omega)}\,d\,t\le C,\qquad \int_0^T\|\nabla f(t)\|^2_{L^2(\Omega)}\,d\,t\le C,$$
$$\int_0^T\|(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)f(t)\|^2_{L^2(\Omega)}\,d\,t\le C.$$

Now taking $\mathbf{a}=\mathbf{A}$ in (49), we infer that

$$\int_\Omega\left[\frac{\sigma}{2}\frac{\partial}{\partial t}|\mathbf{A}|^2+|\nabla\cdot\mathbf{A}|^2+\frac{1}{\mu}|\nabla\times\mathbf{A}|^2\right]\,d\,x$$
$$=-\int_\Omega f(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)\cdot f\mathbf{A}\,d\,x+\int_{\partial\Omega}\mathbf{H}_{ex}\times\mathbf{A}\cdot\mathbf{n}\,d\,a$$
$$\le c\|f(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)\|^2_{L^2(\Omega)}+c\|\mathbf{A}\|_{L^2(\Omega)}\|\mathbf{A}\|_{H^1(\Omega)}\|f\|^2_{L^4(\Omega)}+c\|\mathbf{A}\|_{H^1(\Omega)},$$

which gives

$$\|\mathbf{A}(t)\|^2_{L^2(\Omega)}\le C,\ \int_0^T\left[\|\nabla\times\mathbf{A}(t)\|^2_{L^2(\Omega)}+\|\nabla\cdot\mathbf{A}(t)\|^2_{L^2(\Omega)}\right]\,d\,t\le C.$$

Finally, since

$$\begin{aligned}||f(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)||^2_{L^2(\Omega)}+\frac{1}{\kappa^2}||\nabla f||^2_{L^2(\Omega)}\\ =\frac{1}{\kappa^2}\left[||\nabla(f\cos\theta)||^2_{L^2(\Omega)}+||\nabla(f\sin\theta)||^2_{L^2(\Omega)}\right]\\ -\frac{2}{\kappa}\int_\Omega f\nabla\theta\cdot f\mathbf{A}\,d\,x+\int_\Omega f^2|\mathbf{A}|^2\,d\,x,\end{aligned}$$

we obtain that

$$\textstyle\int_0^T\left[||\nabla(f\cos\theta)(t)||^2_{L^2(\Omega)}+||\nabla(f\sin\theta)(t)||^2_{L^2(\Omega)}\right]\,d\,t\le C.$$

∎

THEOREM 2 *Let $((f,\theta),\mathbf{A})$ be a weak solution to problem (42)-(44) in the sense of the definition 1. Then it is unique.*

PROOF. Suppose that $u_1=(f_1,\theta_1,\mathbf{A}_1)$ and $u_2=(f_2,\theta_2,\mathbf{A}_2)$ are two weak solutions of problem (42)-(44) with the same initial data under the same applied magnetic field. Let us put

$$\delta R=f_1\cos\theta_1-f_2\cos\theta_2,\quad \delta I=f_1\sin\theta_1-f_2\sin\theta_2,\quad \delta\mathbf{A}=\mathbf{A}_1-\mathbf{A}_2.$$

Then the thesis is proved, as a consequence of the Gronwall lemma, if it exists $C(t)\in L^1([0,T])$ such that

$$\int_\Omega\frac{\partial}{\partial t}[(\delta R)^2+(\delta I)^2+||\delta\mathbf{A}||^2]\,d\,x\le C(t)(||\delta R||^2_{L^2}+||\delta I||^2_{L^2}+||\delta\mathbf{A}||^2_{L^2}). \tag{53}$$

By considering the difference of (47)-(49) written for u_1 and u_2 and then choosing respectively $h=\delta R$, $h=\delta I$ and $\mathbf{a}=\delta\mathbf{A}$, we get that

$$\int_\Omega\left[\frac{\gamma}{2}\frac{\partial}{\partial t}(\delta R)^2+\frac{1}{\kappa^2}|\nabla\delta R|^2\right]\,d\,x=\int_\Omega(\delta R)^2\,d\,x+R_1+R_2+R_3+R_4, \tag{54}$$

$$\int_\Omega\left[\frac{\gamma}{2}\frac{\partial}{\partial t}(\delta I)^2+\frac{1}{\kappa^2}|\nabla\delta I|^2\right]\,d\,x=\int_\Omega(\delta I)^2\,d\,x+I_1+I_2+I_3+I_4, \tag{55}$$

$$\int_\Omega\left[\frac{\sigma}{2}\frac{\partial}{\partial t}|\delta\mathbf{A}|^2+(\nabla\cdot\delta\mathbf{A})^2+\frac{1}{\mu}|\nabla\times\delta\mathbf{A}|^2\right]\,d\,x=J_1+J_2, \tag{56}$$

where

$$\begin{aligned}R_1&=\gamma\int_\Omega[f_1\sin\theta_1\nabla\cdot\delta\mathbf{A}+\nabla\cdot\mathbf{A}_2\delta I]\,\delta R\,d\,x,\\ R_2&=-\int_\Omega\left[f_1^2(\delta R)^2+f_2\cos\theta_2(f_1^2-f_2^2)\delta R\right]\,d\,x,\\ R_3&=-\int_\Omega\left[f_1\cos\theta_1(\mathbf{A}_1+\mathbf{A}_2)\cdot\delta\mathbf{A}\delta R+|\mathbf{A}_2|^2(\delta R)^2\right]\,d\,x\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{\kappa}\int_\Omega [\nabla(f_1\sin\theta_1)\cdot\delta\mathbf{A}+\mathbf{A}_2\cdot\nabla(\delta I)]\,\delta R\,d\,x,\\
R_4 &= \frac{1}{\kappa}\int_\Omega \nabla(\delta R)\cdot[\mathbf{A}_1\delta I+f_2\sin\theta_2\delta\mathbf{A}]\;d\,x,\\
I_1 &= -\gamma\int_\Omega [f_1\cos\theta_1\nabla\cdot\delta\mathbf{A}+\nabla\cdot\mathbf{A}_2\delta R]\,\delta I\;d\,x,\\
I_2 &= -\int_\Omega \left[f_1^2(\delta I)^2+f_2\sin\theta_2(f_1^2-f_2^2)\delta I\right]\;d\,x,\\
I_3 &= -\int_\Omega \left[f_1\sin\theta_1(\mathbf{A}_1+\mathbf{A}_2)\cdot\delta\mathbf{A}\delta I+|\mathbf{A}_2|^2(\delta I)^2\right]\;d\,x\\
& +\frac{1}{\kappa}\int_\Omega [\nabla(f_1\cos\theta_1)\cdot\delta\mathbf{A}+\mathbf{A}_2\cdot\nabla(\delta R)]\,\delta I\;d\,x,\\
I_4 &= -\frac{1}{\kappa}\int_\Omega \nabla(\delta I)\cdot[\mathbf{A}_1\delta R+f_2\cos\theta_2\delta\mathbf{A}]\;d\,x,\\
J_1 &= -\int_\Omega \left[f_1^2|\delta\mathbf{A}|^2+\mathbf{A}_2(f_1^2-f_2^2)\cdot\delta\mathbf{A}\right]\;d\,x,\\
J_2 &= -\frac{1}{\kappa}\int_\Omega [f_1(f_1\nabla\theta_1-f_2\nabla\theta_2)+f_2\nabla\theta_2(f_1-f_2)]\cdot\delta\mathbf{A}\;d\,x.
\end{aligned}
$$

By means of theorem 1 it is possible to prove that

$$
\begin{aligned}
|R_1| &\le C(\epsilon)\left(\|\nabla\cdot\delta\mathbf{A}\|^2_{L^2(\Omega)}+\|\nabla(\delta R)\|^2_{L^2(\Omega)}+\|\nabla(\delta I)\|^2_{L^2(\Omega)}\right)\\
&\quad +C(t)\left(\|\delta R\|^2_{L^2(\Omega)}+\|\delta I\|^2_{L^2(\Omega)}\right),\\
|R_2| &\le C(\epsilon)\left(\|\nabla(\delta R)\|^2_{L^2(\Omega)}+\|\nabla(\delta I)\|^2_{L^2(\Omega)}\right)+C(t)\left(\|\delta R\|^2_{L^2(\Omega)}+\|\delta I\|^2_{L^2(\Omega)}\right),\\
|R_3| &\le C(\epsilon)\left(\|\nabla\cdot\delta\mathbf{A}\|^2_{L^2(\Omega)}+\|\nabla\times\delta\mathbf{A}\|^2_{L^2(\Omega)}+\|\nabla(\delta R)\|^2_{L^2(\Omega)}+\|\nabla(\delta I)\|^2_{L^2(\Omega)}\right)\\
&\quad +C(t)\left(\|\delta R\|^2_{L^2(\Omega)}+\|\delta I\|^2_{L^2(\Omega)}+\|\delta\mathbf{A}\|^2_{L^2(\Omega)}\right),\\
|R_4| &\le C(\epsilon)\left(\|\nabla\cdot\delta\mathbf{A}\|^2_{L^2(\Omega)}+\|\nabla\times\delta\mathbf{A}\|^2_{L^2(\Omega)}+\|\nabla(\delta R)\|^2_{L^2(\Omega)}+\|\nabla(\delta I)\|^2_{L^2(\Omega)}\right)\\
&\quad +C(t)\left(\|\delta R\|^2_{L^2(\Omega)}+\|\delta I\|^2_{L^2(\Omega)}+\|\delta\mathbf{A}\|^2_{L^2(\Omega)}\right),
\end{aligned}
$$

where $\epsilon>0$ and $C(\epsilon)$ is a positive constant.

Hence (54) yields

$$
\begin{aligned}
&\int_\Omega\left[\frac{\gamma}{2}\frac{\partial}{\partial t}(\delta R)^2+\frac{1}{\kappa^2}|\nabla\delta R|^2\right]\;d\,x \qquad (57)\\
&\le C(\epsilon)\left(\|\nabla\cdot\delta\mathbf{A}\|^2_{L^2(\Omega)}+\|\nabla\times\delta\mathbf{A}\|^2_{L^2(\Omega)}+\|\nabla(\delta R)\|^2_{L^2(\Omega)}+\|\nabla(\delta I)\|^2_{L^2(\Omega)}\right)\\
&\quad +C(t)\left(\|\delta R\|^2_{L^2(\Omega)}+\|\delta I\|^2_{L^2(\Omega)}+\|\delta\mathbf{A}\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

Proceeding in the same way, from (55) we obtain

$$
\begin{aligned}
&\int_\Omega\left[\frac{\gamma}{2}\frac{\partial}{\partial t}(\delta I)^2+\frac{1}{\kappa^2}|\nabla\delta I|^2\right]\;d\,x \qquad (58)\\
&\le C(\epsilon)\left(\|\nabla\cdot\delta\mathbf{A}\|^2_{L^2(\Omega)}+\|\nabla\times\delta\mathbf{A}\|^2_{L^2(\Omega)}+\|\nabla(\delta R)\|^2_{L^2(\Omega)}+\|\nabla(\delta I)\|^2_{L^2(\Omega)}\right)\\
&\quad +C(t)\left(\|\delta R\|^2_{L^2(\Omega)}+\|\delta I\|^2_{L^2(\Omega)}+\|\delta\mathbf{A}\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

In order to estimate J_1 and J_2, we observe that

$$\begin{aligned} |f_1 - f_2|^2 &\leq |\delta R|^2 + |\delta I|^2, \\ |f_1 \nabla\theta_1 - f_2\nabla\theta_2|^2 &\leq |\nabla(\delta R)|^2 + |\nabla(\delta I)|^2. \end{aligned}$$

As a consequence of the former inequalities and theorem 1, we get that

$$\begin{aligned} |J_i| &\leq C(\epsilon)\left(\|\nabla\cdot\delta\mathbf{A}\|^2_{L^2(\Omega)} + \|\nabla\times\delta\mathbf{A}\|^2_{L^2(\Omega)} + \|\nabla(\delta R)\|^2_{L^2(\Omega)} + \|\nabla(\delta I)\|^2_{L^2(\Omega)}\right) \\ &\quad + C(t)\left(\|\delta R\|^2_{L^2(\Omega)} + \|\delta I\|^2_{L^2(\Omega)} + \|\delta\mathbf{A}\|^2_{L^2(\Omega)}\right), \end{aligned}$$

for $i = 1, 2$ and equation (56) gives

$$\begin{aligned} &\int_\Omega \left[\frac{\sigma}{2}\frac{\partial}{\partial t}|\delta\mathbf{A}|^2 + (\nabla\cdot\delta\mathbf{A})^2 + \frac{1}{\mu}|\nabla\times\delta\mathbf{A}|^2\right] d\,x \\ &\leq C(\epsilon)\left(\|\nabla\cdot\delta\mathbf{A}\|^2_{L^2(\Omega)} + \|\nabla\times\delta\mathbf{A}\|^2_{L^2(\Omega)} + \|\nabla(\delta R)\|^2_{L^2(\Omega)} + \|\nabla(\delta I)\|^2_{L^2(\Omega)}\right) \\ &\quad + C(t)\left(\|\delta R\|^2_{L^2(\Omega)} + \|\delta I\|^2_{L^2(\Omega)} + \|\delta\mathbf{A}\|^2_{L^2(\Omega)}\right). \end{aligned} \tag{59}$$

Finally, by putting together (57), (58), (59) and choosing a suitable $C(\epsilon)$, we obtain the inequality (53). ∎

4. – A new time-dependent model

In this last paragraph we will use the approach, which lead us to system (34)-(36), to show that the differential system developed by Gor'kov and Éliashberg needs to be modified.

To this end, we first consider the divergence of equation (35) and get

$$\nabla\cdot(f^2\mathbf{p}_s) = -\sigma\nabla\cdot\mathbf{E}. \tag{60}$$

On the other hand, thanks to (24), equation (60) can be rewritten as

$$0 = \nabla\cdot\mathbf{J}_s + \nabla\cdot\mathbf{J}_n = \nabla\cdot\mathbf{J}. \tag{61}$$

An application of the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0,$$

tells us that $\dfrac{\partial\rho}{\partial t} = 0$, so that $\rho = \rho_0$, ρ_0 being the initial density. Consequently, from (39) and taking into account $(23)_2$, we have

$$\nabla\cdot(f^2\mathbf{p}_s) = -\frac{\sigma}{\epsilon}\rho_0. \tag{62}$$

This result is in obvious disagreement with (36) since

$$f^2\phi_s = -\frac{\sigma}{\kappa^2\gamma\epsilon}\rho_0 \tag{63}$$

represents a new equation and problem (34)-(36) and (63) becomes overdetermined.

In the following we will assume $\rho_0 = 0$, so that from (62) we have

$$\nabla \cdot (f^2 \mathbf{p}_s) = 0. \tag{64}$$

Therefore we substitute equation (36) with equation (64) and the dynamical problem is governed in $Q = (0,T) \times \Omega$ by the system

$$\gamma \frac{\partial f}{\partial t} = \frac{1}{\kappa^2} \nabla^2 f - f \mathbf{p}_s^2 + f - f^3 \tag{65}$$

$$\frac{1}{\mu} \nabla \times \nabla \times \mathbf{p}_s = -f^2 \mathbf{p}_s - \sigma \left(\frac{\partial \mathbf{p}_s}{\partial t} - \nabla \phi_s \right) \tag{66}$$

$$\nabla \cdot (f^2 \mathbf{p}_s) = 0 \tag{67}$$

together with the boundary and initial conditions (37)-(38).

Finally, it should be noticed that the dynamical model described by the equations (65)-(67) satisfies the Second Law of Thermodynamics if $\sigma \geq 0$ and $\gamma \geq 0$. In fact, under the hypotheses of processes near the transition temperature, it is well known that this law states that the inequality

$$I_\Omega = \oint_0^d \int_\Omega (\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{H} + \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E} + \mathbf{J} \cdot \mathbf{E}) \, d\, x\, d\, t \geq 0$$

must hold for any closed cycle of duration d.

Now, if we take into account the constitutive equations we get

$$I_\Omega = \oint_0^d \int_\Omega \left[\sigma |\mathbf{E}|^2 + f^2 \mathbf{p}_s \cdot (\frac{\partial \mathbf{p}_s}{\partial t} - \nabla \phi_s) \right] \, d\, x\, d\, t.$$

Moreover, thanks to (67), the previous integral reduces to

$$I_\Omega = \oint_0^d \int_\Omega \left[\sigma |\mathbf{E}|^2 + f^2 \mathbf{p}_s \cdot \frac{\partial \mathbf{p}_s}{\partial t} \right] \, d\, x\, d\, t$$

and an integration by parts together with (65) gives

$$I_\Omega = \oint_0^d \int_\Omega \left[\sigma |\mathbf{E}|^2 + \gamma (\frac{\partial f}{\partial t})^2 \right] \, d\, x\, d\, t,$$

which is nonnegative if σ and γ are nonnegative.

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Counterexample to the exponential decay for systems with memory *

Sergio Polidoro †

1. – Introduction

We study the asymptotic behavior of the solution u to the following initial-boundary value problem

$$(1)\qquad \begin{cases} u_{tt} = G_0\Delta u + G' * \Delta u - au_t & \text{in } \Omega \times \mathbb{R}^+ \\ u(\cdot,0) = u_0 \quad u_t(\cdot,0) = u_1 & \text{in } \Omega \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

where Ω is a smooth bounded open subset of $\mathbb{R}^n$, $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega), \Delta$ denotes the Laplace operator in $\mathbb{R}^n$ and "$*$" is the convolution in the variable t:

$$G' * \Delta u(x,t) \equiv \int_0^t G'(t-s)\Delta u(x,s)ds.$$

We assume that

i) G_0, a are real constant coefficients, $G_0 > 0, a \geq 0$,

ii) $G' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+), G' \leq 0$,

iii) $G_\infty \equiv G_0 + \int_0^\infty G'(t)dt > 0$.

This kind of problem arises in linear viscoelasticity and the above assumption on G_0, G' ensure that, when $G' \not\equiv 0$, the system has the "fading memory property" and the problem (1) has a unique weak solution.

In the case $G' \equiv 0$ the asymptotic behavior depends on the coefficient a: when $a = 0$ the energy of the solution u is constant, thus u does not decay; on the other hand, when $a > 0$, the energy of the solution u decays exponentially.

The problem (1), in the case $G' \not\equiv 0$ and $a = 0$, was posed by Volterra and Graffi in their classical works, then it was systematically studied by many authors (we refer to Fabrizio and Morro [4] for a wide bibliography on this topic). Among other

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known results, concerning existence, uniqueness an stability of the solution, let us recall the works by Dafermos [1] and Fabrizio and Lazzari [3], which prove that the exponential decay of G' is a sufficient condition in order to have the exponential decay of the solution u. Our aim is to show that the exponential decay of G' is also a necessary condition for the exponential decay of u and we are able to prove a much stronger result: the exponential decay of G' is necessary also when a is positive.

Our methods also apply to the following problem, which arises in the theory of the heat conduction with memory:

$$(2) \qquad \begin{cases} u_t = K_0 \Delta u + K' * \Delta u - au & \text{in } \Omega \times \mathbb{R}^+ \\ u(\cdot, 0) = u_0 & \text{in } \Omega \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

under the assumptions

i') K_0, a are non-negative constant coefficients,

ii') $K' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+), K' \geq 0$.

As problem (1), also problem (2) has been studied by many authors. Here we refer to the paper by Giorgi and Gentili [2]. Concerning problem (2), we are able to show that the exponential decay of K' is a necessary condition for the exponential decay of u also in this case.

We next give some comments about our results. The aim of the present work is to point out that, although both terms au_t and $G' * \Delta u$ in (1) produce a dissipation of the energy of the system, the presence of the first term is hidden by the presence of the second one. We stress that, when considering problem (2), our result is meaningful also in the case $a = 0$: indeed, if $K' \equiv 0$ and $a = 0$, the solution u does exponentially decay, while for $K' \not\equiv 0$ the asymptotic behavior of u is analogous to the behavior of K'.

However we must say that our results are not completely unexpected. Indeed, let us consider problem (2) for $K' \equiv 0$: it is easy to see that the function $v(x,t) = e^{at}u(x,t)$ is a solution to problem

$$\begin{cases} v_t = K_0 \Delta v & \text{in } \Omega \times \mathbb{R}^+ \\ v(\cdot, 0) = u_0 & \text{in } \Omega \\ v(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

then clearly u decays exponentially. When $K' \not\equiv 0$, the same change of function leads to

$$\begin{cases} v_t = K_0 \Delta v + M' * \Delta v & \text{in } \Omega \times \mathbb{R}^+ \\ v(\cdot, 0) = u_0 & \text{in } \Omega \\ v(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

where $M(t) = e^{at}K'(t)$ does not necessarily satisfy condition *(ii')*, in other words the problem corresponding to $K' \not\equiv 0$ has not the same features of the case $K' \equiv 0$.

This note is organized as follows: in Section 2 we find a counterexample to the exponential decay for a particular case of problem (2). The problem is chosen aiming to avoid as many technical difficulties as possible, however it turns out that the main point (Lemma 1 below) does not requires such a simplification and our results can be extended to more general cases. In Section 3 we state our main results (Theorems 1 and 2) and we outline their proof. Finally, in Section 4, we state some further results concerning the polynomial decay and we give some conclusive remarks.

2. – A counterexample

In this Section we find a counterexample to the exponential decay of the solution to a system with memory. We say that a function $f \in L^1(\mathbb{R}^+)$ has the exponential decay property if a positive α exists such that

$$\int_0^\infty e^{\alpha t}|f(t)|dt < \infty.$$

Aiming to consider the simplest situation, we study problem (2) in one space dimension, namely

$$\begin{cases} u_t = K_0 u_{xx} + K' * u_{xx} - au & \text{in }]0,\pi[\times\mathbb{R}^+, \\ u(x,0) = \sin(x) & \text{for } x \in [0,\pi], \\ u(0,t) = u(\pi,t) = 0 & \text{for } t \in \mathbb{R}^+, \end{cases} \tag{3}$$

under the assumption *(i')* and *(ii')*. Let us note that the solution of the above problem must be a function in the form $u(x,t) = \sin(x)f(t)$, where f is the (unique) classical solution to

$$f'(t) = -K_0 f - K' * f(t) - af(t), \quad f(0) = 1, \tag{4}$$

thus we are allowed to say that u has the exponential decay property if, and only if, f does. Our result is contained in the following

PROPOSITION 1 *Let u be the solution of the problem (3), under the assumptions (i'), (ii'). If u has the exponential decay property, then also K' does.*

In order to prove Proposition 1 we consider the Laplace transform of the function f, that will be denoted as

$$\widehat{f}(z) = \int_0^\infty e^{-zt} f(t)dt.$$

We shall need the following result:

LEMMA 1 *Let $U \subset \mathbb{C}$ be a neighborhood of 0 and let $g : U \to \mathbb{C}$ be a holomorphic function. If $G \in L^1(\mathbb{R}^+)$ is a non-negative function such that $\widehat{G}(z) = g(z)$ for every $z \in U \cap \{\Re e\ z \geq 0\}$, then G has the exponential decay property.*

PROOF. Let us note that, since $G \in L^1(\mathbb{R}^+)$, both functions $\widehat{G}$ and g are holomorphic in the set $\{\Re e\ z > 0\}$ and

$$g^{(k)}(x) = \widehat{G}^{(k)}(x) = (-1)^k \int_0^\infty t^k e^{-xt} G(t)dt$$

for every positive real number $x \in U$ and for any $k \in \mathbb{N} \cup \{0\}$. Since G is non-negative, we can let x go to 0 in the above identity and find

$$g^{(k)}(0) = (-1)^k \int_0^\infty t^k G(t)dt.$$

Using again the fact that g is holomorphic in U we see that a positive r exists such that

$$g(-r) = \sum_{k=0}^\infty \frac{g^{(k)}(0)}{k!}(-r)^k = \sum_{k=0}^\infty \frac{r^k}{k!} \int_0^\infty t^k G(t)dt = \int_0^\infty \sum_{k=0}^\infty \frac{r^k t^k}{k!} G(t)dt = \int_0^\infty e^{rt} G(t)dt,$$

thus G has the exponential decay property. ∎

Proof of Proposition 1. We assume that u has the exponential decay property, hence a positive α exists such that

$$\int_0^\infty e^{\alpha t}|f(t)|dt < \infty,$$

then the Laplace transform of f is defined for any $z \in \mathbb{C}$ such that $\Re e\ z > -\alpha$. By assumption *(ii')* the Laplace transform of K' is defined in $\{\Re e\ z \geq 0\}$ and

$$\widehat{f}(z)(K_0 + \widehat{K'}(z) + z + a) = 1,$$

for every $z \in \{\Re e\ z \geq 0\}$. Note that, by the above identity, $\widehat{f}(0) \neq 0$, then $\widehat{f}(z) \neq 0$ for every z in a suitable neighborhood U of 0. Hence

$$\widehat{K'}(z) = \frac{1}{\widehat{f}(z)} - (K_0 + z + a),$$

for every $z \in U$ such that $\Re e\ z \geq 0$. Our result then follows from Lemma 1. ∎

3. – General results

In this section we give the basic statements concerning the weak solutions to the problems (1) and (2), then we prove the main results.

We say that a function $u \in L^2(\mathbb{R}^+, H_0^1(\Omega)) \cap H^1(\mathbb{R}^+, L^2(\Omega))$ is a weak solution of (1) if $u(\cdot, 0) = u_0$ almost everywhere in Ω and

$$\begin{aligned}(5) \qquad \int_{\Omega\times\mathbb{R}^+} u_t\phi_t dxdt + \int_\Omega u_1\phi(\cdot,0)dx = \\ \int_{\Omega\times\mathbb{R}^+} \left(G_0\langle\nabla u, \nabla\phi\rangle + \langle G' * \nabla u, \nabla\phi\rangle + au_t\phi\right) dxdt\end{aligned}$$

for every $\phi \in L^2(\mathbb{R}^+, H_0^1(\Omega)) \cap H^1(\mathbb{R}^+, L^2(\Omega))$. In (5) ∇ and $\langle\cdot,\cdot\rangle$ are the gradient and the inner product in $\mathbb{R}^n$. We say that a function $u \in L^2(\mathbb{R}^+, H_0^1(\Omega)) \cap H^1(\mathbb{R}^+, L^2(\Omega))$ has the exponential decay property if a positive α exists such that

$$\int_{\Omega\times\mathbb{R}^+} e^{2\alpha t}\left(u_t(x,t)^2 + |\nabla u(x,t)|^2\right) dxdt < \infty.$$

Our first result is contained in the following

THEOREM 1 *Consider the problem (1), under the assumptions (i), (ii), (iii) and let u be a non-trivial solution. If G' does not have the exponential decay property, then u does not have the exponential decay property.*

We next consider a function $u \in L^2(\mathbb{R}^+, H_0^1(\Omega)) \cap L^\infty(\mathbb{R}^+, L^2(\Omega))$ which is a weak solution of (2). This means that

$$\int_{\Omega\times\mathbb{R}^+} u\phi_t dxdt + \int_\Omega u_0\phi(\cdot,0)dx = \int_{\Omega\times\mathbb{R}^+} (K_0\langle\nabla u, \nabla\phi\rangle + \langle K' * \nabla u, \nabla\phi\rangle + au\phi)\, dxdt \tag{6}$$

for every $\phi \in C_0^\infty(\Omega \times [0,\infty[)$ and we say that u has the exponential decay property if a positive α exists such that

$$\int_{\Omega\times\mathbb{R}^+} e^{2\alpha t}|\nabla u(x,t)|^2 dxdt < \infty.$$

Our result about problem (2) is the following one.

THEOREM 2 *Consider the problem (2), under the assumptions (i'), (ii') and let u be a non-trivial solution. If K' does not have the exponential decay property, then u does not have the exponential decay property.*

We next outline the proof of Theorem 1, the proof of Theorem 2 follows the same lines and will be omitted. The details are contained in the paper [5].

PROOF OF THEOREM 1. We assume that G' does not have the exponential decay property but, by contradiction, that the solution u to the problem (1) does. Thus a positive α exists such that

$$\int_{\Omega\times\mathbb{R}^+} e^{2\alpha t}\left(u_t(x,t)^2 + |\nabla u(x,t)|^2\right) dxdt < \infty. \tag{7}$$

As a consequence, the Laplace transform

$$\widehat{u}(x,z) = \int_0^\infty e^{-zt}u(x,t)dt,$$

is defined for almost every $x \in \Omega$ and for any $z \in \mathbb{C}$ such that $\Re e\ z > -\alpha$ and the same assertion is true for the derivatives $\partial_{x_j}u(x,t)$, for $j = 1, ..., n$. We also note that the Laplace transform of G' is defined for every $z \in \{\Re e\ z \geq 0\}$ and that the function $\widehat{u}(\cdot,z)$, is a weak solution of the problem $(G_0 + \widehat{G'}(z))\Delta\widehat{u}(\cdot,z) = (z^2 + az)\widehat{u}(\cdot,z) - u_1 - (a+z)u_0$ in Ω, $u(\cdot,z) = 0$, in $\partial\Omega$, in the sense that $\widehat{u}(\cdot,z) \in H_0^1(\Omega)$ and

$$\begin{aligned}(G_0 + \widehat{G'}(z))\int_\Omega \langle\nabla\widehat{u}(x,z), \nabla\psi(x)\rangle dx + (z^2 + az)\int_\Omega \widehat{u}(x,z)\psi(x)dx &= \\ \int_\Omega (u_1(x) + (a+z)u_0(x))\psi(x)dx,&\end{aligned} \tag{8}$$

for every $\psi \in H_0^1(\Omega)$. We next have to consider two cases.

Suppose that $u_1+au_0 \not\equiv 0$ (here and in the sequel, we mean that $u_1(x)+au_0(x) \neq 0$ for almost any $x \in \Omega$). The identity (8) for $z=0$ is

$$(9) \qquad (G_0+\widehat{G'}(0))\int_\Omega \langle \nabla \widehat{u}(x,0), \nabla \psi(x)\rangle dx = \int_\Omega (u_1(x)+au_0(x))\psi(x)dx,$$

then $\nabla\widehat{u}(\cdot,0) \not\equiv 0$ and, as a consequence, the function

$$g(z) = \int_\Omega \langle \nabla \widehat{u}(x,z), \nabla \widehat{u}(x,0)\rangle dx$$

is non-zero and holomorphic in some neighborhood U of 0. Hence, by using $\widehat{u}(\cdot,0)$ as a test function in (8), we obtain

$$\widehat{G'}(z) = \frac{\int_\Omega (u_1(x)+(a+z)u_0(x)-(z^2+az)\widehat{u}(x,z))\widehat{u}(x,0)dx}{\int_\Omega \langle \nabla \widehat{u}(x,z), \nabla \widehat{u}(x,0)\rangle dx} - G_0,$$

for every $z \in U$ such that $\Re e\ z \geq 0$. Note that the right hand side of the above identity is holomorphic in U then, by Lemma 1 we get a contradiction with the assumption that G' does not have the exponential decay property.

Suppose now that $u_1+au_0=0$ for almost every $x \in \Omega$. In this case the identity (9) yields $\nabla\widehat{u}(\cdot,0) \equiv 0$ and, consequently, $\widehat{u}(\cdot,0) \equiv 0$.

Since u is not the trivial solution of the problem (2), we have $u_0 \not\equiv 0$, then we can choose a test function ψ such that

$$\int_\Omega u_0(x)\psi(x)dx > 0.$$

Aiming to find a contradiction as in the previous case, we compute the derivative of the function $z \mapsto \int_\Omega \langle \nabla\widehat{u}(x,z), \nabla\psi(x)\rangle dx$ at $z=0$: by using the fact that $\widehat{u}(\cdot,0)\equiv 0$ we obtain from (8) that

$$\begin{aligned}\frac{d}{dz}\int_\Omega \langle \nabla \widehat{u}(x,0), \nabla\psi(x)\rangle dx = \lim_{z\to 0}\frac{\int_\Omega \langle \nabla\widehat{u}(x,z), \nabla\psi(x)\rangle dx}{z} = \\ \lim_{z\to 0}\frac{\int_\Omega (u_0(x)-(z+a)\widehat{u}(x,z))\psi(x)dx}{G_0+\widehat{G'}(z)} = \frac{\int_\Omega u_0(x)\psi(x)dx}{G_\infty}\end{aligned}$$

which is positive, by our choice of ψ and by assumption *(iii)*. This means that there exists a neighborhood U of 0 such that the function $z \mapsto \int_\Omega \langle \nabla\widehat{u}(x,z), \nabla\psi(x)\rangle dx$ is holomorphic in U, has a simple zero at $z=0$ and is non zero in $U\backslash\{0\}$. Being $\int_\Omega (u_0(x)-(z+a)\widehat{u}(x,z))\,\psi(x)dx$ holomorphic, we conclude that the function

$$h(z) = \frac{z\int_\Omega (u_0(x)-(z+a)\widehat{u}(x,z))\psi(x)dx}{\int_\Omega \langle \nabla\widehat{u}(x,z), \nabla\psi(x)\rangle dx} - G_0$$

can be extended as a holomorphic function on U. As in the previous case, we can write $\widehat{G'}(z) = h(z)$, for every $z \in U$ such that $\Re e\ z \geq 0$, and Lemma 1 yields a contradiction. ∎

4. – Conclusive remarks

In this section we give some comments which have been inspired by some discussion after the conference I held in Cortona.

The first one concerns the polynomial decay of the solutions to the problems (1) and (2) and has been suggested by J. E. Muñoz Rivera. In order to deal with the problem of the polynomial decay, it is convenient to define *rate of polynomial decay* of a function $g \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ the following number

$$p_0 = \sup\left\{p \in \mathbb{R} : \int_0^\infty (1+t)^{p-1}|g(t)|dt < \infty\right\}$$

provided that it is positive. To motivate our definition, we note that the rate of polynomial decay of the function $g(t) = (1+t)^{-p_0}$ is exactly p_0. Since we also need to consider a function $u \in L^2(\mathbb{R}^+, H^1_0(\Omega))$, we define its rate of polynomial decay as

$$p_0 = \sup\left\{p \in \mathbb{R} : \int_{\Omega\times\mathbb{R}^+} (1+t)^{2p-1}|\nabla u(x,t)|^2 dxdt\right\}.$$

Our results concerning this problem can be summarized as follows

PROPOSITION 2 *Let u be a nontrivial solution to problem (1), under the assumptions (i), (ii), (iii). Denote by p_0 the rate of polynomial decay of G' and by q_0 the rate of polynomial decay of u. If $q_0 > 1$ and $u_1 + au_0 \not\equiv 0$, then $p_0 \geq q_0$.*

PROPOSITION 3 *Let u be a nontrivial solution to problem (2), under the assumptions (i'), (ii'). Denote by p_0 the rate of polynomial decay of G' and by q_0 the rate of polynomial decay of u. If $q_0 > 1$, then $p_0 \geq q_0$.*

The proof of the above results is given in the paper [5].

The second remark has been pointed out by M. Grasselli and V. Pata and concerns a correlation with the following general result on the asymptotic behavior of a strongly continuous semigroup.

THEOREM 3 (see Pazy [6], Ch. 4) *Let $T(t)$ be a C_0 semigroup. If for some $p \in [1,\infty[$*

$$\int_0^\infty \|T(t)x\|^p dt < \infty, \qquad \text{forevery } x \in X,$$

then there are constants $M \geq 1$ and $\mu > 0$ such that $\|T(t)x\| \leq Me^{-\mu t}$.

We recall that Theorem 3 was used by Fabrizio and Lazzari in [3] to prove the asymptotic decay of the solution of the problem (1), under some suitable assumptions on the function G'. In [3] the authors consider the space X of all triples (u, u_t, u^t_d), where $u(\cdot,t) \in H^1_0(\Omega), u_t(\cdot,t) \in L^2(\Omega)$ and u^t_d belongs to a suitable function space Y related to the space of the past histories of the system up to time t. In that setting the norm $\|\cdot\|^2_Y$ is connected with the free energy of the system, the norm of $T(t)x$ is the energy of the system

$$\|T(t)x\|^2 = E(t)^2 = \|u(\cdot,t)\|^2_{H^1_0(\Omega)} + \|v(\cdot,t)\|^2_{L^2(\Omega)} + \|u^t_d(\cdot,t)\|^2_Y$$

and is integrable over $\mathbb{R}^+$.

In our case, Theorem 3 and our Theorem 1 yield that it is impossible to find a function space Y which contains the history of the system, such that the problem (1) can be treated by the semigroup theory in the space $X = H_0^1(\Omega) \times L^2(\Omega) \times Y$ and that

$$\int_0^\infty E(t)^p dt < \infty,$$

for some $p \in [1, \infty[$ and for every initial data.

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Convergence to the Stefan Problem of the Hyperbolic Phase Relaxation Problem and Error Estimates

Vincenzo Recupero *

1. – Introduction

When we consider the evolution of a material contained in a bounded domain Ω of $\mathbb{R}^n$ ($n \in \mathbb{N}$) during the time interval $[0,T]$, the usual equation for the energy balance reads

$$\frac{\partial e}{\partial t} + \operatorname{div} \overline{\mathbf{q}} = g \quad \text{in } Q := \Omega \times (0,T), \tag{1}$$

where e denotes the internal energy of the system, $\overline{\mathbf{q}}$ is the heat flux, and g represents the heat supply.

Let us suppose that the material is homogeneous and assume that it exhibits two phases. Then let us denote by $\overline{\theta}$ the relative temperature and by χ the phase variable (e.g. the concentration of the more energetic phase). Thus a widely used model for describing the phase transition dynamics is obtained from (1) assuming the constitutive laws

$$e = \overline{\theta} + \chi, \tag{2}$$

$$\overline{\mathbf{q}} = -\nabla\overline{\theta} \tag{3}$$

(for simplicity we have normalized to 1 all the physical constants). The relation (3) is usually called *Fourier heat conduction law*. This choice leads to the following equation for the energy balance

$$\frac{\partial(\overline{\theta} + \chi)}{\partial t} - \Delta\overline{\theta} = g \quad \text{in } Q. \tag{4}$$

In order to describe the evolution of the system, we have then to establish a further relation between the temperature and the phase. If $\overline{\theta} = 0$ is the critical temperature of phase transition, we can take the following *equilibrium condition of Stefan type* (cf., e.g., [6, 9, 10] and their references)

$$\chi \in \mathrm{H}(\overline{\theta}) \quad \text{in } Q \tag{5}$$

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where H is the Heaviside graph (i.e., $\mathrm{H}(r) = 0$ if $r < 0$, $\mathrm{H}(0) = [0,1]$, $\mathrm{H}(r) = 1$ if $r > 0$).

The problem of finding $\overline{\theta}$ and χ satisfying (4–5) is the so called *Stefan problem.* It has been widely investigated and several existence and uniqueness results have been proved, provided that (4–5) is coupled with suitable initial and boundary conditions (cf, e.g., [1, 4, 6, 9, 10]).

The inclusion (5) represents an equilibrium condition. If we want to take into account *dissipation phenomena* such as *dynamical supercooling* or *superheating* effects, we have to replace condition (5) by a non-equilibrium one. In [9], Visintin proposed to use the following relaxation dynamics for the phase variable χ, that is,

$$\varepsilon\frac{\partial\chi}{\partial t} + \mathrm{H}^{-1}(\chi) \ni \overline{\theta} \quad \text{in } Q, \tag{6}$$

ε being a small kinetic positive constant. The system (4), (6) is usually called *phase relaxation problem.* In paper [9], after coupling both problems (4–5) and (4), (6) with suitable initial and boundary conditions, it is proved that the last problem is well-posed and its solution converges, in a suitable sense, to the solution of the Stefan problem (4–5) as the parameter ε goes to 0.

Let us observe now that the Fourier law (3) leads to the parabolic equation (4), and it is well known that it allows the thermal disturbances to propagate at infinite speed. The first approach in order to overcome this feature, is due to Cattaneo, which in his work "Sulla conduzione del calore" [2], modified the Fourier law, originating the so–called *Maxwell-Cattaneo law*

$$\alpha\frac{\partial\overline{\mathbf{q}}}{\partial t} + \overline{\mathbf{q}} = -\nabla\overline{\theta} \quad \text{in } Q, \tag{7}$$

in which α represents a small positive relaxation parameter. Observe that a trivial integration of (7) with respect to time gives

$$\overline{\mathbf{q}}(t) = -\frac{1}{\alpha}\int_0^t \exp\left(\frac{s-t}{\alpha}\right)\nabla\overline{\theta}(s)ds, \tag{8}$$

so that (7) can be considered as a particular model of a material with memory (for updated reviews of the theory of Cattaneo see [3] and [8, Chapter 2]). With the constitutive assumptions (2) and (7), the energy balance (1) yields a hyperbolic equation predicting finite speed of propagation for the temperature field. If we couple this equation with the Stefan equilibrium condition we get system (1–2), (5), (7), which is also known as the *hyperbolic Stefan problem.* The existence of solutions of such a system is still an open problem. If we take account of *both* the relaxations (6–7), the following *hyperbolic phase relaxation problem* follows:

$$\frac{\partial(\overline{\theta}+\chi)}{\partial t} + \operatorname{div}\overline{\mathbf{q}} = g \quad \text{in } Q, \tag{9}$$

$$\alpha\frac{\partial\overline{\mathbf{q}}}{\partial t} + \overline{\mathbf{q}} = -\nabla\overline{\theta} \quad \text{in } Q, \tag{10}$$

$$\varepsilon\frac{\partial\chi}{\partial t} + \mathrm{H}^{-1}(\chi) \ni \overline{\theta} \quad \text{in } Q. \tag{11}$$

This system was proposed for the first time in [9], where an existence result has been outlined when (9–11) is coupled with some initial and boundary condition. Moreover it is stated there that its solutions converge to the solution of the analogous problem for (4), (6), as the parameter α goes to zero, whereas ε is fixed.

In [5] and in the present paper, (9–11) is supplied with rather general and meaningful initial-boundary conditions. More precisely, letting Γ_0 and Γ_1 denote two measurable subsets in which the boundary of Ω is partitioned, we take

$$\overline{\theta} = \theta_D \quad \text{on } \Gamma_0 \times (0,T), \tag{12}$$

$$\overline{\mathbf{q}} \cdot \mathbf{n} = \varphi_N \quad \text{on } \Gamma_1 \times (0,T), \tag{13}$$

$$\overline{\theta}(\cdot,0) = \overline{\theta}_0, \ \chi(\cdot,0) = \chi_0, \ \overline{\mathbf{q}}(\cdot,0) = \overline{\mathbf{q}}_0 \quad \text{in } \Omega, \tag{14}$$

where $\theta_D, \varphi_N, \overline{\theta}_0, \chi_0, \overline{\mathbf{q}}_0$ are given functions and $\mathbf{n}$ is the outward unit vector, normal to the boundary of Ω. We assume that θ_D is a sufficiently smooth function defined on the whole Q and that there exists a vector function $\mathbf{q}_N : Q \to \mathbb{R}^n$ such that $\mathbf{q}_N \cdot \mathbf{n} = \varphi_N$ on $\Gamma_1 \times (0,T)$ in a suitable sense. Hence, setting $\theta_0 := \overline{\theta}_0 - \theta_D(0)$ and $\mathbf{q}_0 := \overline{\mathbf{q}}(0) - \mathbf{q}_N(0)$, we rewrite the problem (9–14) in terms of the unknowns $\theta = \overline{\theta} - \theta_D, \chi$, and $\mathbf{q} = \overline{\mathbf{q}} - \mathbf{q}_N$, obtaining the following equations and conditions:

$$\frac{\partial(\theta+\chi)}{\partial t} + \operatorname{div}\mathbf{q} = g - \frac{\partial \theta_D}{\partial t} - \operatorname{div}\mathbf{q}_N \quad \text{in } Q, \tag{15}$$

$$\alpha\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\nabla\theta - \nabla\theta_D - \alpha\frac{\partial \mathbf{q}_N}{\partial t} - \mathbf{q}_N \quad \text{in } Q, \tag{16}$$

$$\varepsilon\frac{\partial \chi}{\partial t} + \mathrm{H}^{-1}(\chi) \ni \theta + \theta_D \quad \text{in } Q, \tag{17}$$

$$\theta = 0 \quad \text{on } \Gamma_0 \times (0,T), \qquad \mathbf{q}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{18}$$

$$\theta(\cdot,0) = \theta_0, \ \chi(\cdot,0) = \chi_0, \ \mathbf{q}(\cdot,0) = \mathbf{q}_0 \quad \text{in } \Omega. \tag{19}$$

This new formulation turns out to be quite convenient to deal with, because of the homogeneous boundary conditions for θ and $\mathbf{q}$ in (18). In the sequel the right hand side of (15) will be denoted by f and, noting that the right hand side of (16) contains a factor α, we will set

$$\mathbf{h}_\alpha = -\nabla\theta_D - \alpha(\partial\mathbf{q}_N/\partial t) - \mathbf{q}_N, \tag{20}$$

$$\mathbf{h} = -\nabla\theta_D - \mathbf{q}_N. \tag{21}$$

However the theorems stated in this paper will be valid for more general data f, $\mathbf{h}_\alpha$, and $\mathbf{h}$.

In most of physical applications the relaxation parameters introduced in (6) and (7) are very small with respect to the used lenght scale, so that the Stefan problem is often considered as approximation for the relaxed systems. Therefore it seems quite important to know the asymptotic behaviour of phase relaxation problems based on (4), (6) as ε goes to zero, as well as the asymptotic investigation of thermodynamic models including (7) as α approaches 0.

In view of these facts it appears quite natural to wonder whether the solutions of the hyperbolic phase relaxation problem (15–19) converge, in a suitable topology, to the solution of the Stefan problem when *both* the two relaxation coefficients α

and ε tend to zero. In paper [5] P. Colli and the author answer affirmatively to this question. After proving in a rigorous way that (15–19) admits at least one solution $(\theta, \chi, \mathbf{q})$, they argue on a triplet $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ which is *any* of its solutions, for $\alpha > 0$ and $\varepsilon > 0$. Then, they show that, as $\alpha, \varepsilon \searrow 0$, the family $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon})$ converges to the solution (θ, χ) of the Stefan problem (cf. (4–5) and (12–14))

$$\frac{\partial(\theta+\chi)}{\partial t} - \Delta\theta = g - \frac{\partial\theta_D}{\partial t} + \Delta\theta_D \quad \text{in } Q, \tag{22}$$

$$\chi \in \mathrm{H}(\theta+\theta_D) \quad \text{in } Q, \tag{23}$$

$$\theta = 0 \quad \text{on } \Gamma_0 \times (0,T), \qquad \partial_{\mathbf{n}}\theta = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{24}$$

$$(\theta+\chi)(\cdot,0) = \theta_0 + \chi_0 \quad \text{in } \Omega, \tag{25}$$

where $\partial_{\mathbf{n}}$ denotes the outward normal derivative to the boundary of Ω and the initial condition (25) is the one agreeing with (22). It is worth noting that in the asymptotic analysis performed in [5], no relation is required between α and ε as they tend to zero.

In the present note we want to extend the asymptotic analysis studied in [5] deducing error estimates for the sequences $\theta_{\alpha\varepsilon} - \theta$ and $\mathbf{q}_{\alpha\varepsilon} - \mathbf{q}$ with respect to the relaxation parameters α and ε. To this aim in Section 2 we give the weak formulations of the Stefan problem and of the hyperbolic Stefan problem. Then we will recall the results obtained in paper [5] and we will exploit them in Section 3 to infer the desired error estimate.

2. – Variational formulations, known theorems and new results

In this section we give the variational formulations of the problems presented in the Introduction and we recall the related existence/uniqueness and convergence results proved in [5]. Finally we will state the theorem which is the object of our note. Concerning the data of the problems we assume that

(H1) Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, with Lipschitz boundary $\Gamma := \partial\Omega$. The outward normal unit vector will be denoted by $\mathbf{n}$.

(H2) Γ_0 and Γ_1 are open subsets of Γ such that $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \Gamma$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is of Lipschitz class.

(H3) $Q := \Omega \times (0,T)$, where T is a positive number.

(H4) α and ε are positive numbers.

(H5) $f \in L^1(0,T;L^2(\Omega)) \cap L^2(0,T;(H^1_{\Gamma_0}(\Omega))')$, where $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_0} = 0\}$.

(H6) $\theta_D \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \equiv H^1(Q)$.

(H7) $\mathbf{h} \in L^2(0,T;(L^2(\Omega))^n)$ and $(\mathbf{h}_\alpha)_{\alpha>0}$ is a family of functions in $L^2(0,T;(L^2(\Omega))^n)$ such that $\mathbf{h}_\alpha \to \mathbf{h}$ in $L^2(0,T;(L^2(\Omega))^n)$ as $\alpha \searrow 0$.

(H8) $\theta_0 \in L^2(\Omega)$, $\chi_0 \in L^\infty(\Omega)$ and $0 \le \chi_0 \le 1$ a.e. in Ω, $\mathbf{q}_0 \in (L^2(\Omega))^n$.

REMARK 1 Concerning the data of the system (15–19), we point out that the assumption (H6) on θ_D and the regularities $g \in L^1(0,T;L^2(\Omega)) \cap L^2(0,T;(H^1_{\Gamma_0}(\Omega))')$ and $\mathbf{q}_N \in H^1(0,T;(L^2(\Omega))^n) \cap L^2(0,T;L^2_{\mathrm{div}}(\Omega))$ actually ensure that (H5)–(H7) hold (the definition of the space $L^2_{\mathrm{div}}(\Omega)$ is recalled just below). Let us also observe that the $(n-1)$-dimensional Hausdorff measure of Γ_0 is not required to be strictly positive.

Concerning the notation, we set $H := L^2(\Omega)$ and $V := H^1_{\Gamma_0}(\Omega)$, endow H and V with the usual inner products, and identify H with its dual space. Then we have $V \subset H \subset V'$ with dense and compact embeddings. We define the operator $A \in \mathcal{L}(V,V')$ by

$$(26) \qquad {}_{V'}\langle Av_1, v_2\rangle_V := \int_\Omega \nabla v_1 \cdot \nabla v_2, \quad v_1, v_2 \in V,$$

where the dot stands for the usual inner product in $\mathbb{R}^n$. Next, we consider the spaces $\mathbf{H} := (L^2(\Omega))^n$ and $L^2_{\mathrm{div}}(\Omega) := \{\mathbf{v} \in \mathbf{H} : \ \mathrm{div}\,\mathbf{v} \in H\}$, the latter endowed with the inner product

$$(27) \qquad (\mathbf{v}_1, \mathbf{v}_2)_{L^2_{\mathrm{div}}(\Omega)} := (\mathbf{v}_1, \mathbf{v}_2)_{\mathbf{H}} + (\mathrm{div}\,\mathbf{v}_1, \mathrm{div}\,\mathbf{v}_2)_H, \quad \mathbf{v}_1, \mathbf{v}_2 \in L^2_{\mathrm{div}}(\Omega).$$

It is well-known that if $\mathbf{v} \in L^2_{\mathrm{div}}(\Omega)$, then $\mathbf{v}\cdot\mathbf{n} \in H^{-1/2}(\Gamma)$ and the restriction $\mathbf{v}\cdot\mathbf{n}|_{\Gamma_1}$ makes sense in $(H^{1/2}_{00}(\Gamma_1))'$ (see, e.g., [7]). In this functional framework we introduce the closed subspace of $L^2_{\mathrm{div}}(\Omega)$

$$(28) \qquad \mathbf{V} := \left\{\mathbf{v} \in L^2_{\mathrm{div}}(\Omega) : \ \mathbf{v}\cdot\mathbf{n}|_{\Gamma_1} = 0\right\}.$$

If we identify $\mathbf{H}$ with its dual space, we get $\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}'$ with dense and continuous embeddings. Moreover, we will consider the operators $B \in \mathcal{L}(\mathbf{H}, V')$ and $\mathbf{L} \in \mathcal{L}(H, \mathbf{V}')$ defined by

$$(29) \qquad {}_{V'}\langle B\mathbf{u}, v\rangle_V := -\int_\Omega \mathbf{u}\cdot\nabla v, \quad \mathbf{u} \in \mathbf{H}, \ v \in V,$$

$$(30) \qquad {}_{\mathbf{V}'}\langle \mathbf{L}u, \mathbf{v}\rangle_{\mathbf{V}} := \int_\Omega u \,\mathrm{div}\,\mathbf{v}, \quad u \in H, \ \mathbf{v} \in \mathbf{V}.$$

We now recall some well-known statements that will be useful in the sequel.

LEMMA 1 *Let* $\mathbf{v}_0 \in \mathbf{H}$. *If there exists a function* $u_0 \in H$ *such that* $B\mathbf{v}_0 = u_0$, *i.e.*,

$$(31) \qquad {}_{V'}\langle B\mathbf{v}_0, v\rangle_V = -\int_\Omega \mathbf{v}_0 \cdot \nabla v = \int_\Omega u_0 v \quad \forall v \in V,$$

then $\mathbf{v}_0 \in \mathbf{V}$, $\mathrm{div}\,\mathbf{v}_0 = u_0$, *and* $\|\mathbf{v}_0\|_{\mathbf{V}} \le \|\mathbf{v}_0\|_{\mathbf{H}} + \|u_0\|_H$.

LEMMA 2 *Let* $u_0 \in H$. *If there is a function* $\mathbf{v}_0 \in \mathbf{H}$ *such that* $\mathbf{L}u_0 = \mathbf{v}_0$, *i.e.*,

$$(32) \qquad {}_{\mathbf{V}'}\langle \mathbf{L}u_0, \mathbf{v}\rangle_{\mathbf{V}} = \int_\Omega u_0 \mathrm{div}\mathbf{v} = \int_\Omega \mathbf{v}_0 \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V},$$

then $u_0 \in V$, $\mathbf{v}_0 = -\nabla u_0$, *and* $\|u_0\|_V \le \|u_0\|_H + \|\mathbf{v}_0\|_{\mathbf{H}}$.

Then the weak formulation of the problem (15–19) reads as follows.

Problem ($\mathbf{P}_{\alpha\varepsilon}$). Find a triplet $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ satisfying the following conditions

$$\theta_{\alpha\varepsilon} \in H^1(0,T;V') \cap L^\infty(0,T;H), \tag{33}$$
$$\chi_{\alpha\varepsilon} \in H^1(0,T;H), \quad 0 \le \chi_{\alpha\varepsilon} \le 1 \quad \text{a.e. in } Q, \tag{34}$$
$$\mathbf{q}_{\alpha\varepsilon} \in H^1(0,T;\mathbf{V}') \cap L^\infty(0,T;\mathbf{H}), \tag{35}$$
$$(\theta_{\alpha\varepsilon} + \chi_{\alpha\varepsilon})' + B\mathbf{q}_{\alpha\varepsilon} = f \quad \text{in } V', \text{ a.e. in } (0,T), \tag{36}$$
$$\alpha\mathbf{q}'_{\alpha\varepsilon} + \mathbf{q}_{\alpha\varepsilon} = \mathbf{L}\theta_{\alpha\varepsilon} + \mathbf{h}_\alpha \quad \text{in } \mathbf{V}', \text{ a.e. in } (0,T), \tag{37}$$
$$\varepsilon\chi'_{\alpha\varepsilon} + \mathrm{H}^{-1}(\chi_{\alpha\varepsilon}) \ni \theta_{\alpha\varepsilon} + \theta_D \quad \text{a.e. in } Q, \tag{38}$$
$$\theta_{\alpha\varepsilon}(0) = \theta_0 \quad \text{in } V', \quad \chi_{\alpha\varepsilon}(0) = \chi_0 \quad \text{in } H, \quad \mathbf{q}_{\alpha\varepsilon}(0) = \mathbf{q}_0 \quad \text{in } \mathbf{V}'. \tag{39}$$

Here and in what follows the symbol " $'$ " will denote the derivative with respect to time of vector-valued functions. The boundary conditions in (18) are not included in (33) and (35), since the spaces V and $\mathbf{V}$, respectively, do not appear there. On the other hand, the analoguos homogeneous boundary conditions for the integrated variables $\int_0^t \theta_{\alpha\varepsilon}$, $\int_0^t \mathbf{q}_{\alpha\varepsilon}$ are collected into equations (36–37), and this can be easily checked with the help of integrations in time and using Lemmas 1–2.

The following existence theorem for Problem ($\mathbf{P}_{\alpha\varepsilon}$) was proved in [5, Theorem 2.1].

THEOREM 1 *Assume that* (H1)–(H8) *hold. Then Problem* ($\mathbf{P}_{\alpha\varepsilon}$) *admits at least one solution. Moreover there exists a constant $C > 0$, independent of α and ε, such that for all solutions $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ of* ($\mathbf{P}_{\alpha\varepsilon}$) *there holds*

$$\begin{aligned} &\|\theta_{\alpha\varepsilon}\|_{L^\infty(0,T;H)} + \|\theta_{\alpha\varepsilon} + \chi_{\alpha\varepsilon}\|_{H^1(0,T;V')} + \alpha^{1/2}\|\mathbf{q}_{\alpha\varepsilon}\|_{L^\infty(0,T;\mathbf{H})} \\ &+ \|\mathbf{q}_{\alpha\varepsilon}\|_{L^2(0,T;\mathbf{H})} + \alpha\|\mathbf{q}'_{\alpha\varepsilon}\|_{L^2(0,T;\mathbf{V}')} + \varepsilon^{1/2}\|\chi_{\alpha\varepsilon}\|_{H^1(0,T;H)} + \|\chi_{\alpha\varepsilon}\|_{L^\infty(Q)} \le C. \end{aligned} \tag{40}$$

We want to stress that a relevant feature of the analysis performed in [5] is that estimate (40) is fulfilled by any solution of Problem ($\mathbf{P}_{\alpha\varepsilon}$).

Now let us state the weak formulation of the Stefan problem.

Problem (P). Find a pair (θ, χ) satisfying the following conditions

$$\theta \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{41}$$
$$\chi \in L^\infty(Q), \tag{42}$$
$$\theta + \chi \in H^1(0,T;V'), \tag{43}$$
$$(\theta + \chi)' + A\theta = f - B\mathbf{h} \quad \text{in } V', \text{ a.e. in } (0,T), \tag{44}$$
$$\chi \in \mathrm{H}(\theta + \theta_D) \quad \text{a.e. in } Q, \tag{45}$$
$$(\theta + \chi)(0) = \theta_0 + \chi_0 \quad \text{in } V'. \tag{46}$$

The next result can be easily deduced by slightly adapting the arguments reported, e.g., in [4] (see also [10, Chapter II]).

THEOREM 2 *Assume that* (H1)–(H8) *hold. Then there exists a unique solution to Problem* **(P)**.

Note that Problem **(P)** can be equivalently formulated saying that a triplet $(\theta, \chi, \mathbf{q})$ is to be found in such a way that (41–43) and (45–46) are satisfied and

$$\mathbf{q} \in L^2(0,T;\mathbf{H}) \tag{47}$$

$$(\theta + \chi)' + B\mathbf{q} = f \quad \text{in } V', \text{ a.e. in } (0,T), \tag{48}$$

$$\mathbf{q} = -\nabla\theta + \mathbf{h} \quad \text{a.e. in } Q. \tag{49}$$

Consequently Theorem 2 can be rephrased according to this equivalent formulation.

As we recalled in the Introduction, the object of paper [5] is the asymptotic behaviour of the solutions of Problem $(\mathbf{P}_{\alpha\varepsilon})$, as α and ε tend to zero. The precise result is the following

THEOREM 3 *Assume that hypoteses* (H1)–(H8) *hold. Let* (θ, χ) *be the unique solution to Problem* **(P)** *and let* $\mathbf{q}$ *be defined by* (49). *Moreover for any pair* $\alpha, \varepsilon > 0$ *let* $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ *denote an arbitrary solution to Problem* $(\mathbf{P}_{\alpha\varepsilon})$. *Then, as* $\alpha, \varepsilon \searrow 0$, *we have that*

$$\theta_{\alpha\varepsilon} \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(0,T;H), \tag{50}$$

$$\chi_{\alpha\varepsilon} \overset{*}{\rightharpoonup} \chi \quad \text{in } L^\infty(Q), \tag{51}$$

$$\mathbf{q}_{\alpha\varepsilon} \rightharpoonup \mathbf{q} \quad \text{in } L^2(0,T;\mathbf{H}). \tag{52}$$

We warn the reader that the part of the previous theorem concerning the convergence of $\theta_{\alpha\varepsilon}$ and $\chi_{\alpha\varepsilon}$ is stated in [5, Theorem 2.2]. Instead the convergence of $\mathbf{q}_{\alpha\varepsilon}$ is proved in Section 5 of paper [5] where is exploited to prove (50–51). However, if we assume (50–51), then (52) can be easily inferred from (37), (40), (H7), and (41).

Let us introduce a general notation which will hold throughout the sequel. For a map $\psi \in L^1(0,T;X)$, where X is a Banach space, we define $\widehat{\psi} : [0,T] \to X$ by

$$\widehat{\psi}(t) := \int_0^t \psi, \quad t \in [0,T]. \tag{53}$$

Now we can state the theorem concerning the error estimate. In order to prove this theorem we need to prescribe a certain rate of convergence for the sequence $\mathbf{h}_\alpha - \mathbf{h}$. Precisely we assume that there exists a constant $C_0 > 0$, independent of α and ε, such that

$$||\mathbf{h}_\alpha - \mathbf{h}||_{L^2(0,T;\mathbf{H})} \leq C_0 \alpha^{1/2} \tag{54}$$

for all $\alpha > 0$. This assumption seems quite reasonable, in view of the fact that in applications the expression of $\mathbf{h}_\alpha$ and $\mathbf{h}$ are given respectively by (20) and (21).

THEOREM 4 *Assume that the hypoteses of Theorem 3 hold and that* (54) *is valid for some positive constant* C_0 *independent of* α *and* ε. *Then there exist a constant* $C_1 > 0$, *independent of* α *and* ε, *such that*

$$||\theta_{\alpha\varepsilon} - \theta||_{L^2(0,T;H)} + ||\widehat{\mathbf{q}}_{\alpha\varepsilon} - \widehat{\mathbf{q}}||_{L^\infty(0,T;\mathbf{H})} \leq C_1(\alpha^{1/4} + \varepsilon^{1/4}). \tag{55}$$

3. – Proof of the error estimate

This section is devoted to proof of the error estimate stated in Theorem 4. For any pair of positive numbers α and ε let us choose an arbitrary solution $(\theta_{\alpha\varepsilon}, \chi_{\alpha\varepsilon}, \mathbf{q}_{\alpha\varepsilon})$ of Problem $(\mathbf{P}_{\alpha\varepsilon})$. In the sequel, the same symbol C will be employed to denote different positive constants which depends only on the data, but not on α and ε. Let us start observing that an integration of the equation (36) shows that

$$z_{\alpha\varepsilon} := \widehat{f} + \theta_0 + \chi_0 - \theta_{\alpha\varepsilon} - \chi_{\alpha\varepsilon} = B\widehat{\mathbf{q}}_{\alpha\varepsilon}, \tag{56}$$

hence, thanks to (H5), (H8), and (40), we deduce that $z_{\alpha\varepsilon}$ is uniformly bounded in $L^\infty(0,T;H)$ with respect to α and ε. Therefore

$$-\int_\Omega \widehat{\mathbf{q}}_{\alpha\varepsilon}(t)\cdot\nabla v = \int_\Omega z_{\alpha\varepsilon}(t)v \quad \forall v \in V \tag{57}$$

and applying Lemma 1 we deduce that $\|\widehat{\mathbf{q}}_{\alpha\varepsilon}\|_{L^\infty(0,T;\mathbf{V})}$ is bounded independently of α, ε, and we see that $\widehat{\mathbf{q}} \in L^\infty(0,T;\mathbf{V})$ and

$$\widehat{\mathbf{q}}_{\alpha\varepsilon} \stackrel{*}{\rightharpoonup} \widehat{\mathbf{q}} \quad \text{in } L^\infty(0,T;\mathbf{V}). \tag{58}$$

Now, for convenience, let us set $\Theta_{\alpha\varepsilon} := \theta_{\alpha\varepsilon}-\theta$, $\mathcal{X}_{\alpha\varepsilon} := \chi_{\alpha\varepsilon}-\chi$, and $\boldsymbol{\Psi}_{\alpha\varepsilon} := \mathbf{q}_{\alpha\varepsilon}-\mathbf{q}$, and let $t \in (0,T)$. Let us integrate in time the difference of equations (36) and (44). We get, thanks to (39) and (46),

$$\Theta_{\alpha\varepsilon} + \mathcal{X}_{\alpha\varepsilon} + B\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon} = 0 \quad \text{in } V', \text{ in } [0,T]. \tag{59}$$

Let us note that $\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon} = \widehat{\mathbf{q}}_{\alpha\varepsilon} - \widehat{\mathbf{q}} \in L^\infty(0,T;\mathbf{V})$ and $B\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon} = \operatorname{div}\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon}$, thus equation (59) is in fact the following identity in H:

$$\Theta_{\alpha\varepsilon} + \mathcal{X}_{\alpha\varepsilon} + \operatorname{div}\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon} = 0 \quad \text{in } H, \text{ in } [0,T]. \tag{60}$$

Let us multiply (60) by $\Theta_{\alpha\varepsilon} \in L^\infty(0,T;H)$ and integrate over $(0,t)\times\Omega$. We get

$$\|\Theta_{\alpha\varepsilon}\|^2_{L^2(0,t;H)} + \int_0^t\int_\Omega \mathcal{X}_{\alpha\varepsilon}\Theta_{\alpha\varepsilon} + \int_0^t\int_\Omega (\operatorname{div}\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon})\Theta_{\alpha\varepsilon} = 0. \tag{61}$$

Now let us subtract the equation (49) from (37). We find that

$$\alpha\mathbf{q}'_{\alpha\varepsilon} + \boldsymbol{\Psi}_{\alpha\varepsilon} = \mathbf{L}\Theta_{\alpha\varepsilon} + \mathbf{h}_\alpha - \mathbf{h} \quad \text{in } \mathbf{V}', \text{ a.e. in } (0,T). \tag{62}$$

Applying equation (62) to $\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon} \in L^\infty(0,T;\mathbf{V})$ and integrating in time we have that

$$\alpha\int_0^t {}_{\mathbf{V}'}\langle \mathbf{q}'_{\alpha\varepsilon}, \widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon}\rangle_{\mathbf{V}} + \frac{1}{2}\|\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon}(t)\|^2_{\mathbf{H}} = \int_0^t\int_\Omega \Theta_{\alpha\varepsilon}\operatorname{div}\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon} + \int_0^t\int_\Omega (\mathbf{h}_\alpha - \mathbf{h})\cdot\widehat{\boldsymbol{\Psi}}_{\alpha\varepsilon}. \tag{63}$$

Finally, observe that the inclusion (45) is equivalent to

$$\mathrm{H}^{-1}(\chi) \ni \theta + \theta_D \quad \text{a.e. in } Q. \tag{64}$$

If we subtract (64) from (38), we get the inclusion

$$\varepsilon\chi'_{\alpha\varepsilon} + \mathrm{H}^{-1}(\chi_{\alpha\varepsilon}) - \mathrm{H}^{-1}(\chi) \ni \Theta_{\alpha\varepsilon} \quad \text{a.e. in } Q. \tag{65}$$

Multiplying (65) by $\mathcal{X}_{\alpha\varepsilon}$ and taking into account of the monotonicity of H^{-1} we infer that

$$(66) \qquad \varepsilon\int_0^t\int_\Omega \chi'_{\alpha\varepsilon}\mathcal{X}_{\alpha\varepsilon} \leq \int_0^t\int_\Omega \Theta_{\alpha\varepsilon}\mathcal{X}_{\alpha\varepsilon}.$$

If we add equations (61), (63), and (66) and we observe that there are two cancellations, we obtain the following inequality:

$$(67) \qquad \begin{aligned} &\|\Theta_{\alpha\varepsilon}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\widehat{\mathbf{\Psi}}_{\alpha\varepsilon}(t)\|^2_{\mathbf{H}} \\ \leq\ & -\alpha\int_0^t {}_{\mathbf{V}'}\langle \mathbf{q}'_{\alpha\varepsilon}, \widehat{\mathbf{\Psi}}_{\alpha\varepsilon}\rangle_{\mathbf{V}} + \int_0^t\int_\Omega (\mathbf{h}_\alpha - \mathbf{h})\cdot\widehat{\mathbf{\Psi}}_{\alpha\varepsilon} - \varepsilon\int_0^t\int_\Omega \chi'_{\alpha\varepsilon}\mathcal{X}_{\alpha\varepsilon}. \end{aligned}$$

Now we are going to estimate the right hand side of (67). Concerning the first integral, observe that $\mathbf{q}_{\alpha\varepsilon} \in H^1(0,T;\mathbf{V}') \cap L^\infty(0,T;\mathbf{H})$ and $\widehat{\mathbf{\Psi}}_{\alpha\varepsilon} \in H^1(0,T;\mathbf{H}) \cap L^\infty(0,T;\mathbf{V})$. Therefore [5, Lemma 5.1] applies and let us deduce that the function $t \mapsto {}_{\mathbf{V}'}\langle \mathbf{q}_{\alpha\varepsilon}(t), \widehat{\mathbf{\Psi}}_{\alpha\varepsilon}(t)\rangle_{\mathbf{V}}$ is absolutely continuous. Then we find a positive constant C such that

$$(68) \qquad \begin{aligned} & -\alpha\int_0^t {}_{\mathbf{V}'}\langle \mathbf{q}'_{\alpha\varepsilon}, \widehat{\mathbf{\Psi}}_{\alpha\varepsilon}\rangle_{\mathbf{V}} \\ =\ & -\alpha {}_{\mathbf{V}'}\langle \mathbf{q}_{\alpha\varepsilon}(t), \widehat{\mathbf{\Psi}}_{\alpha\varepsilon}(t)\rangle_{\mathbf{V}} + \alpha\int_0^t (\mathbf{q}_{\alpha\varepsilon}, \mathbf{\Psi}_{\alpha\varepsilon})_{\mathbf{H}} \\ \leq\ & \alpha\|\mathbf{q}_{\alpha\varepsilon}(t)\|_{\mathbf{H}}\|\widehat{\mathbf{\Psi}}_{\alpha\varepsilon}(t)\|_{\mathbf{H}} + \alpha\|\mathbf{q}_{\alpha\varepsilon}\|_{L^2(0,t;\mathbf{H})}\|\mathbf{\Psi}_{\alpha\varepsilon}\|_{L^2(0,t;\mathbf{H})} \\ \leq\ & \alpha^{1/2}\|\alpha^{1/2}\mathbf{q}_{\alpha\varepsilon}\|_{L^\infty(0,t;\mathbf{H})}\|\widehat{\mathbf{\Psi}}_{\alpha\varepsilon}\|_{L^\infty(0,t;\mathbf{H})} + \alpha^{1/2}\|\alpha^{1/2}\mathbf{q}_{\alpha\varepsilon}\|_{L^2(0,t;\mathbf{H})}\|\mathbf{\Psi}_{\alpha\varepsilon}\|_{L^2(0,t;\mathbf{H})} \\ \leq\ & C\alpha^{1/2}, \end{aligned}$$

the last inequality holding by virtue of (40), (58), and (52). The second integral in (67) can be controlled by observing that, thanks to (54),

$$(69) \qquad \int_0^t\int_\Omega (\mathbf{h}_\alpha - \mathbf{h})\cdot\widehat{\mathbf{\Psi}}_{\alpha\varepsilon} \leq \|\mathbf{h}_\alpha - \mathbf{h}\|_{L^2(0,t;\mathbf{H})}\|\widehat{\mathbf{\Psi}}_{\alpha\varepsilon}\|_{L^2(0,t;\mathbf{H})} \leq C_0\alpha^{1/2},$$

for some positive constant C. Finally, thanks to (40) and to (51) we have that

$$(70) \qquad \begin{aligned} -\varepsilon\int_0^t\int_\Omega \chi'_{\alpha\varepsilon}\mathcal{X}_{\alpha\varepsilon} &\leq \varepsilon\|\chi'_{\alpha\varepsilon}\|_{L^2(0,t;H)}\|\mathcal{X}_{\alpha\varepsilon}\|_{L^2(0,t;H)} \\ &= \varepsilon^{1/2}\|\varepsilon^{1/2}\chi'_{\alpha\varepsilon}\|_{L^2(0,t;H)}\|\mathcal{X}_{\alpha\varepsilon}\|_{L^2(0,t;H)} \\ &\leq C\varepsilon^{1/2}. \end{aligned}$$

Hence collecting (68), (69), and (70), inequality (67) entails that

$$(71) \qquad \|\theta_{\alpha\varepsilon} - \theta\|^2_{L^2(0,T;H)} + \|\widehat{\mathbf{q}}_{\alpha\varepsilon} - \widehat{\mathbf{q}}\|^2_{L^\infty(0,T;\mathbf{H})} \leq C(\alpha^{1/2} + \varepsilon^{1/2}),$$

and (55) is proved.

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On a Thermodynamical Model for Type-II High-T_c Superconductors. Theory and applications.

L. Restuccia* B. T. Maruszewski †

1. – Introduction

Type-II high-T_c superconductor materials find applications in many fundamental technological sectors: in the field of optoelectronics for the realization of wave guides; in the field of electronic sensors, in the processes of fabrication of microwave devices, in the realization of SQUID sensors in a biomedical frame, in the field of electromagnetic screens mainly in biomagnetic measures, in applied computer science in the technology for integrated circuits VLSI (very large scale integration), in the realization of thin superconductor films in order to construct fixed memories and also in the preparation of high critical current superconductors for future applications in the energy transport sector, in particular in preparation of superconductors that create magnetic levitation and suspension. In this survey paper [12-14] some phenomenological aspects of superconductivity are considered and a thermodynamical model for a type-II high-T_c superconductor is presented, describing the properties of Abrikosov flux-line vortices both in lattice and fluid states. A specific stress tensor is created. A method to linearize the obtained field equations has been applied and the magnetomechanical wave propagation problem is considered for vortex lattice both in the solid and liquid states. As an example the dispersion problem in YBCO-ceramics has been studied.

2. – Phenomenological aspects of superconductivity

Superconductors belong generally to two classes of such materials. At applied field strengths less the critical value H_c a type-I superconductor expels magnetic flux from the material and hence it is in the Meissner state. In contrast, a type-II superconductor behaves differently. There are two critical fields for these materials. For applied field strengths less than the lower critical field H_{c_1} that superconductor will exhibit the usual Meissner effect. Applied fields greater than the upper critical field H_{c_2} destroy the superconductivity altogether. Between the lower H_{c_1} and upper

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H_{c_2} magnetic field strengths, the superconductor is in the mixed or vortex state [1-6], and above the critical field H_{c_1} the magnetic field penetrates the superconductor on the London depth frequency-independent (see Figure 1).

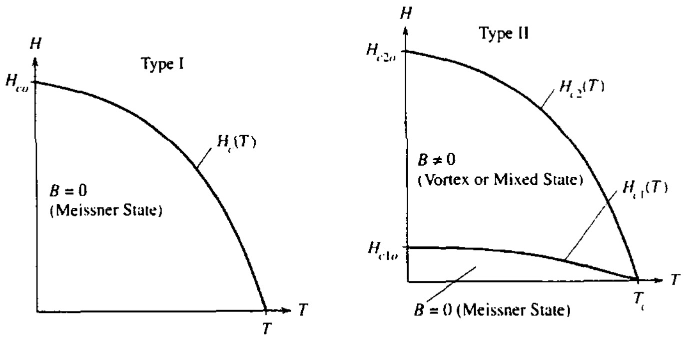

Figure 1: The H - T phase diagrams for type I and type II superconductors.

If the superconductor is limited by a plane and the applied magnetic field is perpendicular to this plane, magnetic flux penetrates the superconductor in the form of Abrikosov vortices (also called flux lines, flux tubes or fluxons) each carrying a quantum of magnetic flux. These tiny vortices of supercurrent tend to arrange themselves in a triangular flux-line lattice, which is more or less perturbed by material inhomogeneities that pin the flux lines. In the first picture of Figure 2 a type-II superconductor has been decorated with magnetic particles in the case that the applied field is perpendicular to the plane of the page.

In classical low-T_c superconductors the vortex lattice is mostly consisted of a parallel straight vortex lines arrangement of which the crossection forms a triangu-

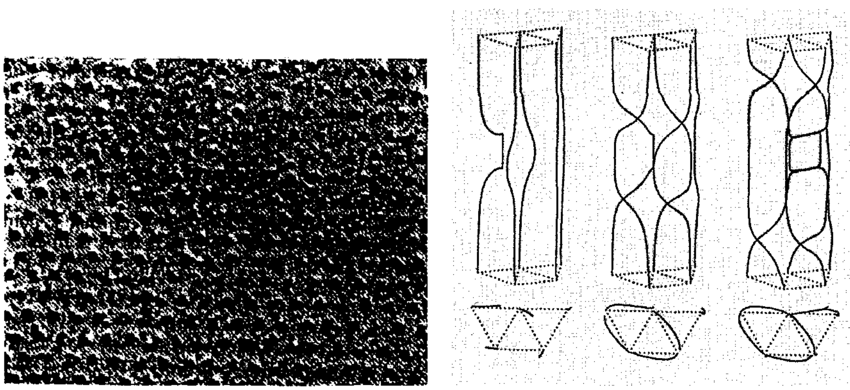

Figure 2: A triangular flux-line lattice (Abrikosov arrangement) and curved vortex lines

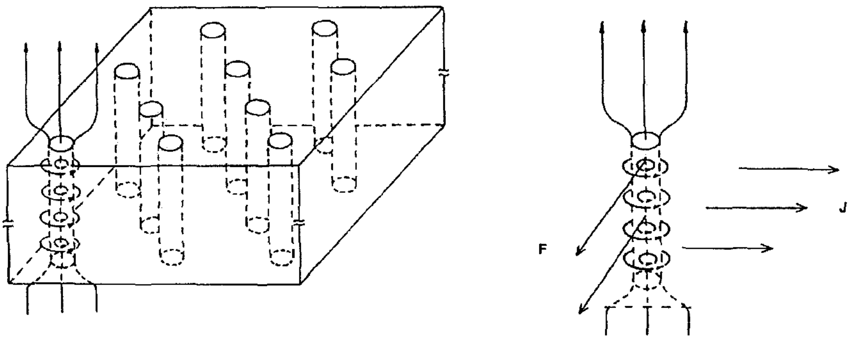

Figure 3: Structure of array of vortices (**F** - Lorentz force, **J** - supercurrent).

lar (also hexagonal) or quadratic symmetry (cf. [3, 4]). However, recent research shows that the vortex lines can be curved or even tangled along the material [5, 6]. Moreover, because the vortex lines can form, among others, sets called twisted triplets, twisted quadruplets, single loops or pairs, the vortex field can be considered within three dimensions. Because of the fact that each vortex line has a sign (has definite vorticity), lines of the opposite signs anihilate. So the density of the vortex field can vary. In the second picture of Figure 2 we see curved vortex lines and their projections to a normal plane.

Inside each vortex there is a core in the normal phase, then outside that there is the superconducting phase. So, the supercurrent flows around each vortex. There exist also the Lorentz force interactions among them. Those interactions form an origin of an additional mechanical (stress, pressure, and the like) field occurring in the type-II superconductor (see Figure 3).

As we have mentioned before vortices (hard vortices) are pinned, in general, on crystal lattice imperfections of the superconductor. However, if the Lorentz force between the vortices is greater than the pinning force, the vortex lattice behaves elastically [4, 5]. The vortex field near the lower critical magnetic intensity limit H_{c_1} forms a lattice of the mentioned above symmetries. The "fluidity" of the vortex array is observed when the applied magnetic field tends to its upper critical limit H_{c_2}. In this way we meet a very interesting situation in a type-II superconductor. We can say, that there coexist two mechanical fields in the medium. One of them is of pure elastic character coming from the properties of crystal lattice of the superconductor . The second one comes from the vortex lattice, which, being of elastic character near the lower magnetic field strength limit H_{c_1}, transfers smoothly into "fluid" form near the upper magnetic field strength H_{c_2}. The vortex motion (creep) is accompanied by an energy dissipation. That motion is damped by a force proportional to the vortex velocity. Hence, except for the elastic properties the vortex field is also of a viscous character. The resistivity in the area of the vortex creep is the same as the resistivity of the current which would flow inside the vortex core. Hence, the viscosity coefficient reads [4]

$$\eta = \frac{\Phi_0 \mu_0 H_{c2}}{\rho_n}, \tag{1}$$

where Φ_o is the magnetic flux, μ_o denotes the permeability of vacuum and ρ_n is the resistivity in the normal state.

3. – Thermodynamical model

In order to obtain an extended-like thermodynamic model for the viscoelastic field of vortices in the type-II superconductor we assume that: *the characteristic volume* of the body is sufficiently large for averaging all the physical quantities taken into consideration (its dimensions are much greater than the London penetration depth λ_o and the coherence length ξ, the diameter of a vortex line), only *depinned (soft) vortices* , averaged in characteristic volume are considered [5] and any creation or anihilation of the vortices is omitted, *the mass density* ρ of the vortex field is defined as the mass of the material in the normal state related to the total volume of the material, *the interaction between vortices* is due only to the Lorentz force, *the energy dissipation* occurs only because of the viscosity η of the vortex field caused by ohmic-like resistivity (normal-state resistivity) inside the vortex core [5], the relaxation feature of *the thermal field* is not taken into account, because of very low temperatures of the considered material, only *small deformations* of the vortex field are considered and they are described by the linear strain tensor $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, where u_i denotes the displacement vector of a vortex, and the velocity of the vortex field point $v_i = \dot{u}_i$ is also small.

So, we confine ourselves to the linear theory of the vortex lattice and to create an extended-like thermodynamic model for the viscoelastic field of vortices we choose the following vector of state (the set of independent variables) (cf. [7, 12]):

$$C = \left\{ \varepsilon_{ij}, \dot{\varepsilon}_{ij}, \varphi, A_i, T, T_{,i}, \psi, \tilde{\psi}, \psi_{,i}, \tilde{\psi}_{,i}, q_i, j_i^S \right\}, \tag{2}$$

where $\dot{\varepsilon}_{ij}$ indicates the viscoelastic character of the vortex field, φ and A_i are the scalar and vector electromagnetic potentials, respectively, T is the absolute temperature, q_i is the heat flux and j_i^S is the supercurrent density, ψ is the order parameter (the wave function of a Cooper pair) and $\tilde{\psi}$ is its complex conjugate. $\psi(\mathbf{r}, t)$ is interpreted as the probability that the single quantum particle would be located at $\mathbf{r}$ at time t, i.e. $\tilde{\psi}(\mathbf{r}, t)\psi(\mathbf{r}, t) = n^S(\mathbf{r}, t)$, where n^S is the local density of superelectrons. The gradients $T_{,i}$, $\tilde{\psi}_{,i}$, $\psi_{,i}$ concern possible nonlocal effects in the material. We have three groups of fundamental laws that govern the set (2). The first group concerns *the mass density balance*, *the momentum balance* (for a nonpolarized and nonmagnetized body) and *the density of the internal energy balance*, respectively [12]

$$\dot{\rho} + \rho v_{k,k} = 0 \tag{3}$$

$$\rho \dot{v}_k - \sigma_{ik,i} - \epsilon_{kij} \left(j_i^N + j_i^S \right) B_j - f_k = 0, \tag{4}$$

$$(5) \qquad \rho\dot{e} - \sigma_{ik}v_{k,i} - q_{k,k} - \left(j_i^N + j_i^S\right)\mathcal{E}_i - \rho r = 0.$$

In (3)–(5) σ_{ik} is the viscoelastic symmetric stress tensor, j_i^N is the normal current which satisfies Ohm's law, j^S is the supercurrent, B_j is the magnetic induction, f_k is a given external body force, e is the internal energy density, $\mathcal{E}_i$ is the electromotive intensity, according to the situation that the observer is in the moving point and r is the heat source distribution.

The second group of laws deals with the electromagnetic field and concerns *Maxwell's equations, the constitutive equations and London's first equation* [3], respectively

$$(6) \qquad \epsilon_{ijk}E_{k,j} = -\frac{\partial B_i}{\partial t}, \qquad B_{k,k} = 0, \qquad \epsilon_{ijk}H_{k,j} = j_i^S + j_i^N, \qquad E_{k,k} = 0,$$

$$(7) \qquad B_i = \mu_0 H_i, \quad D_k = \epsilon_0 E_k, \quad \frac{\partial j_k^S}{\partial t} = \frac{1}{\Lambda}\left(E_k + \epsilon_{kls}v_l B_s\right), \quad \Lambda = \mu_0\lambda_0^2.$$

In (6)-(7) μ_0 is the permeability and ϵ_0 is the permittivity of vacuum, the displacement current and the free charge density have been disregarded. Moreover, the following relations hold true

$$(8) \qquad \begin{aligned} &\mathcal{E}_i = E_i + \epsilon_{ijk}v_j B_k, \quad && E_i = -\varphi_{,i} - \frac{\partial A_i}{\partial t}, \\ &B_i = \epsilon_{ijk}A_{k,j}, && \frac{1}{\mu_0}A_{i,kk} - j_i^N - j_i^S = 0. \end{aligned}$$

The third group of fundamental laws concerns *the time evolution of fluxes and internal variables*

$$(9) \qquad \dot{q}_k - Q_k(C) = 0, \qquad \dot{j}_k^S - J_k^S(C) = 0,$$

$$(10) \qquad \dot{\psi} - \Psi(C) = 0, \qquad \dot{\tilde{\psi}} - \tilde{\Psi}(C) = 0.$$

The use of the second law of thermodynamics in the form of the entropy inequality

$$(11) \qquad \rho\dot{s} + \Phi_{k,k} - \frac{\rho r}{T} \geq 0$$

gives us a possibility to determine all the constitutive functions that in our case form the set

$$(12) \qquad \mathcal{Z} = \left\{\sigma_{ik}, e, s, \Phi_k, j_k^N, Q_k, \Psi, \tilde{\Psi}, J_k^S\right\}.$$

The most effective way to do that is the use of Liu's theorem [9]. For the purpose of the paper we are only interested in the determination of mechanical phenomenological properties of the vortex lattice field versus H ($H_{c_1} < H < H_{c_2}$) and to create a proper constitutive law for the stress tensor σ_{ij}. We choose it as based on isotropic polynomial representation of tensor function of tensor, vector and scalar variables [10, 11]. We assume that the considered lattice is isotropic and we reduce for our purpose the set (2) to

$$C_1 = \{\varepsilon_{ij}, \dot{\varepsilon}_{ij}, B_i\} \tag{13}$$

Then, we assume that σ_{ij} should be of a similar form like a viscoelastic isotropic body, i.e.

$$\sigma_{ij} = 2M\varepsilon_{ij} + L\varepsilon_{kk}\delta_{ij} + 2\mathcal{M}\dot{\varepsilon}_{ij} + \mathcal{L}\dot{\varepsilon}_{kk}\delta_{ij} \tag{14}$$

where M, L, $\mathcal{M}$, $\mathcal{L}$ are the material coefficients. All of them concern both the lattice and fluid states and they are functions of the invariant $B_k B_k$ [12].

Because the definition of σ_{ik} concerns both the viscoelastic lattice and the viscous fluid, we split (14) into the trace and deviatoric parts: $\sigma_o = \sigma_{kk}$, $\tau_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_o\delta_{ij}$. So,

$$\begin{aligned} \sigma_o = {} & (2M + 3L)\,\varepsilon_{kk} + (2\mathcal{M} + 3\mathcal{L})\,\dot{\varepsilon}_{kk}, \\ \tau_{ij} = {} & 2M\varepsilon_{ij} - \frac{2}{3}M\varepsilon_{kk}\delta_{ij} + 2\mathcal{M}\dot{\varepsilon}_{ij} - \frac{2}{3}\mathcal{M}\dot{\varepsilon}_{kk}\delta_{ij}. \end{aligned} \tag{15}$$

Now, we assume that the pure lattice exists only for H_{c1} and the pure fluid exists only for H_{c2}. For magnetic field intensities $H_{c1} < H < H_{c2}$ the coexistence of the lattice and fluid occurs. In this way, the following superposition holds true:

$$\sigma_o = \sigma_o^{lattice} + \sigma_o^{fluid} \qquad , \qquad \tau_{ij} = \tau_{ij}^{lattice} + \tau_{ij}^{fluid}. \tag{16}$$

Hence, introducing the following parameters whose form suggests the invariant $B_s B_s$

$$\alpha = \left(\frac{H_{c_2} - H}{H_{c_2} - H_{c_1}}\right)^2, \qquad \alpha = \begin{cases} 0 & if\ H = H_{c_2} \\ 1 & if\ H = H_{c_1} \end{cases}, \tag{17}$$

$$\beta = \left(\frac{H - H_{c_1}}{H_{c_2} - H_{c_1}}\right)^2, \qquad \beta = \begin{cases} 0 & if\ H = H_{c_1} \\ 1 & if\ H = H_{c_2} \end{cases}, \tag{18}$$

we can rewrite (15)-(16) as follows

$$\sigma_o^{lattice} = \alpha\,(2\mu + 3\lambda)\,\varepsilon_{kk} + \alpha\,(2\mu_L + 3\lambda_L)\,\dot{\varepsilon}_{kk}, \qquad \sigma_o^{fluid} = -\beta p, \tag{19}$$

$$\tau_{ij}^{lattice} = 2\mu\alpha\varepsilon_{ij} - \frac{2}{3}\mu\alpha\varepsilon_{kk}\delta_{ij} + 2\mu_L\alpha\dot{\varepsilon}_{ij} - \frac{2}{3}\mu_L\dot{\varepsilon}_{kk}\delta_{ij} \qquad , \qquad \tau_{ij}^{fluid} = \beta D\dot{\varepsilon}_{ij}, \tag{20}$$

where λ, μ are the Lamé constants of the lattice, λ_L, μ_L are the viscoelastic constants of the lattice, p is the pressure of the fluid and D is the viscosity of the fluid. We have assumed in (19) that the viscosity during compression is negligible so the viscous coefficient by $\dot{\varepsilon}_{kk}$ vanishes. Introducing tilt modulus K and shear modulus G both for elastic and viscous states of the vortex lattice as

$$3K = 2\mu + 3\lambda, \qquad 3K_L = 2\mu_L + 3\lambda_L, \tag{21}$$

$$(22) \qquad G=\mu, \qquad G_L=\mu_L, \qquad K_L=3D=3\eta, \qquad G_L=D=\eta,$$

where η is defined by (1), from (14)-(21) we finally obtain the required form of the stress tensor σ_{ij}:

$$(23) \quad \sigma_{ij}=\left[\left(\frac{1}{3}\alpha K-\frac{2}{3}\alpha G\right)\varepsilon_{kk}-\frac{2}{3}\alpha\eta\dot{\varepsilon}_{kk}-\beta p\right]\delta_{ij}+2\alpha G\varepsilon_{ij}+2\left(\alpha+\beta\right)\eta\dot{\varepsilon}_{ij} \ ,$$

$$(24) \qquad \text{where} \qquad \alpha+\beta=\begin{cases} =1 & if \ H=H_{c_1} \ or \ H=H_{c_2} \\ =f(H) & if \ H_{c_1}<H<H_{c_2} \end{cases} .$$

We have assumed in (23) that the viscosity of the vortex lattice deals only with shear [4]. Therefore, we have put $K_L=0$ in (23). The final form of σ_{ij} for $H=H_{c_1}$ concerns only the vortex lattice and for $H=H_{c_2}$ only the pure fluid. The transfer from the lattice state to the fluid one is continuous, not in the form of a phase transition. That problem is still open because there has not been any convincing proof whether the transfer is smooth or singular [5].

We apply now the proposed tensor (23) to the phenomenological field theory for the vortex field resulting from the thermodynamical model presented. Here,we confine ourselves only to the magnetomechanical features of the considered field. Now, using (4), (8) and the Maxwell equations we obtain (cf. [12])

$$(25) \qquad \sigma_{ik,k}+\epsilon_{ikr}\epsilon_{kls}H_{s,l}B_r=\rho\ddot{u}_i.$$

Eliminating from (6), (7) and (25) E_k, j_k^S, B_k, and σ_{ik} with the help of (23) and utilizing the definition of the linear strain tensor we arrive at the following set of the looked for field equations:

$$(26) \qquad \begin{aligned} &\left[\alpha G+(\alpha+\beta)\,\eta\frac{\partial}{\partial t}\right]u_{i,jj}+\frac{1}{3}\left[(K+G)\,\alpha+(\alpha+3\beta)\,\eta\frac{\partial}{\partial t}\right]u_{j,ij} \\ &\quad+\left[\alpha_{,j}G+(\alpha+\beta)_{,j}\,\eta\frac{\partial}{\partial t}\right](u_{i,j}+u_{j,i}) \\ &+\frac{1}{3}\alpha_{,i}\left(K-2G-2\eta\frac{\partial}{\partial t}\right)u_{k,k}-\beta p_{,i}-\beta_{,i}p-\mu_o\left(H_{r,i}-H_{i,r}\right)H_r=\rho\ddot{u}_i \ , \end{aligned}$$

$$(27) \qquad \lambda_o^2\dot{H}_{i,kk}-\dot{H}_i+\dot{u}_{i,k}H_k-\dot{u}_{k,k}H_i=0.$$

Equations (26), (27) describe dynamics of the vortex field in the type-II superconductor.

Now, we linearize the nonlinear system (3), (26) and (27). Then, we assume that in the interval (H_{c_1} ,H_{c_2}) the amplitude of magnetic field is subjected to a small perturbation starting from a constant magnetic field H_o and also the mass density and the displacement vector modulus of the vortex field are subjected to a small perturbation from the values taken in a uniform and stationary state. So, we have

$$(28) \qquad H=H_0+h \ , \quad |h|<<|H_0| \ , \quad \rho=\rho_0+\tilde{\rho} \ , \quad u=u_0+\tilde{u} \ .$$

In the sequel we continue to indicate $\tilde{\rho}$ and $\tilde{u}$ as ρ and u.

Substituting (28) into (17) and (18) we obtain the linearized forms for α and β (the superimposed dash denotes the linearized parameters):

$$\bar{\alpha} = \frac{a}{b}(a-2h), \qquad \bar{\beta} = \frac{c}{b}(c+2h), \qquad \bar{\alpha}+\bar{\beta} = \frac{1}{b}\left[a^2+c^2+2h(c-a)\right], \tag{29}$$

$$a = H_{c_2} - H_0, \qquad b = (H_{c_2} - H_{c_1})^2, \qquad c = H_0 - H_{c_1}. \tag{30}$$

$\bar{\alpha}$ and $\bar{\beta}$ are not functions of H but are dependent on the parameter H_0 and small perturbation h. Using (23), (29), (30) in (26), (27) we arrive at the following linear form of the set of basic equations:

$$\begin{aligned} &\left[\bar{\alpha}G + (\bar{\alpha}+\bar{\beta})\,\eta\frac{\partial}{\partial t}\right]u_{i,jj} + \frac{1}{3}\left[(K+G)\,\bar{\alpha} + (\bar{\alpha}+3\bar{\beta})\,\eta\frac{\partial}{\partial t}\right]u_{j,ij} \\ &\qquad\qquad -\bar{\beta}p_{,i} - \mu_0\left(h_{r,i} - h_{i,r}\right)H_{0r} = \rho\ddot{u}_i \\ &\dot{u}_{i,k}H_{0k} - \dot{u}_{k,k}H_{0i} + \lambda_0^2 h_{i,kk} - \dot{h}_i = 0, \\ &\dot{\rho} + \rho_0 v_{k,k} = 0. \end{aligned} \tag{31}$$

4. – Applications: magnetoacoustic waves in a vortex lattice and in a vortex fluid

The set of equations (31) form a very good starting point to analyze magnetomechanical wave propagation along the vortex array both in the "lattice" and "fluid" states [13, 14]. Let us assume that the applied magnetic field is taken as $H_0 = [0, 0, H_{03}]$ very close to the limiting values H_{c_1} and H_{c_2}. Supposing that the superconducting body occupies the whole space and that vortices are parallel one to another in x_3 direction we consider a propagation of magnetomechanical waves along x_1 direction.

First, we confine ourselves only to the vortex lattice state of the vortex array. Taking into account $H_0 \simeq H_{c_1}$, $c = 0$, $a = H_{c_2} - H_{c_1}$ using (20), (21) in (31), the linear field equations can be rewritten into the form

$$\begin{aligned} &\mu u_{i,jj} + \eta\dot{u}_{i,jj} + (\lambda+\mu)\,u_{j,ij} + \frac{1}{3}\eta\dot{u}_{j,ij} - \mu_0\left(h_{r,i} - h_{i,r}\right)H_{0r} - \rho\ddot{u}_i = 0 \\ &\dot{u}_{i,k}H_{0k} - \dot{u}_{k,k}H_{0i} + \lambda_0^2 h_{i,kk} - \dot{h}_i = 0. \end{aligned} \tag{32}$$

Stating that the looked for solutions are convergent in time, from dispersion relation two modes of propagation (both dispersive and damped) are obtained: a coupled longitudinal magnetomechanical wave and an uncoupled transverse wave (where there is no interaction between the elastic field and magnetic field). Solutions of (32) are looked for in the form

$$u_1 = u_{01}e^{ik(x_1-vt)}, \qquad u_2 = u_{02}e^{ik(x_1-vt)}, \qquad h_3 = h_{03}e^{ik(x_1-vt)}. \tag{33}$$

Hence, the final form of the field equations is the following

$$c_L^2 u_{1,11} + \frac{4\eta}{3\rho}\dot{u}_{1,11} - \frac{\mu_o}{\rho}h_{3,1}H_{03} - \ddot{u}_1 = 0, \qquad c_T^2 u_{2,11} + \frac{\eta}{\rho}\dot{u}_{2,11} - \ddot{u}_2 = 0, \tag{34}$$

$$(35)\qquad -u_{1,1}H_{03}+\lambda_0^2 h_{3,11}-h_3=0, \qquad \left(\text{where } c_L^2=\frac{\lambda+2\mu}{\rho}\quad,\quad c_T^2=\frac{\mu}{\rho}\right).$$

The dispersion relation for the coupled magnetomechanical u_1 wave is the following

$$(36)\qquad v^2+\frac{4\eta}{3\rho}ikv-\left(c_L^2+\frac{\mu_0 H_{03}^2}{\rho\left(\lambda_0^2k^2+1\right)}\right)=0.$$

Solving (36) we have

$$(37)\qquad \begin{aligned} Re\ v &= \frac{\sqrt{\rho\left[9H_{03}^2\mu_0+9c_L^2\rho\theta-4\eta^2k^2\rho\theta\right]}}{3\rho\sqrt{\theta}}, \\ Im\ v &= -\frac{2\eta k}{3} \qquad \left(\text{where } \theta=k^2\lambda_0^2+1\right). \end{aligned}$$

The dispersion relation for noncoupled mode u_2 and its solutions are the following

$$(38)\qquad v^2+\frac{i\eta}{\rho k}v-c_T^2=0,$$

$$(39)\qquad Re\ v=\frac{\sqrt{4c_T^2K^2\rho^2-\eta^2}}{2k\rho}\quad,\qquad Im\ v=-\frac{\eta}{2K\rho}.$$

In the case of long wave approximation the mode u_1 is not damped and dispersed because

$$(40)\qquad Re\ v=\frac{\sqrt{\rho\left[H_{03}^2\mu_0+c_L^2\rho\right]}}{\rho}\quad,\qquad Im\ v=0.$$

It is logical that the shortest possible wave u_1 can propagate only if $k=\frac{2\pi}{\lambda_0}$ with the velocity

$$(41)\qquad \begin{aligned} Re\ v &= \frac{\sqrt{\rho\left[9H_{03}^2\mu_0\lambda_0^2+9c_L^2\rho\lambda_0^2\Pi-16\pi^2\eta^2\rho\Pi\right]}}{3\rho\sqrt{\Pi}}, \\ Im\ v &= -\frac{4\pi v}{3\lambda_0}, \qquad \text{where}\quad \Pi=4\pi^2+1. \end{aligned}$$

The mode u_2 behaves differently. There is no case of long wave for u_2 because the damping in time is infinite (see (39)). But in the case $k\to\infty$ we observe a pure elastic-like transverse wave as follows:
$Re\ v=c_T,\quad Im\ v=0.$

Now, we confine ourselves only to the fluid state of the vortex array. The basic set of field equations comes from (31), taking into account that $H_0\approx H_{c_2}$, $a=0$ and $c=H_{c_2}-H_{c_1}$:

$$(42)\qquad \dot\rho+\rho_0 v_{k,k}=0, \qquad \eta v_{i,jj}+\frac{1}{3}\eta v_{j,ij}-p_{,i}-\mu_0\left(h_{r,i}-h_{i,r}\right)H_{or}=\rho_0\dot v_i$$

$$\lambda_0^2 \dot{h}_{i,kk} - \dot{h}_i + v_{i,k} H_{0k} - v_{k,k} H_{0i} = 0. \tag{43}$$

Solutions of (42)-(43) are looked for in the form

$$v_1 = v_{10} e^{ik(x_1 - vt)} \qquad \rho = \rho_{00} e^{ik(x_1 - vt)} \qquad h_3 = h_{30} e^{ik(x_1 - vt)} \quad . \tag{44}$$

In (42) $p = p(\rho)$, $p_{,i} = \frac{dp}{d\rho}\rho_{,i}$. So, to linearize (42) we expand $\frac{dp}{d\rho}$ into Taylor's series around the rest state of the fluid (indicated with the index "$_0$") and we confine ourselves only to the first term of the expansion $\left(\frac{dp}{d\rho}\right)_0$, which defines the acoustic wave speed called c_0. Hence, we have

$$\frac{4\eta}{3\rho_0}\dot{\rho}_{,11} + \mu_0 h_{3,11} H_{03} + \left(\frac{dp}{d\rho}\right)_0 \rho_{,11} = \ddot{\rho}, \qquad \lambda_0^2 h_{3,11} - h_3 + \frac{1}{\rho_0}\rho H_{03} = 0. \tag{45}$$

Now, we calculate $\left(\frac{dp}{d\rho}\right)_0$. If the vortex field is modeled as an ideal gas, we use the constitutive law $p = RT_o\rho$, where R is the gas constant and $T_0 < T_c$ (see cross section in the plane $(x_1,\, x_2)$ in the first picture of Figure 4).

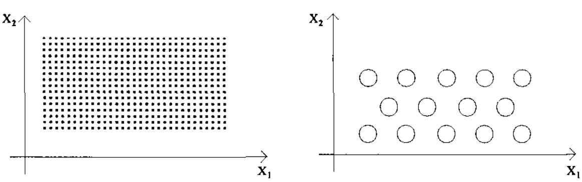

Figure 4: Ideal gas and van der Waals gas models

If we assume that the flow of vortices is adiabatic we have

$$\begin{aligned} &\frac{p}{\rho^{\mathcal{K}}} = \frac{p_0}{\rho_0^{\mathcal{K}}} \quad , \qquad \text{where} \quad \mathcal{K} = \frac{c_p}{c_v}, \\ &\frac{dp}{d\rho} = \mathcal{K}\rho^{\mathcal{K}-1}, \qquad \frac{p_0}{\rho_0^{\mathcal{K}}} = \frac{p}{\rho}\mathcal{K}, \qquad \left(\frac{dp}{d\rho}\right)_0 = c_{0g}^2 = \mathcal{K}RT_0 \quad . \end{aligned} \tag{46}$$

If $\mathcal{K}$=1 the processes in the fluid are not adiabatic ones. If we model the vortex field by van der Waals gas (see cross section in the plane $(x_1,\, x_2)$ in the second picture of Figure 4), we consider the following equation $p = RT_0\rho\,(1 - b_v\rho)^{-1} - a_v\rho^2$, where a_v is the pressure correction coming from interactions between vortex lines and b_v is the volume correction coming from the sizes of the vortex cores and we obtain

$$\left(\frac{dp}{d\rho}\right)_0 = c_{0v}^2 = RT_0\,(1 - b_v\rho_0)^{-2} - 2a_v\rho_0. \tag{47}$$

For the liquid state of the vortex field we can also determine the acoustic wave speed from the general constitutive relations (7), (8). Putting $\beta = 1$, $\alpha = 0$ and assuming that there are no shear components in the total stress tensor, we obtain that the pure acoustic wave speed for the fluid c_{of} is equal to the longitudinal elastic wave velocity, i.e.

$$c_{0f}^2 = \frac{c_{11}}{\rho_0} = \frac{\lambda + 2\mu}{\rho_0} = \frac{1}{c_k \rho_0}, \tag{48}$$

where c_k denotes the compressibility of the fluid and c_{11} is the elastic bulk modulus. Now, using (44) in (45), we arrive at the following equation for amplitudes of the considered waves

$$\left(\frac{4\eta}{3\rho_0} ikc - c_0^2 + c^2\right) \rho_{00} - \mu_0 H_{03} h_{30} = 0 \quad , \tag{49}$$

$$\frac{H_{03}}{\rho_0} \rho_{00} - \left(1 + \lambda_0^2 k^2\right) h_{30} = 0. \tag{50}$$

From the dispersion relation it follows that only a real value of the speed of such waves leads to a convergent solution (representing dispersive and damped wave) and such a value depends on London's penetration depth. In fact, we have

$$\begin{gathered} \left(c^2 - c_0^2 + \frac{4\eta}{3\rho_0} ikc\right) \left(1 + \lambda_0^2 k^2\right) - \frac{\mu_0 H_{03}^2}{\rho} = 0, \\ c^2 = \frac{c_A^2}{1 + \lambda_0^2 k^2} + c_0^2 - \frac{4\eta}{3\rho_0} ikc, \end{gathered} \tag{51}$$

$$c_A = \left(\frac{\mu_0 H_{03}^2}{\rho_0}\right)^{\frac{1}{2}} \tag{52}$$

(where c_A denotes Alfvén velocity).

Hence, we obtain

$$\begin{gathered} Re\,(c) = \frac{|H_0| \sqrt{-\mu_0 \left(4k^4 \phi_0^2 \mu_0 \lambda_0^2\right) + k^2 \left(4\phi_0^2 \mu_0 - 9\rho_n^2 \rho_0 \lambda_0^2\right) - 18\rho_n^2 \rho_0}}{3\rho_n \rho_0 \sqrt{k^2 \lambda_0^2 + 1}}, \\ Im\,(c) = \frac{2iH_0 k \phi_o \mu_0}{3\rho_n \rho_o}. \end{gathered} \tag{53}$$

Now, we introduce a ratio w defined in the following way

$$w = \frac{\text{volume of material in the normal state}}{\text{total volume of material}}. \tag{54}$$

We assume that all vortices in the liquid state touch one to another forming the triangular structure of their circular cross sections. In the picture shown below we suggest how to calculate the ratio w. The left black figure concerns volume of the material in the normal state. Then the right black figure deals with the total volume of the material. Then, we have $w = \frac{\pi}{2\sqrt{3}} = 0.918$. The radius of the circumference in the crossection shown in Figure 5 is the coherence length ξ. So the density ρ_0 of the vortex field must be calculated as follows: $\rho_0 = \rho_{YBCO} \cdot w$.

If w$\rightarrow$ 1 , $\rho_0 = \rho_{YBCO}$ and the vortex cores are strongly overlapped.

Quantity	Value	References
$\mu_0 H_{03}$	$120\ T$	[4]
λ_0	$4 \cdot 10^{-7} m$	[4]
ξ	10^{-9} m	[4]
T_0	$\leq 92\ K$	[4]
ρ_n	$6 \cdot 10^{-5}\Omega\cdot$m	[4]
ρ_0	$5 \cdot 10^3 kg/m^3$	[4]
μ_0	$4\pi \cdot 10^{-7} T \cdot m/A$	
Φ_0	$2.07 \cdot 10^{-15} T \cdot m^2$	[4]
c_{11}	$120 \cdot 0,955 \cdot 10^8 T \cdot A \cdot m^{-1}$	[6]
H_{03}	$0.955 \cdot 10^8 A/m$	

Curves	H_{03} values	w
1	H_{c_2}	1
2	$0.8\ H_{c_2}$	1
3	$0.6\ H_{c_2}$	1
4	$0.4\ H_{c_2}$	1
5	$0.2\ H_{c_2}$	1
6	$0.2\ H_{c_2}$	$\pi/2\sqrt{3}$

Tables: Quantity values and the family of dispersion curves.

Now, the obtained results have been applied to the ceramic material $YBa_2Cu_3O_7$ (YBCO). A family of dispersion curves $c = c(k)$ are obtained for different values of H_{03} very close to H_{c_2}. For the acoustic wave speed $c_0 = c_{of} = 1.51 \cdot 10^3 m/s$ and various values of w (see right table) the curves are shown in Figure 6. It is seen that the dispersion is observed only for long magnetoacoustic waves (for big k values the dispersion disappears) and that for $H_{03} > 0, 2H_{c_2}$ the vortex cores overlap and c velocity of the waves decreases (if the applied magnetic field is weakened by interactions among the vortices).

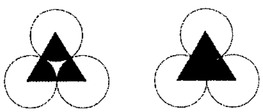

Figure 5: Crossection of the vortex array. Determination of w.

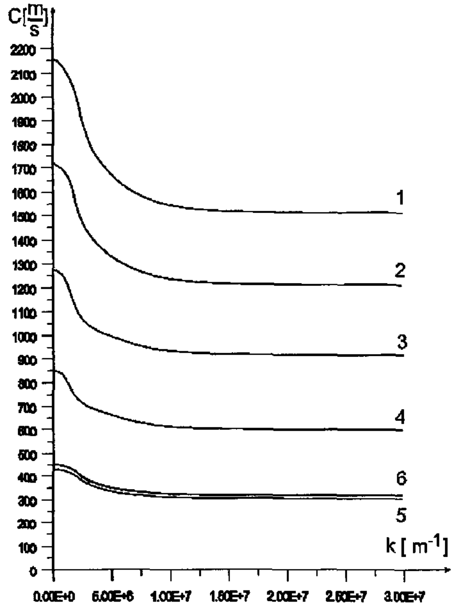

Figure 6: Dispersion curves

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Decay of the energy to partially viscoelastic materials *

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1. – Introduction

Let us consider a n-dimensional body which in its reference configuration is homogeneous and occupies the open bounded set $\Omega \subset I\!R^n$ with smooth boundary Γ. Let $x \mapsto u(x,t)$ be the position of the material particle x at time t. Then the viscoelastic equation of motion is given by

$$\rho u_{tt} - \kappa\Delta u + \int_0^\infty g(s) \text{ div } \{a(x)\nabla u(\cdot, t-s)\} \, ds = f \quad \text{in} \quad \Omega\times]0,\infty[$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega\times]0,\infty[=\Gamma\times]0,\infty[$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_t(x), \quad \text{in} \quad \Omega.$$

where ρ is the mass density function, g is the relaxation function and f denotes the body force. Here we are mainly interested on the asymptotic behaviour of the solution u when t tends to infinity. Note that the above model is dissipative, and the dissipation is given by the memory term, where $a \geq 0$. The memory is effective only in a part of the body Ω where $a > 0$. In this paper we also consider locally distributed dissipation; but this dissipation does not appear by the introduction of any artificial mechanism. On the contrary, it arises because of the mixed structure of the material. That is, we consider a body consisting of an elastic and a viscoelastic part. So, the dissipation is due to the memory effect which works only over a portion of the material.

Denoting by σ the stress and by $*$ the convolution product $g * f = \int_0^t g(t-\tau)f(\tau)\, d\tau$, the constitutive law we use in this paper is given by

$$\sigma = \kappa\nabla u + a(x) g * \nabla u,$$

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so that, the corresponding motion equation for $\kappa = 1$ may be written as

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\text{div }\{a(x)\nabla u\}\ d\tau = 0, \quad \text{in} \quad \Omega\times]0,\infty[\tag{1}$$

with initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega. \tag{2}$$

and Dirichlet's boundary condition

$$u(x,t) = 0, \quad \text{on} \quad \Gamma\times]0,\infty[. \tag{3}$$

For materials with memory the stress depends not only on the present values but also on the entire temporal history of the motion. Therefore, we have also to prescribe the history of u prior to 0 as the initial data. Here we assume that u vanishes identically for $t < 0$, that is

$$u(x,t) = 0, \quad \text{for} \quad t < 0.$$

Let $x \mapsto a(x)$ be a non negative C^2-function defined over Ω and let us denote by ω_ϵ the set

$$\omega_\epsilon = (\cup_{x\in\Gamma_0} B_\epsilon(x)) \cap \Omega,$$

where $B_\epsilon(x) = \{x \in I\!R^n; \quad \|x\| \leq \epsilon\}$ and Γ_0 is given by

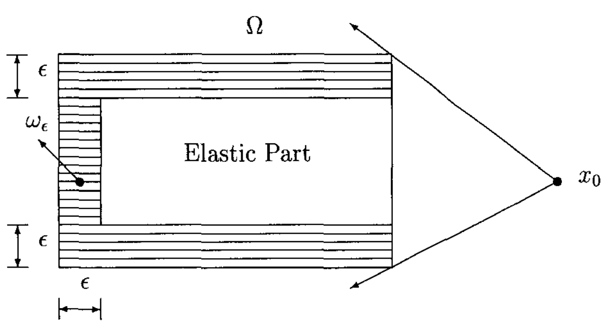
$$\Gamma_0 = \{x \in \Gamma; \quad (x - x_0)\cdot\nu \geq 0\},$$

where ν is the unitary external normal defined over Γ and x_0 any point of $I\!R^n$. For the one dimensional case we have that $\omega_\epsilon =]L-\epsilon, L[$. The hypotheses we use on a are the following

$$x \mapsto a(x) \in C^2(\overline{\Omega}); \qquad a(x) = \begin{cases} 0 & \text{on} \quad \Omega \setminus \omega_\epsilon \\ 1 & \text{on} \quad \omega_{\epsilon/2}. \end{cases} \tag{4}$$

$$|\nabla a(x)|^2 \leq c|a(x)|. \tag{5}$$

In the next picture Ω is a rectangle, the set ω_ϵ denotes the viscoelastic part of the body. Note that ω_ϵ is behind the part that an observer can see when situated on x_0. This is a particular case in which the geometric requeriments of Bardos et al. [1] and Ralston [8, 9] are satisfied



Let us mention some other papers related to the problems we address. Dafermos in [2] proved that the solution to viscoelastic system goes to zero as time goes to infinity; but without giving explicit rate of decay. Lagnese in [3] considered the linear viscoelastic plate equation obtaining uniform rates of decay but introducing additional damping terms acting on the boundary. Uniform rates of decay for the solutions of linear viscoelastic system with memory were obtained recently by M. Rivera et al [5]. Unfortunately the method used to achieve uniform rates of decay in those works are based on second order estimates, which are time depending in our problem. Thus, the methods that have been used for establishing uniform rates of decay fail in the case of partially viscoelastic equation. Therefore a new asymptotic technique has to be devised. For nonlinear models see [6].

The aim of this paper is to show that in the geometrial setting above, the energy decays exponential provided the kernel g also decays exponentially. More especifically, we assume that g satisfies

$$g \in C^3(]0,\infty[), \qquad g(t) > 0, \qquad |g''(t)| \le cg(t), \qquad |g'''(t)| \le cg(t) \tag{6}$$

$$-\kappa_0 g(t) \le g'(t) \le \kappa_1 g(t) \tag{7}$$

$$\alpha := 1 - \int_0^\infty g(\tau)\, d\tau < 1 \tag{8}$$

To facilitate our analysis, we introduce the following binary operators

$$\begin{aligned} g \,\square\, \nabla u &= \int_0^t g(t-\tau) \int_\Omega a(x)|\nabla u(x,t) - \nabla u(x,\tau)|^2\, dx d\tau \\ g \,\square\, u &= \int_0^t g(t-\tau) \int_\Omega a(x)|u(x,t) - u(x,\tau)|^2\, dx d\tau \end{aligned}$$

Under the above conditions the main result of this paper is given by

THEOREM 1 *Under the above assumptions on Ω, ω and a and with the kernel g satisfying (6)–(8), the weak solution of the viscoelastic equation (1)–(3), decays exponentially as time goes to infinity. That is, there exist positive constants c and γ that do not depend on the initial data, such that*

$$E(t) \le cE(0)e^{-\gamma t}$$

where by $E(t)$ we are denoting the first order energy

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 + \left\{1 - a(x)\int_0^t g(\tau)\, d\tau\right\} |\nabla u|^2\, dx + \frac{1}{2} g \,\square\, \nabla u.$$

The method we use here is based on the construction of a functional $\mathcal{L}$ for which an inequality of the form

$$\frac{d}{dt}\mathcal{L}(t) \le -c\mathcal{L}(t)$$

holds, with $c > 0$.

2. – Existence results and preliminaries

Our starting point is given by the following Lemma

LEMMA 1 *For any $v \in C^1(0,T;H^1(0,L))$ we get*

$$\begin{aligned}
\int_\Omega \int_0^t g(t-\tau)a(x)\nabla v\, d\tau \cdot \nabla v_t\, dx &= -\frac{1}{2}g(t)\int_\Omega a(x)|\nabla v|^2\, dx + \frac{1}{2}g' \,\Box\, \nabla v \\
&\quad -\frac{1}{2}\frac{d}{dt}\left\{g \,\Box\, \nabla v - \left(\int_0^t g\, d\tau\right)\int_\Omega a(x)|\nabla v|^2\, dx\right\} \\
\int_\Omega \int_0^t g(t-\tau)a(x)v\, d\tau \cdot v_t\, dx &= -\frac{1}{2}g(t)\int_\Omega a(x)|v|^2\, dx + \frac{1}{2}g' \,\Box\, v \\
&\quad -\frac{1}{2}\frac{d}{dt}\left\{g \,\Box\, v - \left(\int_0^t g\, d\tau\right)\int_\Omega a(x)|v|^2\, dx\right\}
\end{aligned}$$

PROOF. It is easy to see that

$$\begin{aligned}
\frac{d}{dt}\{g \,\Box\, \nabla v\} &= g' \,\Box\, \nabla v - 2\int_\Omega \int_0^t g(t-\tau)a(x)\nabla v(\tau)d\tau \cdot \nabla v_t(x,t)\, d\tau dx \\
&\quad +2\int_0^t g(t-\tau)\, d\tau \int_\Omega a(x)\nabla v \nabla v_t\, dx \\
&= g' \,\Box\, \nabla v - 2\int_\Omega \int_0^t g(t-\tau)a(x)\nabla v(\tau)\, d\tau \cdot \nabla v_t\, dx \\
&\quad +\frac{d}{dt}\left\{\int_0^t g(\tau)\, d\tau \int_\Omega a(x)|\nabla v|^2\, dx\right\} - g(t)\int_\Omega a(x)|\nabla v|^2\, dx
\end{aligned}$$

This shows our result, the proof of the other identity is similar. ■

It is not difficult to show that there exists only one solution to equation (1)–(3). We summarize the existence result in the following theorem

THEOREM 2 *Let us suppose that g is a C^0-function and that the initial data satisfies*

$$(u_0, u_1) \quad \in \quad H_0^1(\Omega) \times L^2(\Omega),$$

then, there exists only one weak solution u to equation (1)-(3) with the following regularity,

$$u \in L^\infty(0,\infty;H_0^1(\Omega)), \quad u_t \in L^\infty(0,\infty;L^2(\Omega)).$$

In addition, if $g \in C^1$ and

$$(u_0, u_1) \quad \in \quad H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega);$$

then, there exist only one strong solution u of equation (1)-(3) satisfying

$$u \in C^{2-i}(0,\infty;H_0^1(\Omega) \cap H^i(\Omega)), \quad i = 1,2. \quad u \in C^2(0,\infty;L^2(\Omega)).$$

The dissipative property of the viscoelastic equation is summarized in the following Lemma:

LEMMA 2 *Any strong solution of (1)-(3) satisfies*

$$\frac{d}{dt}E(t) = \frac{1}{2}g' \Box \nabla u - \frac{1}{2}g(t) \int_{\Omega} a(x)|\nabla u|^2 \, dx.$$

PROOF. Multiplying equation (1) by u_t and integration over Ω yields

$$\frac{d}{dt}\int_{\Omega}\left(|u_t|^2 + |\nabla u|^2\right) \, dx = \int_{\Omega} a(x) g * \nabla u \cdot \nabla u_t \, dx.$$

From Lemma 1 our conclusion follows. ∎

Lemma 2 tells us that the dissipation given by the memory term is effective only on the support of the function a. We will show in section 4 that such dissipation is enough to produce the exponential decay of the energy, as time goes to infinity.

3. – Regularity of the convolution

Let us denote by $\|\cdot\|_{C^0}$ the norm in $C^0(\overline{\Omega})$. The following Lemma will play an important role in the sequel.

LEMMA 3 *Let us suppose that g is a positive function satisfying condition (8), $a \in C^0(\overline{\Omega})$ is such that $\|a\|_{C^0} \leq 1$ and finally let us take $f \in L^p(0,T;L^2(\Omega))$ with $1 \leq p < \infty$. In this conditions we have that there exists only one solution v of the Volterra's equation*

$$v(x,t) - \int_0^t g(t-\tau)a(x)v(\cdot,\tau) \, d\tau = f(x,t), \quad a.e. \quad (x,t) \in \Omega \times]0,T[$$

satisfying

$$v \in L^p(0,T;L^2(\Omega)).$$

Besides, there exists a positive constant c independent of T, such that

$$\|v\|_{L^p(0,T;L^2)} \leq c\|f\|_{L^p(0,T;L^2)}.$$

PROOF. (See [7]) ∎

LEMMA 4 *Let us suppose that $0 \leq a(x) \leq 1$ satisfies conditions (6)-(8) and that g is a positive function satisfying (8). If u is a weak solution of (1)-(3) satisfying*

$$u_t \in L^\infty(0,T;L^2(\Omega)), \quad u \in L^\infty(0,T;H_0^1(\Omega)),$$

then we have that

$$g * u \in L^2(0,T;H^2(\Omega))$$

and

$$\|g * u\|_{L^2(0,T;H^2)} \leq C\int_0^T E(t) \, dt + CE(0). \tag{9}$$

where C is a positive constant independent of T.

PROOF. Applying convolution to equation (1) we have

$$-\Delta g * u + g * g * div\,\{a(x)\nabla u\} = -g * u_{tt}.$$

Performing an integration by parts over $]0,t[$ we get

$$g * u_{tt} = g(0)u_t - g(t)u_t(\cdot,0) + \int_0^t g'(t-\tau)u_t\,d\tau := -F.$$

From the hypotheses we conclude that $F \in L^2(0,T;L^2(\Omega))$. Denoting by $v = g*\Delta u$, and using the elliptic regularity our conclusion follows. ∎

4. – Exponential Decay

To show the exponential decay of the solution let us introduce the following functional

$$\begin{aligned} I(t) := &\int_\Omega a(x)\left\{u_t(g*u)_t - \frac{1}{2}g(0)|u|^2 - \int_0^t g\,d\tau|u|^2\,dx\right\}\,dx \\ &-\frac{1}{2}g'' \,\square\, u + \frac{1}{2}\int_\Omega a^2(x)|g*\nabla u|^2\,dx. \end{aligned}$$

In these conditions we have:

LEMMA 5 *Under the above conditions and for $g \in C^3$, satisfying conditions (6)–(8), we have that for any $\delta > 0$ there exists C_δ satisfying*

$$\begin{aligned} \frac{d}{dt}I(t) \leq &-g(0)\int_\Omega a(x)|u_t|^2\,dx + \delta\int_\Omega a(x)|\nabla u|^2\,dx + C_\delta g\,\square\,\nabla u \\ &+C_\delta g(t)\int_\Omega a(x)|\nabla u|^2\,dx + C_\delta\int_{\omega_\epsilon}\int_0^t g(t-\tau)|u(x,t)-u(x,\tau)|^2\,dxdt \\ &+C_\delta\int_{\omega_\epsilon}|u(x,t)|^2\,dx. \end{aligned}$$

Here $C_\delta \to \infty$ when $\delta \to 0$.

PROOF. Multiplying equation (1) by $a(x)(g*u)_t$ we get:

$$\underbrace{\int_\Omega u_{tt}a(x)(g*u)_t\,dx}_{:=I_1} - \underbrace{\int_\Omega \Delta u a(x)(g*u)_t\,dx}_{:=I_2}$$

$$-\underbrace{\int_\Omega a(x)g*\nabla u\cdot\nabla\left\{a(x)(g*u)_t\right\}\,dx}_{:=I_3} = 0,$$

from where we have:

$$\begin{aligned}
I_1 &= \frac{d}{dt}\int_\Omega u_t a(x)(g*u)_t\,dx - \int_\Omega u_t a(x)(g*u)_{tt}\,dx \\
&= \frac{d}{dt}\int_\Omega u_t a(x)(g*u)_t\,dx - \int_\Omega u_t a(x)(g(0)u+g'*u)_t\,dx \\
&= \frac{d}{dt}\int_\Omega u_t a(x)(g*u)_t\,dx - g(0)\int_\Omega a(x)|u_t|^2\,dx \\
&\quad -\int_\Omega a(x)u_t\left\{g'(0)u+g''*u\right\}\,dx \\
&= \frac{d}{dt}\int_\Omega u_t a(x)(g*u)_t\,dx - g(0)\int_\Omega a(x)|u_t|^2\,dx - \frac{g'(0)}{2}\frac{d}{dt}\int_\Omega a(x)|u|^2\,dx \\
&\quad -\int_\Omega a(x)u_t g''*u\,dx.
\end{aligned}$$

Using Lemma 1 we get that

$$\begin{aligned}
\int_\Omega a(x)u_t g''*u\,dx &= \frac{1}{2}g'''\,\Box\,u - \frac{1}{2}\int_\Omega a(x)|u|^2\,dx \\
&\quad -\frac{1}{2}\frac{d}{dt}\left\{g''\,\Box\,u - \int_0^t g\,d\tau\int_\Omega a(x)|u|^2\,dx\right\},
\end{aligned}$$

from where it follows that

$$I_1(t) = \frac{d}{dt}I_0(t) - g(0)\int_\Omega a(x)|u_t|^2\,dx + \frac{1}{2}g''(t)\int_\Omega a(x)|u|^2\,dx - \frac{1}{2}g'''\,\Box\,u,$$

where by I_0 we are denoting

$$I_0 = \int_\Omega a(x)\left\{u_t(g*u)_t - \frac{g'(0)}{2}|u|^2\right\}\,dx - \frac{1}{2}g''\,\Box\,u - \int_0^t g\,d\tau\int_\Omega a(x)|u|^2\,dx.$$

Using similar approach we estimate the others terms. ∎

LEMMA 6 *With the same hypotheses as Lemma 5 we have that the solution of equation (1)–(3) satisfies,*

$$\frac{d}{dt}\int_\Omega auu_t\,dx \le \int_\Omega a(x)|u_t|^2\,dx - c_0\int_\Omega a(x)|\nabla u|^2\,dx + c\int_{\omega_\epsilon}|u|^2\,dx$$
$$+C_\delta\int_0^t\int_{\omega_\epsilon} g(t-\tau)|u(\cdot,t)-u(\cdot,\tau)|^2\,dxdt.$$

PROOF. Let us multiply equation (1) by $a(x)u(x,t)$ and using $|\Delta a(x)| \le C$ our conclusion follows. ∎

LEMMA 7 *Let us denote by q_k a C^1-function, then any strong solution ($u \in C^i(0,T;H^{2-i}(\Omega))$ for $i=0,1,2$) of the wave equation*

$$u_{tt} - \Delta u = f, \tag{10}$$

$$u(x,t)=0, \quad on \quad \Gamma\times]0,\infty[,$$

satisfies the following identity

$$\frac{d}{dt}\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\,dx = \int_\Omega f q_k \frac{\partial u}{\partial x_k}\,dx + \frac{1}{2}\int_\Gamma q_k \nu_k |\frac{\partial u}{\partial \nu}|^2\,d\Gamma$$

$$+\int_\Gamma \frac{\partial u}{\partial \nu} q_k \frac{\partial u}{\partial x_k}\,d\Gamma + \frac{1}{2}\int_\Omega \frac{\partial q_k}{\partial x_k}\left\{|u_t|^2+|\nabla u|^2\right\}dx + \int_\Omega \nabla u\cdot\nabla q_k \frac{\partial u}{\partial x_k}\,dx.$$

PROOF. (See [4]) ∎

LEMMA 8 *Let us take $q_k = a^2(x)h_k$, where $h_k \in C^2(\overline{\Omega})$ is such that $h_k = \nu_k$ on Γ. Then we have that*

$$-\frac{d}{dt}\int_\Omega a^2(x)u_t h_k \frac{\partial u}{\partial x_k}\,dx \le -\frac{1}{2}\int_{\Gamma_0} |\frac{\partial u}{\partial \nu}|^2\,d\Gamma + C\int_\Omega a(x)\left\{|u_t|^2+|\nabla u|^2\right\}dx$$

$$+C\int_\Omega a(x)|h(x)\ \mathrm{div}\{ag*\nabla u\}|^2\,dx,$$

for any solution of equation (1)-(3)

PROOF. From Lemma 7 applied to $q_k = a^2(x)h_k$ we have

$$-\frac{d}{dt}\int_\Omega a^2 h_k u_t \frac{\partial u}{\partial x_k}\,dx = -\int_\Omega f a^2(x)h_k \frac{\partial u}{\partial x_k}\,dx = \int_\Gamma a^2(x)h_k\nu_k|\frac{\partial u}{\partial \nu}|^2\,dx$$

$$+\frac{1}{2}\int_\Omega \frac{\partial\{a^2(x)h_k\}}{\partial x_k}\left\{|u_t|^2-|\nabla u|^2\right\}dx + \int_\Omega \nabla u\cdot\nabla a^2 h_k \frac{\partial u}{\partial x_k}\,dx.$$

Since $a=1$ on Γ_0 and $h_k=\nu_k$ on Γ, using the Cauchy-Schwarz inequality, our conclusion follows. ∎

Let us denote by $\mathcal{L}_N(t)$ the functional

$$\mathcal{L}_N(t) = NE(t) + I(t) + \frac{g(0)}{2}\int_\Omega a(x)uu_t\,dx - \delta_0\int_\Omega a^2(x)u_t h_k \frac{\partial u}{\partial x_k}\,dx.$$

In these conditions, and using Lemma 5 and Lemma 6 we arrive to

LEMMA 9 *Under the above notations we have*

$$\begin{aligned}(11)\quad \frac{d}{dt}\mathcal{L}_N(t) \ \le\ & -\kappa_0\int_\Omega a(x)\left\{|u_t|^2+|\nabla u|^2\right\}dx \\ & -\frac{N}{2}\left\{g\,\Box\,\nabla u + g(t)\int_\Omega a(x)|\nabla u|^2\,dx\right\} - \frac{\delta_0}{2}\int_\Gamma |\frac{\partial u}{\partial \nu}|^2\,d\Gamma \\ & +C\delta_0\int_\Omega a(x)|h(x)\ \mathrm{div}\{ag*\nabla u\}|^2\,dx + C\int_{\omega_\epsilon}|u|^2\,dx \\ & +C\int_0^t\int_{\omega_\epsilon} g(t-\tau)|u(x,t)-u(x,\tau)|^2\,dxd\tau.\end{aligned}$$

LEMMA 10 *Let us suppose that u is the weak solution of (1)-(3), then there exists a positive constant C, independent of T, such that*

$$\text{(12)} \quad \int_0^T \int_\Omega a(x)|\operatorname{div}\{ag * \nabla u\}|^2 \, dxdt \leq C \int_0^T \int_\Omega a(x)\left\{|u_t|^2 + |\nabla u|^2\right\} dxdt + C\int_0^T g(t)\, dtE(0),$$

$$\text{(13)} \quad \int_0^T \int_\Omega |\operatorname{div}\{ag * \nabla u\}|^2 \, dxdt \leq C\int_0^T E(t)dt + C\int_0^T g(t)\, dtE(0).$$

PROOF. Note that

$$\operatorname{div}\ \{ag * \nabla u\} = \nabla a \cdot \nabla u + ag * \Delta u.$$

As in the proof of Lemma 4, $v = \sqrt{a}g * \Delta u$ satisfies

$$-v + a(x)g * v = \sqrt{a}G.$$

Using similar arguments, we conclude that

$$\begin{aligned} \|v\|^2_{L^2(0,T;L^2)} &\leq \int_0^T \int_\Omega a(x)|G|^2 \, dxdt \\ &\leq \int_0^T \int_\Omega a(x)\left\{|u_t|^2 + |\nabla u|^2\right\} dxdt + C\int_0^T g\, dtE(0). \end{aligned}$$

Therefore it follows that

$$\int_0^T \int_\Omega a(x)|\operatorname{div}\ \{ag * \nabla u\}|^2 \, dxdt \leq C\int_0^T \int_\Omega a(x)\left\{|u_t|^2 + |\nabla u|^2\right\} dxdt + C\int_0^T g(t)\, dtE(0),$$

for a positive constant C. The proof is now complete ∎

LEMMA 11 *Let us suppose that φ is a weak solution of the wave equation*

$$\varphi_{tt} - \Delta\varphi = 0$$

$$\varphi(x,0) = \varphi_0, \quad \varphi_t = \varphi_1$$

$$\varphi(x,t) = 0, \qquad on \qquad \Sigma = \Gamma\times]0,\infty[.$$

Then, for $x_0 \in \mathbb{R}^n$ and $T > 2R(x_0)$, there exists a positive constant $C > 0$ for which we have

$$E(0) \leq C\int_0^T \int_\omega |\varphi_t|^2 + |\nabla\varphi|^2 \, dxdt$$

for any $(\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega)$ where

$$R(x_0) = \max_{x\in\overline{\Omega}} |\sum_{k=1}^n (x_k - x_k^0)^2|^{1/2}$$

PROOF. See [4] Lemma 2.3, Chapter VIII, pag 411. ∎

Our next step is to estimate the term $\int_\omega |u|^2\,dx$. To do this we will use the following Lemma

LEMMA **12** *Let us suppose that u is a weak solution of (1)–(3), then for any $\epsilon > 0$ there exist a positive constant C_ϵ for which we have*

$$\begin{aligned}\int_0^T \int_\Omega |u|^2\,dxdt &\le C_\epsilon\left\{\int_0^T g(t)\int_\Omega a(x)|\nabla u|^2\,dxdt + \int_0^T g\,\Box\,\nabla u\,dt\right\}\\ &\quad +\epsilon\int_0^T \int_\Omega a(x)\left\{|\nabla u|^2 + |u_t|^2 + |\operatorname{div} ag * \nabla u|^2\right\}dxdt,\end{aligned}$$

$$\begin{aligned}\int_0^T \int_\Omega |g*\nabla u|^2\,dxdt &\le C_\epsilon\left\{\int_0^T g(t)\int_\Omega a(x)|\nabla u|^2\,dxdt + \int_0^T g\,\Box\,\nabla u\,dt\right\}\\ &\quad +\epsilon\int_0^T \int_\Omega a(x)\left\{|\nabla u|^2 + |u_t|^2 + |\operatorname{div} ag * \nabla u|^2\right\}dxdt,\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \int_0^t \int_\Omega g(\sigma - t)|u(x,\sigma) - u(x,t)|^2\,dxd\sigma dt \le\\ &\qquad C_\epsilon\left\{\int_0^T g(t)\int_\Omega a(x)|\nabla u|^2\,dxdt + \int_0^T g\,\Box\,\nabla u\,dt\right\}\\ &\qquad +\epsilon\int_0^T \int_\Omega a(x)\left\{|\nabla u|^2 + |u_t|^2 + |\operatorname{div} ag * \nabla u|^2\right\}dxdt,\end{aligned}$$

provided T is large enough.

PROOF. We argue by contradiction. Suppose that there exists $\epsilon_0 > 0$ and a sequence of functions such that

$$\begin{aligned}\int_0^T \int_\Omega |u^\nu|^2\,dxdt &\ge \nu\left\{\int_0^T g(t)\int_\Omega a(x)|\nabla u^\nu|^2\,dxdt + \int_0^T g\,\Box\,\nabla u^\nu\,dt\right\}\\ &\quad +\epsilon_0\int_0^T \int_\Omega a(x)\left\{|\nabla u^\nu|^2 + |u_t^\nu|^2 + |\operatorname{div} ag * \nabla u|^2\right\}dxdt,\end{aligned}\tag{14}$$

for $\nu \to \infty$. By the linearity of the problem we may suppose that

$$\int_0^T \int_\Omega |u^\nu|^2\,dxdt = 1, \quad \forall \nu \in I\!N.\tag{15}$$

So, we get that

$$g(t)a(x)|\nabla u^\nu|^2 + \int_0^t a(\cdot)g(t-\tau)|u^\nu(\cdot,\tau) - u^\nu(\cdot,t)|^2\,d\tau \to 0 \text{ strongly in } L^1(]0,\infty[\times\Omega).\tag{16}$$

Let us decompose u^ν into:

$$u^\nu = w^\nu + v^\nu,$$

where

$$w^\nu_{tt} - \Delta w^\nu = -div\,\{ag * \nabla u^\nu\} \quad \text{(bounded in} \quad L^2(0,T;L^2(\Omega))).$$

$$w^\nu(x,0) = 0, \quad w^\nu_t(x,0) = 0, \quad \text{in} \quad \Omega$$
$$w^\nu(x,t) = 0, \quad \text{on} \quad \Gamma\times]0,\infty[,$$

and

$$v^\nu_{tt} - \Delta v^\nu = 0,$$
$$v^\nu(x,0) = u^\nu(x,0), \quad v^\nu_t(x,0) = u^\nu_t(x,0), \quad \text{in} \quad \Omega$$
$$v^\nu(x,t) = 0, \quad \text{on} \quad \Gamma\times]0,\infty[$$

From (14) and (15) it follows that u^ν is bounded in

$$W^{1,\infty}(0,T;L^2(\omega)) \cap L^\infty(0,T;H^1(\omega)).$$

Note that w^ν is also bounded in

$$W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)).$$

Thereby, we conclude that $v^\nu = u^\nu - w^\nu$ satisfies

$$v^\nu_t \quad \text{is bounded in} \quad L^2(0,T;L^2(\omega)),$$

$$v^\nu \quad \text{is bounded in} \quad L^2(0,T;H^1(\omega)).$$

Using Lemma 11 we have

$$(u^\nu(\cdot,0), u^\nu_t(\cdot,0)) = (v^\nu(\cdot,0), v^\nu_t(\cdot,0)), \quad \text{is bounded in} \quad H^1_0(\Omega)\times L^2(\Omega).$$

which implies that

$$v^\nu \quad \text{is bounded in} \quad W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)).$$

Hence

$$u^\nu = w^\nu + v^\nu \quad \text{is bounded in} \quad W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)).$$

Therefore there exists a subsequence (which we still denote in the same way) and a function $u \in W^{1,\infty}(0,T;L^2(\Omega))$ such that

$$u^\nu \quad \rightarrow \quad u \quad \text{weak * in} \quad W^{1,\infty}(0,T;L^2(\Omega))$$

and satisfying

$$u_{tt} - \Delta u = 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in} \quad \Omega$$
$$u(x,t) = 0, \quad \text{on} \quad \Gamma\times]0,T[$$

From (16) we conclude that

$$u = 0 \quad \text{on} \quad \omega_\epsilon \times]0,T[$$

Using the Holmgren's Theorem for $T > 2\text{diam}\,(\Omega \backslash \omega_\epsilon)$ we get that $u = 0$ on $\Omega \times]0,T[$. But this is contradictory with (15) since due to the compactness of the embedding $H^1(\Omega \times]0,T[) \subset L^2(\Omega \times]0,T[)$, the sequence u^ν converges strongly in $L^2(\Omega \times]0,T[)$. This contradiction proves the first inequality. To prove the other we use similar arguments. Thereby, our conclusion follows. ∎

Using the inequalities (11), (13), Lemma 12 and taking $\epsilon > 0$ small enough we arrive at

$$\mathcal{L}_N(T) - \mathcal{L}_N(0) \le -\kappa_0 \int_0^T \mathcal{M}(t)\, dt + C\epsilon E(0) + C\epsilon \int_0^T E(t)\, dt \tag{17}$$

for $N > 2C$; where by $\mathcal{M}$ we are denoting

$$\mathcal{M}(t) = \int_\Omega a(x) \left\{ |u_t|^2 + |\nabla u|^2 \right\} dx + g \,\square\, \nabla u + \int_{\Gamma_0} |\frac{\partial u}{\partial \nu}|^2 \, d\Gamma.$$

Now we are in conditions to prove the main result of this paper.

Proof of Theorem 1. We will suppose that the initial data belongs to $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Our conclusion will follow using standard density arguments. Using Lemma 7 for $q = x - x^0$ we conclude that

$$\begin{aligned} -\frac{d}{dt}\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\, dx &= -\int_\Omega f q_k \frac{\partial u}{\partial x_k}\, dx + \frac{1}{2}\int_\Omega \frac{\partial q_k}{\partial x_k}\left\{ |u_t|^2 - |\nabla u|^2 \right\} dx \\ &\quad + \int_\Omega \nabla u \cdot \nabla q_k \frac{\partial u}{\partial x_k}\, dx - \frac{1}{2}\int_\Gamma q_k \nu_k |\frac{\partial u}{\partial \nu}|^2, \end{aligned}$$

from where it follows

$$\begin{aligned} -\frac{d}{dt}\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\, dx &= -\int_\Omega f q_k \frac{\partial u}{\partial x_k}\, dx + \frac{n}{2}\int_\Omega \left\{ |u_t|^2 - |\nabla u|^2 \right\} dx \\ &\quad + \int_\Omega |\nabla u|^2\, dx - \frac{1}{2}\int_\Gamma q_k \nu_k |\frac{\partial u}{\partial \nu}|^2, \end{aligned}$$

which implies that

$$\begin{aligned} -\frac{d}{dt}\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\, dx &= -\int_\Omega f q_k \frac{\partial u}{\partial x_k}\, dx + \frac{n-1}{2}\int_\Omega \left\{ |u_t|^2 - |\nabla u|^2 \right\} dx \\ &\quad + \frac{1}{2}\int_\Omega |u_t|^2 + |\nabla u|^2\, dx - \frac{1}{2}\int_\Gamma q_k \nu_k |\frac{\partial u}{\partial \nu}|^2. \end{aligned} \tag{18}$$

Multiplying by u equation (1) we get

$$\frac{d}{dt}\int_\Omega u u_t\, dx = \int_\Omega \left\{ |u_t|^2 - |\nabla u|^2 \right\} dx + \int_\Omega a g * \nabla u \cdot \nabla u\, dx.$$

Inserting this identity into (18) we have

$$\begin{aligned}
-\frac{d}{dt}\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\, dx &= -\int_\Omega f q_k \frac{\partial u}{\partial x_k}\, dx + \frac{n-1}{2}\frac{d}{dt}\int_\Omega u u_t\, dx \\
&\quad -\frac{n-1}{2}\int_\Omega ag * \nabla u \cdot \nabla u\, dx + \frac{1}{2}\int_\Omega |u_t|^2 + |\nabla u|^2\, dx \\
&\quad -\frac{1}{2}\int_\Gamma q_k \nu_k |\frac{\partial u}{\partial \nu}|^2,
\end{aligned}$$

from where we have

$$\begin{aligned}
\frac{d}{dt}\underbrace{\left\{-\int_\Omega u_t q_k \frac{\partial u}{\partial x_k}\, dx - \frac{n-1}{2}\int_\Omega u u_t\, dx\right\}}_{:=X(t)} &= -\int_\Omega f q_k \frac{\partial u}{\partial x_k}\, dx \\
-\frac{n-1}{2}\int_\Omega ag * \nabla u \cdot \nabla u\, dx + \frac{1}{2}\int_\Omega |u_t|^2 + |\nabla u|^2\, dx &- \frac{1}{2}\int_\Gamma q_k \nu_k |\frac{\partial u}{\partial \nu}|^2.
\end{aligned}$$

Integrating over $[0,T]$ we get

$$\begin{aligned}
(19) X(T) - X(0) &= -\int_0^T\int_\Omega f q_k \frac{\partial u}{\partial x_k}\, dx\, dt - \frac{n-1}{2}\int_0^T\int_\Omega ag * \nabla u \cdot \nabla u\, dx\, dt \\
&\quad +\frac{1}{2}\int_0^T\int_\Omega |u_t|^2 + |\nabla u|^2\, dx\, dt - \frac{1}{2}\int_0^T\int_\Gamma q_k \nu_k |\frac{\partial u}{\partial \nu}|^2\, dt.
\end{aligned}$$

Since

$$X(T) \le CE(T), \quad X(0) \le CE(0),$$

and using

$$\int_0^T E(t)\, dt \le C\left\{\int_0^T\int_\Omega |u_t|^2 + |\nabla u|^2\, dx\, dt + \int_0^T g \,\Box\, \nabla u\, dt\right\},$$

together with inequality (19) we conclude that

$$\int_0^T E(t)\, dt \le C\int_0^T \mathcal{M}(t)\, dt + C\left\{E(T) + E(0)\right\}.$$

From the energy identity we get

$$E(0) \le E(T) + \int_0^T \mathcal{M}(t)\, dt. \tag{20}$$

Therefore, there exist a positive constant C_1 such that

$$\int_0^T E(t)\, dt \le C_1\int_0^T \mathcal{M}(t)\, dt + C_1 E(T). \tag{21}$$

Since $E(t)$ is a decreasing function we have that

$$E(T) \le \frac{1}{T}\int_0^T E(t)\, dt.$$

Inserting the above inequality into (21) we get

$$(1-\frac{C}{T})\int_0^T E(t)\,dt \le C_1\int_0^T \mathcal{M}(t)\,dt. \tag{22}$$

On the other hand, it is not difficult to see that

$$c_0E(t) \le \mathcal{L}(t) \le c_1E(t). \tag{23}$$

Therefore using (17), (22), and (23) we conclude that

$$\begin{aligned}
\mathcal{L}(T)-\mathcal{L}(0) &\le -\kappa_0\int_0^T \mathcal{M}(t)\,dt + C\epsilon E(0) + C\epsilon\int_0^T E(t)\,dt \qquad \text{(using (20))}\\
&\le -\kappa_0\int_0^T \mathcal{M}(t)\,dt + C\epsilon\left\{E(T)+\int_0^T \mathcal{M}(t)\,dt\right\} + C\epsilon\int_0^T E(t)\,dt\\
&\le -\kappa_1\int_0^T \mathcal{L}(t)\,dt.
\end{aligned}$$

provided ϵ small enough. Using inequality (23) we can establish that

$$\int_0^T \mathcal{L}(t)\,dt \ge c\int_0^T E(t)\,dt \ge cTE(T) \ge c_2T\mathcal{L}(T),$$

which implies that

$$\mathcal{L}(T)-\mathcal{L}(0) \le -\kappa_1cT\mathcal{L}(T).$$

This is equivalent to

$$\mathcal{L}(T) \le \frac{1}{1+CT}\mathcal{L}(0).$$

From where our conclusion follows. ∎

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Some Remarks on the Conserved Penrose-Fife Phase Field Model with Memory Effects

E. Rocca *

1. – Introduction

This note is concerned with the study of the following initial-boundary value problem in the cylindrical domain $Q := \Omega \times (0,T)$, where $\Omega \subset \mathbf{R}^N$ ($N \leq 3$) is a bounded connected domain with a smooth boundary Γ and $T > 0$. Find a pair $(\vartheta, \chi) : Q \to \mathbf{R}^2$ satisfying

$$\partial_t(\vartheta + \lambda(\chi)) - \Delta(\psi(\vartheta) + k * \alpha(\vartheta)) = g \quad \text{in } Q, \tag{1}$$

$$-\partial_\nu(\psi(\vartheta) + k * \alpha(\vartheta)) = \gamma(\psi(\vartheta) + k * \alpha(\vartheta) - h) \quad \text{on } \Sigma := \Gamma \times (0,T), \tag{2}$$

$$\vartheta(\cdot, 0) = \vartheta^0 \quad \text{in } \Omega, \tag{3}$$

$$\partial_t\chi - \Delta\Big(-\Delta\chi + \xi + \sigma'(\chi) + \frac{\lambda'(\chi)}{\vartheta}\Big) = 0 \quad \text{in } Q, \tag{4}$$

$$\xi \in \beta(\chi), \quad \text{in } Q, \tag{5}$$

$$\partial_\nu\chi = 0, \quad \partial_\nu\Big(-\Delta\chi + \xi + \sigma'(\chi) + \frac{\lambda'(\chi)}{\vartheta}\Big) = \ell \quad \text{on } \Sigma, \tag{6}$$

$$\chi(\cdot, 0) = \chi^0 \quad \text{in } \Omega, \tag{7}$$

with $\partial_t = \partial/\partial t$, being Δ the Laplacian with respect to the space variables, and ∂_ν denoting the outward normal derivative on Γ.

In (1) there is a memory term given by the convolution product with respect to time, that is

$$(a * b)(t) := \int_0^t a(s)b(t-s)ds, \quad t \in [0,T], \tag{8}$$

where a and b may also depend on the space variables.

The system given by the partial differential equations (1) and (4) provides a quite general version of the phase-field model proposed by Penrose and Fife in [5-6] for the kinetics of phase transitions.

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This model describes the evolution of a material, with constant latent heat of fusion-solidification process (being it $\lambda'(\chi)$), exhibiting two different phases (e.g., solid-liquid in melting phoenomena), in terms of the absolute temperature $\vartheta : Q \to (0, +\infty)$ and the order parameter $\chi : Q \to \mathbf{R}$ (representing, for instance, the fraction of one of the phases). The data g, h and ℓ (which is required to be here a zero-mean value function) stand for the heat supply, the outer temperature and the mass flux on the boundary; the functions λ and σ come from the smooth part of the free energy, while the multivalued map β is derived from its non smooth but convex part (usually β is the inverse of the Heaviside graph). To be more precise, the sum $\beta + \sigma'$ stands for the derivative of the double-well part of a Ginzburg-Landau free energy potential (see e.g. [3, 6]).

We may observe that we have coupled the second equation, which rules the evolution of the order parameter χ, with Neumann homogeneous boundary condition on χ and Neumann non-homogeneous boundary condition for the chemical potential $w := -\Delta\chi + \xi + \sigma'(\chi) + \frac{\lambda(\chi)}{\vartheta}$. This seems indeed to be of some physical interest. In any case, we have to take the "natural" homogeneous Neumann boundary condition for the concentration χ (see [5] for a justification).

In [7] the analogous system of equations, with Neumann homogeneous boundary conditions for the chemical potential w, was studied.

Finally, $k : [0, T] \to \mathbf{R}$ is an integration kernel, $\alpha : (0, +\infty) \to \mathbf{R}$ is a concave function, which will be specified in the sequel, ψ is a maximal monotone function and it is linked to α as detailed below, and γ is a positive constant coefficient.

The term $-\Delta(\psi(\vartheta) + k * \alpha(\vartheta))$ in (1) represents the divergence of the heat flux, which is given by

$$\mathbf{q} = -\nabla(\psi(\vartheta) + k * \alpha(\vartheta)). \tag{9}$$

In this work we are going to give an existence result of a weak solution to (1-7). For the proof of this result we will refer to [7], in which analogous statement are given for the same problem with Neumann-homogeneous boundary condition on w and with a more regular memory kernel k. Here we will see as this proof may be adapted to the case of a less regular k and Neumann non-homogeneous boundary condition on w.

More regularity on the data and on the memory kernel is required here in order to get the uniqueness of the solution.

2. – Main results

Consider the initial-boundary value problem (1-7). We make the following general assumptions on the data of the system

(10) β is the subdifferential of a non-negative, proper, convex,

$$\text{and l.s.c. function } \widehat{\beta} : \mathbf{R} \to [0, +\infty] \text{ satisfying}$$

$$\widehat{\beta}(0) = 0, \text{ and } D(\beta) \text{ denotes its domain,}$$

$$\sigma \in C^2(\mathbf{R}), \quad \sigma'' \in L^\infty(\mathbf{R}), \tag{11}$$

$$\lambda : \mathbf{R} \to \mathbf{R}, \text{ and } \lambda(r) = Lr + L', \ \forall r \in \mathbf{R}, \quad \text{for some constant } L, \ L' > 0, \tag{12}$$

$$\psi : (0, +\infty) \to \mathbf{R} \text{ a maximal monotone function,} \tag{13}$$

$$\alpha : (0, +\infty) \to \mathbf{R}, \text{ and } \alpha(r) = -\frac{C_1}{r} + C_2 r, \quad \text{for some positive constants } C_1, \ C_2 \text{ and } \forall r \in (0, +\infty), \tag{14}$$

$$\alpha \circ \psi^{-1}, \ \psi \circ \alpha^{-1} : \mathbf{R} \to \mathbf{R} \text{ are Lipschitz continuous,} \tag{15}$$

$$k \in L^2(0,T) \cap B^{\eta}_{1,\infty}(0,T), \text{ for some } \eta \in (0,1), \tag{16}$$

$$g \in L^2(0,T; L^2(\Omega)), \quad h \in L^2(0,T; H^{\frac{1}{2}}(\Gamma)), \tag{17}$$

$$\ell \in L^2(\Sigma), \text{ and } \int_\Gamma \ell(t) = 0 \quad \text{for a.e. } t \in (0,T), \tag{18}$$

$$\vartheta^0 \in L^2(\Omega), \quad \vartheta^0 > 0 \quad \text{a.e. in } \Omega, \quad u^0 := \alpha(\vartheta^0) \in L^2(\Omega), \tag{19}$$

$$\chi^0 \in H^1(\Omega), \quad \widehat{\beta}(\chi^0) \in L^1(\Omega). \tag{20}$$

Let us now remark some properties of such an α as in (14), that will be useful in the sequel

$$\alpha' \geq C_2 > 0; \tag{21}$$

$$\lim_{r \searrow 0} r^2 \alpha'(r) = C_1; \tag{22}$$

$$\lim_{r \searrow 0} \alpha(r) = -\infty \text{ and } \lim_{r \to +\infty} \alpha(r) = +\infty. \tag{23}$$

Moreover, since α is invertible, we can set

$$\rho := \alpha^{-1} : \mathbf{R} \to (0, +\infty), \text{ that is increasing and Lipschitz continuous,} \tag{24}$$

because (21) gives $\rho' \leq 1/C_2$.

Now let us give a variational formulation of (1-7). To this end, we denote by $(\cdot,\cdot)$ both the scalar product in $H := L^2(\Omega)$ and in $(L^2(\Omega))^N$, also denoted by H, and by $|\cdot|$ the corresponding norm. For the sake of convenience, $V := H^1(\Omega)$ will be endowed with the inner product $((\cdot,\cdot))$, defined by

$$((v_1, v_2)) := \int_\Omega \nabla v_1 \nabla v_2 + \gamma \int_\Gamma v_1 v_2, \quad \forall v_1, v_2 \in V, \tag{25}$$

where γ is the positive constant appearing in the boundary condition (2). Define $W := H^2(\Omega)$ and let us also indicate by $\langle \cdot,\cdot \rangle$ the duality pairing between V' and V. We identify H with a subspace of V', as usual, so that $\langle u, v \rangle = (u, v)$ for all $u \in H$ and for all $v \in V$.

Next, we define the Riesz isomorphism $J : V \to V'$, and the scalar product in V', respectively, by

$$\langle J v_1, v_2 \rangle := ((v_1, v_2)), \quad \forall v_1, v_2 \in V, \tag{26}$$

$$((w_1, w_2))_* := \langle w_1, J^{-1} w_2 \rangle, \quad \forall w_1, w_2 \in V'. \tag{27}$$

Let us observe that the norm in V related to the inner product defined above (which will be indicated as $\|\cdot\|$) is equivalent to the usual norm in V. Similar considerations holds also for V' and we term $\|\cdot\|_*$ the norm in V' related to the inner product (27).

Remark 1 Thanks to (14), if we set $u := \alpha(\vartheta)$, it is possible to write the right hand side of (4) in the form $\frac{L}{C_1}(u - C_2\rho(u))$. Indeed, the role played by (14) is fundamental in view of the resolution of (1-7), and, in the following variational formulation, it is convenient to write the equations in terms of u rather than of ϑ.

Then our problem can be stated as follows

Problem (P). *Find a pair (ϑ, χ) and (w, ξ) such that*

$$\vartheta \in L^2(0,T;V) \cap C^0([0,T];H), \quad \vartheta > 0 \ a.e. \ in \ Q; \tag{28}$$

$$u := \alpha(\vartheta) \in L^2(0,T;V), \quad k * u \in L^2(0,T;V); \tag{29}$$

$$\chi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;W), \quad \chi \in D(\beta) \ a.e. \ in \ Q; \tag{30}$$

$$\xi \in L^2(0,T;H); \tag{31}$$

$$w \in L^2(0,T;V); \tag{32}$$

$$\xi \in \beta(\chi) \ a.e. \ in \ Q; \tag{33}$$

$$\partial_t(\rho(u) + \lambda(\chi)) + J(\psi(\rho(u))) + J(k * u) = f \ in \ V', \ a.e. \ in \ (0,T); \tag{34}$$

$$\langle \partial_t \chi, v \rangle + \int_\Omega \nabla w \nabla v = \int_\Gamma \ell\, v, \quad \forall v \in V, \ a.e. \ in \ (0,T); \tag{35}$$

$$\langle w, v \rangle = \int_\Omega \nabla \chi \nabla v + \left\langle \xi + \sigma'(\chi) - \frac{L}{C_1}(u - C_2\rho(u)), v \right\rangle, \tag{36}$$

$$\forall v \in V, \ a.e. \ in \ (0,T);$$

$$\vartheta(\cdot, 0) = \vartheta^0, \quad \chi(0) = \chi^0, \quad a.e. \ in \ \Omega. \tag{37}$$

Let us now state our main results, which will be proven in the following sections.

Theorem 1 *Suppose that (10-20) are satisfied and assume that the mean value of χ^0 is an interior point of $D(\beta)$, i.e.,*

$$m_0 := \frac{1}{|\Omega|}\langle \chi^0, 1 \rangle \in int(D(\beta)). \tag{38}$$

Then, Problem (P) admits at least one solution.

Concerning the uniqueness of solution, we have the following result

Theorem 2 *Suppose that (10-20) and (38) are satisfied. Assume in addition that*

$$f \in W^{1,1}(0,T;V'), \quad \ell \in W^{1,1}(0,T;H^{-\frac{1}{2}}(\Gamma)) \tag{39}$$

$$u^0 \in V, \tag{40}$$

$$\chi^0 \in H^3(\Omega), \ \partial_\nu \chi^0 = 0 \ on \ \Gamma, \ \exists \xi^0 \in V \ s.t. \ \xi^0 \in \beta(\chi^0) \ a.e. \ in \ \Omega, \tag{41}$$

$$k \in W^{1,1}(0,T), \tag{42}$$

then there exists a solution $(\vartheta, \chi, w, \xi)$ to Problem (P) satisfying the further regularity

$$\vartheta \in H^1(0,T;H), \tag{43}$$

$$u := \alpha(\vartheta) \in L^\infty(0,T;V), \tag{44}$$

$$\chi \in W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{45}$$

$$\xi \in L^\infty(0,T;H), \tag{46}$$

$$w \in L^\infty(0,T;V), \tag{47}$$

and the components ϑ and χ of such a solution are unique.

REMARK 2 Let us observe that (17) and (39) are satisfied if $g \in W^{1,1}(0,T;H)$ and $h \in L^2(0,T;H^{1/2}(\Omega)) \cap W^{1,1}(0,T;H^{-\frac{1}{2}}(\Gamma))$. Moreover (40) yields $\vartheta^0 = \rho(u^0) \in V$, because ρ is Lipschitz continuous.

REMARK 3 Let us also note that testing (35) with $v = 1$ yields, thanks to (18),

$$\frac{d}{dt}\langle\chi, 1\rangle = 0 \quad \text{in } (0,T).$$

This means that

$$\langle\chi(t), 1\rangle = \langle\chi_0, 1\rangle \quad \forall t \in [0,T],$$

i.e., the mean value of χ is conserved. This fact is often used in the sequel.

3. – Existence-Uniqueness

In this section, we present an implicit time discretization scheme for (28-37). As a first step, we prepare some results in the direction of a discrete convolution procedure.

We start by fixing a partition of the time interval $[0,T]$. To this end, we choose a constant time step $\tau := T/n$, $n \in \mathbf{N}$. Let us assume $\tau \leq 1$. Our next aim is to introduce a discrete version of the convolution product in $(0,t)$, for $t \in (0,T)$. Hence, we recall (cf., e.g., [7]) the following

DEFINITION 1 Let $\underline{a} = \{a_i\}_{i=1}^n \in E^n$, and $\underline{b} = \{b_i\}_{i=1}^n \in E^n$, where E stands for a real linear space. Then we define the vector $\{(\underline{a} *_\tau \underline{b})_i\}_{i=0}^n \in E^{n+1}$ as

$$(\underline{a} *_\tau \underline{b})_i := \begin{cases} 0 & \text{if } i = 0 \\ \tau \sum_{j=1}^i a_{i-j+1} b_j & \text{if } i = 1, \ldots, n. \end{cases} \tag{48}$$

We note that an equivalent definition is the one that calls $(\underline{a} *_\tau \underline{b})_i := \tau \sum_{j=1}^i a_{i-j+1} b_j$ for any $i = 0, \ldots, n$, with the convention (widely used in the sequel) that it is equal to zero when the sum is done on an empty set of indices. We stress also that, in the definition of $(\underline{a} *_\tau \underline{b})_i$ only the values $\{a_j\}_{j=1}^i$ and $\{b_j\}_{j=1}^i$ are involved.

Let us now introduce some convenient notations.

For the $(n+1)$-tuple $\{z_i\}_{i=0}^n \in E^{n+1}$, let the functions $\overline{z}_\tau$, $z_\tau : (0,T) \to E$ be specified by

$$\overline{z}_\tau(t) := z_i, \quad z_\tau(t) := \alpha_i(t) z_i + (1 - \alpha_i(t)) z_{i-1}, \tag{49}$$

$$\text{where} \quad \alpha_i(t) := (t - (i-1)\tau)/\tau, \ \text{for } t \in ((i-1)\tau, i\tau], \ i = 1, \ldots, n.$$

Let us also set

(50) $$\delta z_i := \frac{z_i - z_{i-1}}{\tau}, \quad \text{for } i = 1, \ldots, n.$$

Now a proof of Theorem 1, we may exactly follow [7, Section 3-4], except for the approximation of the kernel k and for [7, Lemma 3.2]. Indeed all the estimates can be repeated, taking as approximation for k the following one

(51) $$k_i := \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} k(s)ds, \quad \text{for } i = 1, \ldots, n,$$

and [7, Lemma 3.2] may be rewritten as done in the following Lemma 1 (see also [7, Remark 4.1]).

In view of giving this Lemma 1, let we state this

REMARK 4 We may observe that the hypothesis (16) (in particular $k \in B^{\eta}_{1,\infty}(0,T)$, for some $\eta \in (0,1)$) may be rewritten in this form

(52) $$k \in (L^1(0,T),\, BV(0,T))_{\eta,\infty}, \text{ for some } \eta \in (0,1),$$

which means that k belongs to the interpolation space between $L^1(0,T)$ and $BV(0,T)$, of place η, and $BV(0,T)$ denotes the space of the functions with bounded total variation (see e.g. [2]). Recalling also (49), this mens that

$$k = \overline{k}_\tau + (k - \overline{k}_\tau), \text{ where}$$
$$\overline{k}_\tau \in BV(0,T),$$
$$(k - \overline{k}_\tau) \in L^1(0,T),$$
$$\|k - \overline{k}_\tau\|_{L^1(0,T)} \le c\tau^\eta,$$
$$\|\overline{k}_\tau\|_{BV(0,T)} \le \frac{C'}{\tau^{(1-\eta)}},$$

for some positive constants C, C', and with $\eta \in (0,1)$.

Now we are ready to give the following

LEMMA 1 *Let (16) hold and $\{\sigma_i\}_{i=1}^n \in E^n$, where E denotes a linear space endowed with the norm $\|\cdot\|_E$. Moreover, let $\{k_i\}_{i=0}^n$, $\overline{\sigma}_\tau$, and $\{(\underline{k} *_\tau \underline{\sigma})_i\}_{i=1}^n$ be defined as in (51), (49) and (48), respectively. Then, there exists a positive constant C, independent of τ, such that*

(53) $$\|\overline{(\underline{k} *_\tau \underline{\sigma})}_\tau - k * \overline{\sigma}_\tau\|_{L^1(0,T;E)} \le C\tau^\eta \|\overline{\sigma}_\tau\|_{L^1(0,T;E)},$$
for some $\eta \in (0,1)$.

Proof. We have that

(54) $$\|\overline{(\underline{k} *_\tau \underline{\sigma})} - k * \overline{\sigma}_\tau\|_{L^1(0,T;E)}$$
$$\le \|\overline{(\underline{k} *_\tau \underline{\sigma})} - \overline{k}_\tau * \overline{\sigma}_\tau\|_{L^1(0,T;E)} + \|(\overline{k}_\tau - k) * \overline{\sigma}_\tau\|_{L^1(0,T;E)}.$$

For the second term, Young theorem ensures that

$$\|(\overline{k}_\tau - k) * \overline{\sigma}_\tau\|_{L^1(0,T;E)} \le \|\overline{k}_\tau - k\|_{L^1(0,T)} \|\overline{\sigma}_\tau\|_{L^1(0,T;E)} \le \tau^\eta \|\overline{\sigma}_\tau\|_{L^1(0,T;E)}, \text{ for some } \eta \in (0,1),$$

thanks also to (16) and Remark 4

As regards the first term in the right hand side of (54), we may see that, thanks to (48), (50), (16), and Remark 4,

$$\begin{aligned}
&\|\overline{(\underline{k} *_\tau \underline{\sigma})} - \overline{k}_\tau * \overline{\sigma}_\tau\|_{L^1(0,T;E)} \\
&= \sum_{i=1}^{n} \int_{(i-1)\tau}^{i\tau} \Big\|\tau k_1 \sigma_i + \sum_{j=1}^{i-1} \tau (k_{i-j+1} - k_{i-j})\sigma_j\Big\|_E \, dt \\
&\leq \tau |k_1| \|\overline{\sigma}_\tau\|_{L^1(0,T;E)} + \tau \|\overline{(\underline{\delta k} *_\tau \underline{\sigma})}_\tau\|_{L^1(0,T;E)} \\
&\leq \tau |k_1| \|\overline{\sigma}_\tau\|_{L^1(0,T;E)} + \tau \|\overline{(\delta k)}_\tau\|_{L^1(0,T)} \|\overline{\sigma}_\tau\|_{L^1(0,T;E)} \\
&\leq C\tau^\eta \|\overline{\sigma}_\tau\|_{L^1(0,T;E)}, \text{ for some } \eta \in (0,1).
\end{aligned}$$

This concludes the proof of Lemma 1.

Regarding f and ℓ, we set

$$f_i := \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} f(t)\, dt \in V', \text{ for } i = 1 \dots, n, \tag{55}$$

$$\ell_i := \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} \ell(t)\, dt \in L^2(\Gamma), \text{ for } i = 1 \dots, n, \tag{56}$$

Note that

$$\|\overline{f}_\tau\|_{L^2(0,T;V')} \leq \|f\|_{L^2(0,T;V')}, \tag{57}$$

$$\|\overline{\ell}_\tau\|_{L^2(\Sigma)} \leq \|\ell\|_{L^2(\Sigma)}, \tag{58}$$

Then, the approximation scheme may be formulated by making use of an auxiliary unknown $\xi_i = \beta_\tau(\chi_i)$, where

(59) β_τ, for $\tau > 0$, is the Yosida approximation of β, with constant $\tau^{1/4}$,

so that β_τ is Lipschitz continuous with constant $\tau^{-1/4}$.

Then, the approximated problem takes the form

$$\tau^{1/4} \frac{u_i - u_{i-1}}{\tau} + \frac{\rho(u_i) - \rho(u_{i-1})}{\tau} + L \frac{\chi_i - \chi_{i-1}}{\tau} \tag{60}$$

$$+ J(\psi(\rho(u_i))) + J(\underline{k} *_\tau \underline{u})_i = f_i, \text{ in } V', \quad \text{for } i = 1, \dots, n;$$

$$\langle \frac{\chi_i - \chi_{i-1}}{\tau}, v\rangle + \int_\Omega \nabla w_i \nabla v = \int_\Gamma \ell_i v, \quad \forall v \in V, \text{ for } i = 1, \dots, n; \tag{61}$$

$$\langle w_i, v\rangle = \int_\Omega \nabla \chi_i \nabla v + \int_\Omega \xi_i v + \int_\Omega \sigma'(\chi_i) v - \frac{L}{C_1} \int_\Omega (u_i - C_2 \rho(u_i)) v, \tag{62}$$

$$\forall v \in V, \text{ for } i = 1, \dots, n;$$

$$\xi_i = \beta_\tau(\chi_i), \text{ for } i = 1, \dots, n; \tag{63}$$

$$u_0 = u^0, \quad \chi_0 = \chi^0. \tag{64}$$

Next, we state and an existence and uniqueness result for the solution to scheme (60-64).

THEOREM 3 *Let assumptions (10-20) and (55-56) hold. Let the time step τ be small enough. Then, there exists a unique quadruplet of vectors $\{\vartheta_i, \chi_i, w_i, \xi_i\}_{i=0}^n \in H^{4(n+1)}$, which fulfills relations (60-64).*

A proof of this Lemma may be done following exactly the proof of [7, Lemma 3.1].

Then, all the estimates done in [7, Section 4] may be repeated in this case, thanks to (15-16), (see also [7, Remark 4.1]). So, (as in [7, (4.24-4.31)]) one can infer that there exists at least a subsequence of time steps (still denoted by τ), and some functions ϑ, u, χ, φ, such that

$$\rho(\overline{u}_\tau) \overset{*}{\rightharpoonup} \vartheta \quad \text{in } L^\infty(0,T;H), \tag{65}$$

$$\overline{u}_\tau \rightharpoonup u \quad \text{in } L^2(0,T;V), \tag{66}$$

$$\chi_\tau \rightharpoonup \chi \quad \text{in } H^1(0,T;V'), \tag{67}$$

$$\overline{\chi}_\tau \overset{*}{\rightharpoonup} \chi \quad \text{in } L^\infty(0,T;V), \tag{68}$$

$$\overline{(\underline{k} *_\tau \underline{u})_\tau} \rightharpoonup \varphi \quad \text{in } L^2(0,T;V). \tag{69}$$

In addition, the generalized Ascoli theorem (see [8, Cor. 4, Sec. 8]) ensures that, thanks to (67-68), at least for a subsequence of $\tau \searrow 0$,

$$\overline{\chi}_\tau \to \chi \quad \text{in } C^0([0,T];H), \tag{70}$$

$$\tau^{1/4}\overline{u}_\tau + \rho(\overline{u}_\tau) \to \vartheta \quad \text{in } C^0([0,T];H), \tag{71}$$

$$\rho(\overline{u}_\tau) \to \vartheta \quad \text{in } L^2(0,T;V) \quad \text{and so a.e. in } Q. \tag{72}$$

Now, to deduce that $\rho(u) = \vartheta$ and $\psi(\rho(u)) = \psi(\vartheta)$ we can use [1, Prop. 1.1, p.42], with the maximal monotonicity of α and ψ, and (72).

Moreover, Lemma 1 and (66) lead to

$$\overline{(\underline{k} *_\tau \underline{u})_\tau} - k * \overline{u}_\tau \to 0 \quad \text{in } L^1(0,T;V), \quad \text{and} \tag{73}$$

$$k * \overline{u}_\tau - k * u \rightharpoonup 0 \quad \text{in } L^2(0,T;V). \tag{74}$$

Thus, $\varphi = k * u$.

Then, exactly like in [7, (4.34-4.36)], we may also recover

$$\|\overline{\xi}_\tau\|_{L^2(0,T;H)} \le C, \tag{75}$$

$$\|\overline{w}_\tau\|_{L^2(0,T;V)} \le C, \tag{76}$$

$$\|\overline{\chi}_\tau\|_{L^2(0,T;W)} \le C. \tag{77}$$

Note that assumption (38) is used at this step.

Thus, we can still take convergent subsequences by compactness, letting $\tau \searrow 0$. Finally, on account of (65-68) and (74-77), passing to the limit in (60-64), as $\tau \searrow 0$, we immediately recover (34-36) and the regularity (28-32).

By (70) and (72), we get also (37).

Next, we note that $\{\beta_\tau(\overline{\chi}_\tau)\}_\tau$ and $\{\overline{\chi}_\tau\}_\tau$ converge to some ξ and χ weakly in H, for instance, and we have to deduce (6). This can be done using [1, Prop. 1.1, p.

42], and the strong convergence of $\{\overline{X}_\tau\}$, given in (70). This concludes the proof of Theorem 2.1.

A proof of Theorem 2.2 may be given following exactly [7, Section 5].

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Local solution to Frémond's full model for irreversible phase transitions

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1. – Introduction

In this work, we aim to analyze the following system of PDE's:

$$\theta_t - \theta\chi_t - \Delta\theta = \chi_t^2, \tag{1}$$

$$\chi_t + \alpha(\chi_t) - \Delta\chi + \beta(\chi) \ni \theta - \theta_c. \tag{2}$$

The system above describes the *irreversible phase transition* process of a homogeneous substance located inside a bounded container $\Omega \subset \mathbb{R}^3$. The evolution of this material is ruled by two state variables, i.e., the *absolute* temperature θ and the *phase field* χ. We note that, in (1–2), $\theta_c > 0$ is the assigned phase transition temperature and $\alpha, \beta \subset \mathbb{R} \times \mathbb{R}$ are suitable *maximal monotone graphs* yielding the desired constraints for the behavior of χ. More in detail, we assume that $\alpha = \partial I_{[0,+\infty)}$, so that $\chi_t \geq 0$, and $\beta = \partial I_{[0,1]}$, that gives $0 \leq \chi \leq 1$. Indeed, $\chi = 0$ ($\chi = 1$) is assumed to stand for the pure solid phase (pure liquid, respectively) and $0 < \chi < 1$ denotes the presence of a mixture. Of course, the irrevesibility of the process is accounted for by the constraint on χ_t.

The above model was derived by Frémond et al. [2, 4] starting from the consideration that the microscopic movements of molecula may significantly influence the phase transition process described by the *macroscopic* variables θ, χ. Indeed, the papers quoted above present a detailed derivation of (1–2) starting from physical considerations and in particular give a proof of its thermodynamic consistence via the Clausius-Duhem inequality.

Although much work has been devoted to the analysis of (1–2) and several variants of it [1, 2, 3, 5, 8], up to now the existence of a (global in time) solution is known just in one space dimension [7] (see also [10] for the reversible case, i.e., for $\alpha \equiv 0$). The result of [7] is based on a physically meaningful approximation of (1–2), where a finite maximum speed $\lambda > 0$ is imposed to the phase change process by taking $\alpha = \partial I_{[0,\lambda]}$ in (2). Such a modified system is more accessible from the mathematical point of view since the above choice for α guarantees the uniform boundedness of χ_t that is a useful tool for deriving suitable a priori estimates. Actually, a global

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existence result for the λ-system is shown in the paper [9]. Starting from this approximation, the authors of [7] prove global existence for the original system (1–2) by letting $\lambda \to +\infty$ and mainly relying on a sharp λ-independent estimate, originally devised to Dafermos and Hsiao [6] for the study of thermoelasticity, that allows them to get a global L^2 boundness for θ despite of the quadratic growth of the right hand side of (2). Unfortunately, this argument (also exploited in [10]) is strongly dependent on the choice of the one dimensional setting and can not be adapted to our case.

Hence, in this paper, we come back to the λ-regularized system and derive local-in-time a priori estimates, independent of λ, for the solution of the approximating problem. The key point is the control of the high power terms resulting from the quadratic nonlinearities in (1); this is reached by performing an accurate choice of the test functions for (2) that permits us to express such terms in the right norms and control them for small times by an extended Gronwall inequality [12].

The rest of the paper is organized as follows: in the next section we present some mathematical preliminaries and detail the precise hypotheses of the problem and the existence theorem. Then, the proof is achieved in Section 3 by a priori estimates and a compactness argument.

2. – Preliminaries and main result

We start by fixing some notations. Let $\Omega \subset \mathbb{R}^3$ be a smooth and bounded domain and $T > 0$ be a final time. Then, set $Q_t := \Omega \times (0,t)$ for all $t \in (0,T]$ and $Q := Q_T$. Letting $\mathbf{n}$ stand for the outer normal unit vector to $\partial\Omega$, we set $H := L^2(\Omega)$, $V := H^1(\Omega)$, $W := \{u \in H^2(\Omega)$ such that $\partial_{\mathbf{n}} u = 0$ on $\partial\Omega\}$, endowed with the usual scalar products, and we denote by $\|\cdot\|$ the norm in H and by $\|\cdot\|_E$ the norm of the generic normed space E. Finally, we let V^* be the dual of V.

Next, we introduce our assumptions on data by requiring that

(3) $\quad \theta_c > 0 \quad$ is a prescribed constant,

(4) $\quad \alpha = \partial I_{[0,+\infty)}, \qquad \alpha_\lambda = \partial I_{[0,\lambda]} \quad$ for $\lambda > 0$,

(5) $\quad \beta = \partial I_{[0,1]}$,

(6) $\quad \theta_0 \in V \cap L^\infty(\Omega), \quad \theta_0 > 0 \quad$ a.e. in Ω,

(7) $\quad \chi_0 \in W^{2,\infty}(\Omega), \quad 0 \le \chi_0 \le 1 \quad$ a.e. in Ω,

(8) $\quad$ there exists $\eta_0 \in L^\infty(\Omega)$ such that $\eta_0 \in \beta(\chi_0)$ a.e. in Ω.

Our main task is the proof of the following existence theorem:

THEOREM 1 *There exists a final time T_0, with $0 < T_0 \le T$, and a quadruple $(\theta, \chi, \xi, \eta)$ of functions satisfying*

(9) $\quad \theta \in H^1(0,T_0;H) \cap C^0([0,T_0];V) \cap L^2(0,T_0;W)$,

(10) $\quad \chi \in W^{1,\infty}(0,T_0;L^3(\Omega)) \cap L^\infty(0,T_0;W)$,

(11) $\quad \xi \in L^\infty(0,T_0;H), \qquad \eta \in L^\infty(0,T_0;H)$,

(12) $\quad 0 \le \theta, \quad 0 \le \chi \le 1, \quad$ *and* $\quad 0 \le \chi_t \qquad$ *a.e. in* Q_{T_0}

and such that the following relations hold a.e. in Q_{T_0}*:*

$$\theta_t + \theta\chi_t - \Delta\theta = \chi_t^2, \tag{13}$$

$$\chi_t + \xi - \Delta\chi + \eta = \theta - \theta_c, \tag{14}$$

$$\xi \in \alpha(\chi_t) \quad \text{and} \quad \eta \in \beta(\chi). \tag{15}$$

Moreover, the following initial conditions hold:

$$\theta(x,0) = \theta_0(x) \quad \text{and} \quad \chi(x,0) = \chi_0(x) \qquad \text{a.e. in } \Omega. \tag{16}$$

REMARK 1 We note that the regularity properties of the solution stated in (9–10) are not optimal and one might easily improve them by deriving new a priori estimates and using bootstrap arguments. However, we prefer not to insist on this point and focus our attention just on the existence of a solution.

Let us now recall the main theorem of [9]. Actually, we state it in the slighty modified version proved in [7, Thm. 3.1]. We note that, although this result was originally presented in the one dimensional setting, by looking at the proof one can easily convince that – for fixed $\lambda > 0$ – it also holds in three space dimensions:

THEOREM 2 *There exists a quadruple* $(\theta_\lambda, \chi_\lambda, \xi_\lambda, \eta_\lambda)$ *of functions satisfying*

$$\theta_\lambda \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{17}$$

$$\chi_\lambda \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{18}$$

$$\xi_\lambda \in L^\infty(0,T;H), \qquad \eta_\lambda \in L^\infty(0,T;H), \tag{19}$$

$$0 \le \theta_\lambda, \quad 0 \le \chi_\lambda \le 1, \quad \text{and} \quad 0 \le \chi_{\lambda t} \le \lambda \qquad \text{a.e. in } Q \tag{20}$$

and such that the following relations hold a.e. in Q*:*

$$\theta_{\lambda t} + \theta_\lambda \chi_{\lambda t} - \Delta\theta_\lambda = \chi_{\lambda t}^2, \tag{21}$$

$$\chi_{\lambda t} + \xi_\lambda - \Delta\chi_\lambda + \eta_\lambda = \theta_\lambda - \theta_c, \tag{22}$$

$$\xi_\lambda \in \alpha_\lambda(\chi_{\lambda t}) \quad \text{and} \quad \eta_\lambda \in \beta(\chi_\lambda). \tag{23}$$

Moreover, the following initial conditions hold:

$$\theta_\lambda(x,0) = \theta_0(x) \quad \text{and} \quad \chi_\lambda(x,0) = \chi_0(x) \qquad \text{a.e. in } \Omega. \tag{24}$$

3. – Proof of Theorem 1

We consider a family $(\theta_\lambda, \chi_\lambda, \xi_\lambda, \eta_\lambda)$ of solutions to the λ-approximated problem with the regularities stated in Theorem 2 and derive some a priori estimates, independent of λ, in order to pass to the limit as $\lambda \to +\infty$. We point out that some estimates turn out to be formal in this setting; however, they could be made rigorous by performing them, e.g., in the framework of a time discretization argument (cf. [7, Section 3]). Furthermore, we note that in the following the symbol c will stand for possibly different positive constants, that are assumed to depend only on $\Omega, T, \theta_c, \theta_0, \chi_0, \eta_0$. In particular, c is not allowed to depend on λ.

First estimate. Multiply (21) by 1 and (22) by $\chi_{\lambda t}$, sum the results and integrate in time over $(0,t)$ for $t \le T$. Owing to the nonnegativity of θ_λ and on the constraint $0 \le \chi_\lambda \le 1$, and proceeding, e.g., as in [7, Subsec. 4.1], we get

$$\|\theta_\lambda\|_{L^\infty(0,T;L^1(\Omega))} + \|\chi_\lambda\|_{L^2(0,T;V)} \le c. \tag{25}$$

Key estimate. Differentiate formally (22) with respect to time. Of course, this procedure can be made rigorous at a discrete level (or by using a difference quotiens argument). It follows

$$\chi_{\lambda tt} + \xi_{\lambda t} - \Delta\chi_{\lambda t} + \eta_{\lambda t} = \theta_{\lambda t}. \tag{26}$$

Then, multiply (21) by θ_t and (26) by $\chi_{\lambda t}^2$, sum the results and integrate in time over $(0,t)$ for $t \le T$. Noting that by the monotonicity of β and the nonnegativity of $\chi_{\lambda t}$ it is $\eta_{\lambda t}\chi_{\lambda t}^2 \ge 0$ a.e. in Q_t, easy computations yield

$$\begin{aligned}
&\|\theta_{\lambda t}\|^2_{L^2(Q_t)} + \frac{1}{2}\|\nabla\theta_\lambda(t)\|^2 + \frac{1}{3}\|\chi_{\lambda t}(t)\|^3_{L^3(\Omega)} + \frac{8}{9}\|\nabla(\chi_{\lambda t}^{3/2})\|^2_{L^2(Q_t)} \\
&\quad \le \frac{1}{2}\|\nabla\theta_0\|^2 + \frac{1}{3}\|\chi_{\lambda t}(0)\|^3_{L^3(\Omega)} + 2\int_0^t\int_\Omega \theta_{\lambda t}\chi_{\lambda t}^2 \\
&\quad - \int_0^t\int_\Omega \theta_\lambda\theta_{\lambda t}\chi_{\lambda t} - \int_0^t\int_\Omega \xi_{\lambda t}\chi_{\lambda t}^2,
\end{aligned} \tag{27}$$

so that we have to control the five terms on the right hand side. Of course, the first one is bounded by (6). For the second, we compute formally (22) for $t = 0$ and note that

$$\chi_{\lambda t}(0) \in (\mathrm{id} + \alpha_\lambda)^{-1}(\Delta\chi_0 - \eta_0 + \theta_0 - \theta_c) \qquad \text{a.e. in } \Omega.$$

Hence, noting that $(\mathrm{id} + \alpha_\lambda)^{-1}$ is a contraction, by (6–8) we derive that the L^3 norm of $\chi_{\lambda t}(0)$ is uniformly bounded in λ.

Then, we have to work with the three integral terms. By the elementary Young inequality and Sobolev's embedding theorem, the first integral term gives

$$2\int_0^t\int_\Omega \theta_{\lambda t}\chi_{\lambda t}^2 \le \frac{1}{2}\|\theta_{\lambda t}\|^2_{L^2(Q_t)} + 2\int_0^t\int_\Omega \chi_{\lambda t}^4$$

and

$$\begin{aligned}
2\int_0^t\int_\Omega \chi_{\lambda t}^4 &\le 2\int_0^t \|\chi_{\lambda t}^{3/2}\|_V \|\chi_{\lambda t}^{5/2}\|_{V^*} \\
&\le \frac{1}{3}\int_0^t \|\chi_{\lambda t}^{3/2}\|_V^2 + c\int_0^t \|\chi_{\lambda t}^{5/2}\|^2_{L^{6/5}(\Omega)} \\
&\le \frac{1}{3}\|\chi_{\lambda t}\|^3_{L^3(Q_t)} + \frac{1}{3}\|\nabla(\chi_{\lambda t}^{3/2})\|^2_{L^2(Q_t)} + c\int_0^t \|\chi_{\lambda t}\|^5_{L^3(\Omega)}.
\end{aligned} \tag{28}$$

As for the second integral on the right hand side of (27), we get

$$\left|\int_0^t\int_\Omega \theta_\lambda\theta_{\lambda t}\chi_{\lambda t}\right| \le \frac{1}{4}\|\theta_{\lambda t}\|^2_{L^2(Q_t)} + 2\int_0^t\int_\Omega \chi_{\lambda t}^4 + \frac{1}{8}\int_0^t\int_\Omega \theta_\lambda^4,$$

whence the second term can be treated as in (28), while the third one yields

$$\frac{1}{8}\int_0^t\int_\Omega \theta_\lambda^4 \le c\int_0^t \|\theta_\lambda\|_V^4 \le c\int_0^t \|\theta_\lambda\|^4 + c\int_0^t \|\nabla\theta_\lambda\|^4.$$

Thus, using a three dimensional Gagliardo-Nirenberg inequality [11, p. 125] and recalling the first of (25),

$$c\int_0^t \|\theta_\lambda\|^4 \le c\int_0^t \|\theta_\lambda\|^{8/5}_{L^1(\Omega)}\|\nabla\theta_\lambda\|^{12/5} + c\int_0^t \|\theta_\lambda\|^4_{L^1(\Omega)}$$
$$\le c\int_0^t \|\theta_\lambda\|^{8/5}_{L^\infty(0,T;L^1(\Omega))}\|\nabla\theta_\lambda\|^{12/5} + c \le c\int_0^t \|\nabla\theta_\lambda\|^4 + c.$$

Finally, we work on the latter term in (27) and note that, a.e., in Q, it is

$$\chi^2_{\lambda t} \in (\alpha_\lambda^{-1})^2(\xi_\lambda),$$

where $(\alpha_\lambda^{-1})^2$ is easily seen to be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$. Integrating by parts and proceeding similarly as in [7, Subsec. 4.4], we derive

$$-\int_0^t\int_\Omega \xi_{\lambda t}\chi^2_{\lambda t} = -\lambda^2\int_\Omega \xi^+_\lambda(t) + \lambda^2\int_\Omega \xi^+_\lambda(0),$$

with $(\,\cdot\,)^+$ standing for the positive part function. Now, the first integral on the right hand side above is clearly nonpositive, while the latter one can be controlled this way. First, note that, by formally computing (22) for $t=0$, we have

$$\xi_\lambda(0) \in \alpha_\lambda(\chi_\lambda(0)) = \alpha_\lambda[(\mathrm{id}+\alpha_\lambda)^{-1}(\Delta\chi_0 - \eta_0 + \theta_0 - \theta_c)].$$

Then, by (6–8) it follows that the term inside the square brackets is bounded in $L^\infty(\Omega)$ independently of λ, so that for sufficiently large λ, $\xi_\lambda(0)$ is nonpositive and its positive part is zero.

Now, taking all the above considerations into account, it is not difficult to deduce from (27) that, for λ large enough, it is

$$\frac{1}{4}\|\theta_{\lambda t}\|^2_{L^2(Q_t)} + \frac{1}{2}\|\nabla\theta_\lambda(t)\|^2 + \frac{1}{3}\|\chi_{\lambda t}(t)\|^2_{L^3(\Omega)} + \frac{2}{9}\|\nabla(\chi^{3/2}_{\lambda t})\|^2_{L^2(Q_t)}$$
$$\le c + \frac{2}{3}\int_0^t \|\chi_{\lambda t}\|^3_{L^3(\Omega)} + c\int_0^t \|\chi_{\lambda t}\|^5_{L^3(\Omega)} + c\int_0^t \|\nabla\theta_\lambda\|^4. \tag{29}$$

Then, from the relation above one sees that the extended Gronwall lemma in the form of, e.g., [12, Thm. 7.1, p. 33] applies to the function

$$t \mapsto \|\chi_{\lambda t}(t)\|^3_{L^3(\Omega)} + \|\nabla\theta_\lambda(t)\|^2$$

and this yields a finite time $T_0 > 0$, possibly with $T_0 < T$, such that the following bounds hold independently of λ:

$$\|\theta_{\lambda t}\|_{L^2(0,T_0;H)} + \|\theta_\lambda\|_{L^\infty(0,T_0;V)} \le c, \tag{30}$$

$$\|\chi_{\lambda t}\|_{L^\infty(0,T_0;L^3(\Omega))} + \|\nabla(\chi^{3/2}_{\lambda t})\|_{L^2(0,T_0;H)} \le c. \tag{31}$$

Third estimate. Multiply (22) by $(-\Delta\chi_\lambda + \eta_\lambda)_t$ and integrate over $(0,t)$, as before. Owing to the first bound in (30), one can proceed exactly as in [7, Subsec. 4.6] to get the bounds

$$\|\chi_\lambda\|_{L^\infty(0,T_0;W)} + \|\xi_\lambda\|_{L^\infty(0,T_0;H)} + \|\eta_\lambda\|_{L^\infty(0,T_0;H)} \le c. \tag{32}$$

Fourth estimate. It remains to achieve the complete parabolic regularity (9) for θ. With this aim, note that (21) can be rewritten as

$$\theta_{\lambda t} - \Delta\theta_\lambda = -\theta_\lambda \chi_{\lambda t} + \chi^2_{\lambda t}, \tag{33}$$

and we just need a λ-uniform bound of the right hand side in $L^2(0, T_0; H)$. As for the first term, this is an immediate consequence of the second of (30), the first of (31), and the Sobolev embedding $V \subset L^6(\Omega)$. The latter term is controlled upon noticing that (31) yields a bound of $\chi^2_{\lambda t}$ in

$$L^\infty(0, T_0; L^{3/2}(\Omega)) \cap L^{3/2}(0, T_0; L^{9/2}) \subset L^4(0, T_0; H),$$

where of course the inclusion, given by elementary interpolation, is continuous.

Passage to the limit. This last step of the procedure can be performed exactly as in [7, Sec. 5], since we have the same bounds on the approximating functions, albeit they are local in time in our setting. This concludes the proof of Theorem 1.

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A scalar model of viscoelasticity with singular memory

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1. – Introduction

We shall consider a scalar partial differential equation with time delay

$$(1 + K*)\, u_{,tt} - \nabla \cdot \left(n(\mathbf{x})^2 \nabla u\right) = 0 \tag{1}$$

with the initial conditions

$$u(t, \mathbf{x}) = 0 \quad \text{for } t < 0 \tag{2}$$

$$u(0, \mathbf{x}) = 0,\; u_{,t}(0, \mathbf{x}) = \delta(\mathbf{x}) \tag{3}$$

where the asterisk denotes the Volterra convolution operator

$$K * u(t, \mathbf{x}) := \int_0^\infty K(\tau)\, u(t - \tau, \mathbf{x})\, \mathrm{d}\tau \tag{4}$$

The spatial part can also be replaced by a more general operator A on a Banach space Y (Sec. 6). The Laplace transform

$$\tilde{u}(s, \mathbf{x}) = \int_0^\infty \mathrm{e}^{-s\,t} u(t, \mathbf{x})\, \mathrm{d}t$$

of the solution satisfies the equation

$$\left[1 + \tilde{K}(s)\right] \left[s^2 \tilde{u} - \delta(\mathbf{x})\right] - \nabla \cdot \left[n(\mathbf{x}) \nabla \tilde{u}\right] = 0 \tag{5}$$

The operator $1 + K*$ is invertible and equation (1) can be recast in the form

$$u_{,tt} - (1 + R*) \nabla \cdot \left(n(\mathbf{x})^2 \nabla u\right) = \left[\delta(t) + R(t)\right] \delta(\mathbf{x}) \tag{6}$$

where $(1 + R*)(1 + K*)f = f$ or $1 + \tilde{R}(s) = [1 + \tilde{K}(s)]^{-1}$

Eq. (6) is common in the theory of viscoelasticity but eq. (1) is more convenient for our purposes, as will be seen below. The function $R(t)$ is the time derivative of the relaxation function [27, 10].

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In this paper we shall be interested in weakly singular convolution kernels $K(t) \sim c t^{-\gamma} + \ldots$ for $t \to 0$, with $0 < \gamma < 1$. It follows that $\tilde{K}(s) \sim c\Gamma(1-\gamma)\, s^{\gamma-1}$ for $s \to \infty$ [9, 31]. Clearly, $\tilde{R}(s) \sim -c\Gamma(1-\gamma)\, s^{\gamma-1}$. Assuming that R is monotonic and non-increasing, $R(t) \sim -ct^{-\gamma}$ for $t \to 0$ by a Tauberian theorem [9, 31].

An example of such a kernel is provided by a Bagley-Torvik model in polymer rheology [1] with the Laplace transform of the memory kernel R

$$\tilde{R}(s) = -\frac{R_0\,(s/\omega_c)^{-\alpha}}{1 + (s/\omega_c)^{-\alpha}} \tag{7}$$

where ω_c is a characteristic frequency.

As a rule, *solutions of equations* (1) *are infinitely smooth at the characteristics.* Intuitively, this results from the fact that the rate of exponential attenuation is unbounded in the high-frequency limit and the propagator spectrum decays exponentially. A rigorous statement of the smoothness property requires however some *additional restrictions on the convolution kernels. Rigorous results for specific classes* of equations (1) can be found in [25, 22, 23, 5]. Related results for linear viscoelasticity can be found in [28, 20, 7, 6]. Little is known about non-linear equations with weakly singular memory, but smoothing has been demonstrated for scalar viscoelasticity with completely monotonic kernels [16] as well as for fractional conservation laws [15, 4, 14]. In some cases though non-linearity can be an obstacle to smoothing of discontinuous initial data [13].

Our considerations will make use of a class of functions which we call basis functions [19]. These functions are a generalization of totally skewed stable probability distributions [29]. They appeared for the first time in asymptotic solutions of viscoelastic equations in one spatial dimension in the paper of Buchen and Mainardi [3] and, independently, in an analysis of exact solutions of eq. (1) in [23], particularly in the case of half-integer expansions. Furthermore, a connection between viscoelastic fluid dynamics and stable probability distributions was made in [21]. In [17, 18, 19] exact and asymptotic solutions for scalar partial differential equations with singular kernels and systems thereof were studied in some detail.

In Sec. 2 a definition of the basis functions is given and their properties are studied. Some explicit expressions for basis functions are presented in Secs 3 and 4. In Sec. 5 equations (1) with explicit solutions are discussed. In Sec. 6 more general equations with the Laplacian replaced by a more general linear operator are studied by the methods of abstract Volterra equations. Well-posedness results are obtained.

2. – Basis functions: General results

In [17, 19] *we constructed asymptotic solutions of equations* (1) *of the form*

$$\tilde{u}(s,\mathbf{x}) \sim \exp\left(-s\, S^{(0)}(\mathbf{x}) - \sum_{r=1}^{N} s^{\gamma_r}\, S^{(r)}(\mathbf{x})\right) \sum_{n=0}^{\infty} s^{-\alpha_n}\, u^{(n)}(\mathbf{x}) \tag{8}$$

where $\tilde{u}(s,\mathbf{x})$ denotes the Laplace transform of $u(\mathbf{x},t)$ and the variable s is identified with $-\mathrm{i}\omega$,

$$\gamma_0 = 1,\ 0 < \gamma_r < \gamma_1 < 1 \text{ for } r > 1$$

$$0 < \alpha_n < \alpha_{n+1} \text{ for } n > 0$$

$$S^{(1)}(\mathbf{x}) \geq 0$$

Equation (8) can be understood in the sense that

$$\tilde{u}(s, \mathbf{x}) / \exp\left(-s\, S^{(0)}(\mathbf{x}) - \sum_{r=1}^{N} s^{\gamma_r}\, S^{(r)}(\mathbf{x})\right) \sim \sum_{n=0}^{\infty} s^{-\alpha_n}\, u^{(n)}(\mathbf{x})$$

is an asymptotic expansion in the sense of Poincaré-Watson [8]. Asymptotic expansions with a similar phase function were introduced for the first time in [3] in a study of one-dimensional wave motion in a homogeneous viscoelastic medium.

The termwise inverse Laplace transform of eq. (8) has the form of a formal expansion

$$(9) \qquad u(t, \mathbf{x}) = \sum_{n=0}^{\infty} f_{\alpha_n}(t - S^{(0)}(\mathbf{x}), \gamma, S(\mathbf{x}))\, u^{(n)}(\mathbf{x})$$

where $\gamma = \{\gamma_1, \ldots, \gamma_N\}$, $S(\mathbf{x}) = \{S^{(r)}(\mathbf{x}) \mid r = 1, \ldots, N\}$, and the basis functions f_α are defined by the formula

$$(10) \qquad f_\alpha(t, \gamma, \lambda) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{B}} \mathrm{e}^{s\,t}\, s^{-\alpha} \exp\left(-\sum_{r=1}^{N} \lambda_r\, s^{\gamma_r}\right) \mathrm{d}s$$

where $\alpha \in \mathbb{R}_+$, $\gamma \in]0, 1[^{\times N}$, $\lambda \in \mathbb{R}^N$, $0 < \gamma_k \leq \gamma_1$, $\lambda_1 > 0$. The fractional powers $s^{-\alpha}$, s^{γ_r} are taken in $-\pi < \arg s < \pi$ and $\mathcal{B}$ denotes the Bromwich contour $\operatorname{Re} s = \epsilon > 0$. Since $\operatorname{Re} s^{\gamma_1} > 0$ on $\mathcal{B}$, the functions f_α are defined for $\lambda_1 > 0$.

In this paper the basis functions are used to construct exact solutions of initial and initial-boundary value problems.

For $t = 0$ the Bromwich contour can be closed in the right half of the complex plane, where the integrand of

$$(11) \qquad \frac{\partial^n f_\alpha}{\partial t^n} = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{B}} \mathrm{e}^{s\,t}\, s^{n-\alpha} \exp\left(-\sum_{r=1}^{N} \lambda_r\, s^{\gamma_r}\right) \mathrm{d}s$$

$n \in \mathbb{N}$, does not have singularities. Consequently, we have

THEOREM 1

$$f_\alpha(\cdot, \gamma, \lambda) \in \mathcal{C}^\infty(\mathbb{R})$$

and

$$f_\alpha(t, \gamma, \lambda) = 0 \quad \textit{for } t < 0$$

where $\lambda = \{\lambda_1, \ldots, \lambda_N\}$

The following theorem follows easily from the definition:

THEOREM 2

$$\int_0^\infty f_0(t, \gamma, \lambda)\, \mathrm{d}t = 1$$

Let

$$\chi_\beta(t) = \begin{cases} t^{\beta-1}/\Gamma(\beta) & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

The left-hand side is a holomorphic function of $\beta \in \mathbb{C}$ and $\chi_{-n} = \delta^{(n)}$ for $n \in \mathbb{Z}_+ \cup \{0\}$, where $\delta^{(n)}$ denotes the derivative of n-th order of δ[12].

THEOREM 3 *For* $\alpha \in \mathbb{R}_+$

$$f_\alpha(t, \gamma, \lambda) \to \chi_\alpha(t)$$

in the sense of distributions for $\lambda_r \to 0+$, $r = 1, \ldots, N$.

PROOF. Let $\phi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of test functions. By Parseval's theorem

$$\begin{aligned} &\int_{-\infty}^{\infty} f_\alpha(t, \gamma, \lambda)\, \phi(t)\, \mathrm{d}t \\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega\, t}\, (-\mathrm{i}\omega)^{-\alpha} \exp\left(- \sum_{r=1}^{N} \lambda_r\, |\omega|^{\gamma_r}\, \mathrm{e}^{-\mathrm{i}\,\pi\, \gamma_r/2} \right) \hat{\phi}(\omega)\, \mathrm{d}\omega \end{aligned}$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ, $\hat{\phi} \in \mathcal{S}(\mathbb{R})$. We note that the argument of exp is non-positive for $0 < \gamma_r < 1$, $\lambda_r \geq 0$, $r = 1, \ldots, N$. By the Lebesgue dominated convergence theorem the right-hand side tends to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega\, t}\, (-\mathrm{i}\omega)^{-\alpha}\, \hat{\phi}(\omega)\, \mathrm{d}\omega = \langle \chi_\alpha, \phi \rangle$$

for $\lambda_r \to 0$, $r = 1, \ldots, N$, where the angular parentheses denote the pairing between $\mathcal{S}$ and $\mathcal{S}'$. Hence $f_\alpha(\cdot, \gamma, \lambda) \to \chi_\alpha$ in $\mathcal{S}'(\mathbb{R})$. ■

THEOREM 4 *The basis functions satisfy the following recurrence relations*

$$\frac{\partial f_\alpha}{\partial \lambda_r} = -f_{\alpha - \gamma_r} \tag{13}$$

and

$$t f_\alpha(t, \gamma, \lambda) = \alpha f_{\alpha+1}(t, \gamma, \lambda) + \sum_{r=1}^{N} \lambda_r \gamma_r f_{\alpha - \gamma_r + 1}(t, \gamma, \lambda) \tag{14}$$

PROOF. Equation (14) can be proved by integration by parts in equation (10). ■

Non-negativity, monotonicity and unimodality are important properties of the basis functions.

THEOREM 5 *Let $\lambda_k \geq 0$ and $0 < \gamma_k \leq \gamma_1 < 1$ for $k = 1, \ldots, N$, $\lambda_1 > 0$.*
For $\alpha \geq 0$

$$f_\alpha(t, \gamma, \lambda) \geq 0$$

For $\alpha \geq 1$ the functions $f_\alpha(\cdot, \gamma, \lambda)$ are monotonic (increasing).
For $0 \leq \alpha < 1$ with $\lambda_k > 0$ for some $1 \leq k \leq N$ and $\lambda_l = 0$ for $1 \leq l \leq N$, $l \neq k$ the function $f_\alpha(\cdot, \gamma, \lambda)$ is unimodal.

PROOF. Let

$$\phi(t) = \sum_{r=1}^{N} \lambda_r \frac{t^{-\gamma_r}}{\Gamma(1 - \gamma_r)}$$

so that

$$s\,\tilde{\phi}(s) = \sum_{r=1}^{N} \lambda_r\, s^{\gamma_r}$$

If we prove that

$$I_n := (-1)^n \frac{\mathrm{d}^n\, \mathrm{e}^{-s\,\tilde{\phi}(s)}}{\mathrm{d}s^n} \geq 0 \tag{15}$$

then the inequalities $f_1 \geq 0$ and $f_0 = \partial f_1/\partial t \geq 0$ follow from Bernstein's theorem [9, 31]. Hence, for arbitrary $\alpha > 0$

$$f_\alpha(t, \gamma, \lambda) = \chi_\alpha * f_0(t, \gamma, \lambda) \geq 0$$

For $\alpha \geq 1$

$$\frac{\partial f_\alpha}{\partial t} = f_{\alpha-1} \geq 0$$

whence f_α, $\alpha \geq 1$, are monotonic.

We now prove eq. (15). For $n = 0$ eq. (15) is obvious. For $n \geq 1$ we shall use the following inequality

$$(-1)^m \frac{\mathrm{d}^{m+1} s\left[\tilde{\phi}(s)\right]}{\mathrm{d}s^{m+1}} \geq 0 \quad \text{for } m \in \mathbb{Z}_+ \cup \{0\} \tag{16}$$

Indeed, for $m = 0$ we have

$$\left(s\,\tilde{\phi}(s)\right)' = -\int_0^\infty t\,\phi'(t)\,\mathrm{e}^{-st}\,\mathrm{d}t \geq 0 \tag{17}$$

because

$$s\,\tilde{\phi}(s) = \lim_{\varepsilon\to 0} s \int_\varepsilon^\infty \phi(t)\,\mathrm{e}^{-st}\,\mathrm{d}t = \lim_{\varepsilon\to 0} \left[\phi(\varepsilon)\,\mathrm{e}^{-s\varepsilon} + \int_\varepsilon^\infty \phi'(t)\,\mathrm{e}^{-st}\,\mathrm{d}t\right]$$

$t\,\phi'(t)$ is integrable near 0 and $\varepsilon\,\phi(\varepsilon) = \mathrm{o}[1]$ and $\phi'(t) \leq 0$. Eq (16) is now proved by repeated differentiations.

The n-th derivative of $\mathrm{e}^{-s\,\tilde{\phi}(s)}$ has the form $\mathrm{e}^{-s\,\tilde{\phi}(s)}\,Y_n$, where Y_n is a sum of products of derivatives of $-s\,\tilde{\phi}(s)$ whose orders add up to n. Eq. (16) implies (15).

$f_0(\cdot,\gamma,0,\ldots,\lambda_k,0,\ldots,0)$ is a totally skewed γ_k-stable probability distribution [29, 30], and it is known to be unimodal (*ibidem*). For $0<\alpha<1$ the function χ_α is strongly unimodal [30], hence

$$f_\alpha(t,\gamma,0,\ldots,0,\lambda_k,0,\ldots,0) = f_0(t,\gamma,0,\ldots,0,\lambda_k,0,\ldots,0) * \chi_\alpha(t)$$

is unimodal. ∎

For $N=1$ and $\gamma_1 = 1/2$ the basis functions are either unimodal ($0<\alpha<1$) or monotonic ($\alpha \geq 1$).

3. – Basis functions for half-integer power expansions

The basis functions $f_\alpha(t,1/2,\lambda)$ can be expressed in terms of elementary functions and the complementary error function

$$f_0(t,1/2,\lambda) = \frac{\lambda}{2\sqrt{\pi}}\, t^{-3/2}\, \mathrm{e}^{-\lambda^2/(4t)} \tag{18}$$

$$f_{1/2}(t,1/2,\lambda) = \frac{1}{\sqrt{\pi}}\, t^{-1/2}\, \mathrm{e}^{-\lambda^2/(4t)} \tag{19}$$

$$f_1(t,1/2,\lambda) = \operatorname{erfc}\left(\lambda/\left(2\sqrt{t}\right)\right) \tag{20}$$

$$f_{3/2}(t,1/2,\lambda) = \frac{2}{\sqrt{\pi}}\, t^{1/2}\, \mathrm{e}^{-\lambda^2/(4t)} + \lambda \operatorname{erfc}\left(\lambda/\left(2\sqrt{t}\right)\right) \tag{21}$$

for $t>0$, with $f_\alpha(t,1/2,\lambda)=0$ for $t<0$.

The functions $f_{n/2}$ for $n>3$ can be calculated from the recurrence relations (14).

4. – Basis functions for expansions in inverse powers of $s^{1/3}$

For memory kernels with $\tau^{-1/3}$ and $\tau^{-2/3}$ singularities

$$\tilde{K}(s) \sim \sum_{\mu=0}^{\infty} s^{-\mu/3}\, K^{(\mu)} \tag{22}$$

the solution is constructed in the form of the asymptotic expansion (9) with

$$f_{\mu/3}(t,2/3,1/3,\lambda_1,\lambda_2) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{B}} s^{-\mu/3}\, \mathrm{e}^{s\,t}\, \mathrm{e}^{-\lambda_1 s^{2/3} - \lambda_2 s^{1/3}}\, \mathrm{d}s \tag{23}$$

The functions $f_0(t,2/3,1/3,\lambda)$, $f_{1/3}(t,2/3,1/3,\lambda)$, $f_{2/3}(t,2/3,1/3,\lambda)$ can be calculated in an explicit form. Substituting $\mu=2$, $\tau = s^{1/3} - \lambda_1/3t$ in eq. (23) we have

$$\begin{aligned} f_{2/3}(t,2/3,1/3,\lambda_1,\lambda_2) &= \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{B}} s^{-2/3}\, \mathrm{e}^{st}\, \mathrm{e}^{-\lambda_1 s^{2/3} - \lambda_2 s^{1/3}}\, \mathrm{d}s \\ &= \frac{3^{2/3}}{2\pi\, t^{1/3}} \mathrm{e}^{-\beta} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\tau^3/3 + \gamma\tau)}\, \mathrm{d}\tau \end{aligned} \tag{24}$$

with

$$\gamma = (3t)^{-1/3}\left(\lambda_2 + \frac{1}{3}\frac{\lambda_1^2}{t}\right), \; \beta = \frac{\lambda_1}{3t}\left[\lambda_2 + \frac{2\lambda_1^2}{9t}\right]$$

whence

$$f_{2/3}(t, 2/3, 1/3, \lambda_1, \lambda_2)$$

(25)
$$= \frac{3^{2/3}}{t^{1/3}}\left[\mathrm{Ai}\left((3\,t)^{-1/3}\left(\lambda_2 + \frac{\lambda_1^2}{3t}\right)\right)\exp\left(-\frac{\lambda_1}{3t}\left(\lambda_2 + \frac{2\lambda_1^2}{9t}\right)\right)\right]$$

Two other basis functions can be determined from the formulae

(26)
$$f_0(t, 2/3, 1/3, \lambda_1, \lambda_2) = -\frac{\partial f_{2/3}}{\partial \lambda_2}$$

(27)
$$f_{1/3}(t, 2/3, 1/3, \lambda_1, \lambda_2) = -\frac{\partial f_{2/3}}{\partial \lambda_1}$$

(28)
$$\frac{\partial f_1(t, 2/3, 1/3, \lambda_1, \lambda_2)}{\partial t} = f_0(t, 2/3, 1/3, \lambda_1, \lambda_2)$$

The basis functions $f_{\mu/3}$ for $\mu > 0$ can be calculated by using the identity

(29)
$$\mu\, f_{(\mu+3)/3} = 3\,t\, f_{\mu/3} - 2\,\lambda_1\, f_{(\mu+1)/3} - \lambda_2\, f_{(\mu+2)/3}$$

which can be derived by integrating eq. (23) by parts.

5. – Exact fundamental solutions for scalar equations

Explicit fundamental solutions of eqs (1) for odd spatial dimensions can be constructed provided the coefficients of the convolution kernel satisfy appropriate constraints.

The solution is constructed with the following ansatz

(30)
$$\tilde{u}(s, \mathbf{x}) = w(s)\, U(s, r)$$

where

(31)
$$U(s, r) = \mathrm{e}^{s\,r\,\varphi(s)}/(4\pi\, r)$$

Substituting (30) in the Laplace-transformed equation (1) and noting that

(32)
$$\nabla^2 U = -\delta(\mathbf{x}) + s^2\,\varphi(s)^2\, U$$

yields the equation

(33)
$$s^2\left[1 + \tilde{K} - \varphi(s)^2\right]\tilde{u} + \left[w(s) - 1 - \tilde{K}(s)\right]\delta(\mathbf{x}) = 0$$

whence

$$\varphi(s)^2 = 1 + \tilde{K}(s) = w(s) \tag{34}$$

The following ansätze lead to explicitly computable original functions:

$$\varphi(s) = 1 + a\,s^{-1/2}; \quad \tilde{K}(s) = 2a\,s^{-1/2} + a^2\,s^{-1} \tag{35}$$

$$\begin{aligned} &\varphi(s) = 1 + a\,s^{-1/3} + b\,s^{-2/3}; \\ &\tilde{K}(s) = 2a\,s^{-1/3} + (2b + a^2)\,s^{-2/3} + 2a\,b\,s^{-1} + b^2\,s^{-4/3} \end{aligned} \tag{36}$$

Substitution of eqs (35) or (36) in eq. (30) followed by an inverse Laplace transformation yields the fundamental solution [18]:

$$\begin{aligned} &u(t, \mathbf{x}) \\ &= \frac{1}{4\pi\,r}\left[f_0(t - r, 1/2, a\,r) + 2a\,f_{1/2}(t - r, 1/2, a\,r) + a^2\,f_1(t - r, 1/2, a\,r)\right] \end{aligned} \tag{37}$$

for the convolution kernel (35) and

$$\begin{aligned} u(t, \mathbf{x}) = \frac{1}{4\pi\,r}\big[&f_0(t - r, 2/3, 1/3, a\,r, b\,r) + 2a\,f_{1/3}(t - r, 2/3, 1/3, a\,r, b\,r) \\ &+ (2b + a^2)\,f_{2/3}(t - r, 2/3, 1/3, a\,r, b\,r) \\ &+ 2a\,b\,f_1(t - r, 2/3, 1/3, a\,r, b\,r) + b^2\,f_{4/3}(t - r, 2/3, 1/3, a\,r, b\,r)\big] \end{aligned} \tag{38}$$

for the kernel (36).

6. – Well-posedness, regularity and solutions for more general equations

In this section we shall consider eq. (1) with a special class of convolution kernels K and a rather general spatial operator. Some methods of abstract Volterra equations [26] will be applied. An abstract Volterra equation on a Banach space Y

$$u(t) = g(t) + \int_0^t h(t - \tau)\,A\,u(\tau)\,\mathrm{d}\tau \tag{39}$$

with a closed densely defined linear operator A on Y is said to be well-posed if for every constant $y \in \mathcal{D}(A)$ there is a unique solution $u(t; y)$ and $u(t, y) \to 0$ for $y \to 0$ uniformly on compact subsets of $t \in \mathbb{R}_+$. The integrals appearing here are Bochner integrals. Solutions $u(t) \in \mathcal{D}(A)$ of eq. (39) are called strong.

The resolvent $S(t)$ is a strongly continuous one-parameter family of bounded operators on Y satisfying the conditions

(i) $S(0) = I$;

(ii) $S(t)\mathcal{D}(A) \subset \mathcal{D}(A)$ and $A\,S(t)w = S(t)\,A\,w$ for every $w \in \mathcal{D}(A)$, $t \geq 0$;

(iii)

$$S(t)\,w = w + \int_0^t h(t - \tau)\,A\,S(\tau)w\,\mathrm{d}\tau \tag{40}$$

Eq. (39) is well-posed iff it has a resolvent [26].

A is an infinitesimal generator of a cosine family $C(t)$ [11] if $C(t)$ satisfies equation (40) with $h(t) \equiv t$.

We shall consider the problem

$$u_{,tt} + K(t) * u_{,tt} = A\,u \tag{41}$$

$$u(t) = 0 \quad \text{for } t \leq 0; \quad u_{,t}(0+, \mathbf{x}) \;=\; v_0(\mathbf{x}) \tag{42}$$

in $Y = \mathcal{L}^2(\mathbb{R}^3)$, with a linear densely defined operator A on Y. An example of A is a closed extension of $\nabla \cdot\, [n(\mathbf{x})^2\, \nabla \cdot]$ on a bounded or unbounded domain in $\mathbb{R}^3$, with appropriate boundary conditions. Such an operator is an infinitesimal generator of a cosine family if the associated initial-boundary value problem with $K = 0$ is well-posed in Y. The corresponding abstract Volterra equation is

$$u(t) = t\,v_0 + \int_0^t \mathrm{d}\tau\,(t-\tau)\,\left[A\,u(\tau) + \int_0^\tau R(\tau-\sigma)\,A\,u(\sigma)\,\mathrm{d}\sigma\right] \tag{43}$$

where $R(t)$ is defined by the formula $1 + \tilde{R}(s) = \left[1 + \tilde{K}(s)\right]^{-1}$.

Theorem 6 *Let $1 + \tilde{K}(s) = [1 + \phi(s)]^2$ and suppose that $\psi(s) = s\,[1 + \phi(s)]\,, s \in \mathbb{R}_+$ is a Bernstein function. Let A be the infinitesimal generator of a cosine family $\{C(t) \mid t \geq 0\}$.*

There is a non-negative function $v : \mathbb{R}_+^2 \to \mathbb{R}$, non-decreasing with respect to the first argument such that

$$\exp\left(-\tau\psi(s)\right)/s = \int_0^\infty \mathrm{e}^{-t\,s}\,v(t,\tau)\,\mathrm{d}t \tag{44}$$

and $v(0,\tau) = 0$. The problem (43) is well posed and its resolvent is given by the Stieltjes integral

$$S(t) = -\int_0^\infty C(\tau)\,\mathrm{d}_\tau v(t,\tau) \tag{45}$$

Proof. Since ψ is a Bernstein function, the function $s \to \exp\left(-\tau\,\psi(s)\right)$ is completely monotonic [26]. By the Bernstein theorem [31] there is a non-negative function $v(t,\tau)$, non-decreasing with respect to the first argument, such that eq. (44) holds and $v(0,\tau) = 0$.

By Corollary 4.5 of [26] eq. (39) has a resolvent $S(t)$ satisfying eq. (45). Consequently, eq. (39) is well-posed. ∎

Corollary 1 *Let $1 + \tilde{K} = [1 + \phi(s)]^2$,*

$$\phi(s) = \sum_{n=1}^{N} a_n\, s^{\gamma_n - 1}$$

with $0 < \gamma_n < 1$, $a_n > 0$ for $n = 1, \ldots, N$.

Eq. (43) *is well-posed, eq.* (45) *holds with*

$$v(t,\tau) = f_1(t-\tau, \tau a, \gamma)$$

$$S(t) = \int_0^t \left[f_0(t-\tau, \tau a, \gamma) + \sum_{n=1}^{N} a_n\, f_{\gamma_n}(t-\tau, \tau a, \gamma) \right] C(\tau)\, \mathrm{d}\tau \tag{46}$$

where $\tau\, a = \{\tau\, a_n \mid n = 1, \dots N\}$, $\gamma = \{\gamma_n \mid n = 1, \dots N\}$.

$$S \in \mathcal{C}^\infty\left(\mathbb{R}_+; \mathcal{B}(Y)\right)$$

where $\mathcal{B}(Y)$ *denotes the Banach space of bounded operators on* Y.

PROOF. $\psi(s) = s + \sum_{n=1}^{N} a_n\, s^{\gamma_n}$ is clearly a Bernstein function, hence Theorem 6 applies with v expressed in terms of f_1 as above. Eq. (46) is obtained by working out the derivative $\mathrm{d}f_1(t-\tau, \tau a, \gamma)/\mathrm{d}\tau$. The explicit form of the resolvent shows its smoothness. ∎

For A = closure of ∇^2 in $Y = \mathcal{L}^2\left(\mathbb{R}^3\right)$ we have

$$\left(C(t)\, v_0\right)(\mathbf{x}) = \frac{1}{4\pi r}\, \left[\delta(t-r) - \delta(t+r)\right] *_x\, v_0(\mathbf{x})$$

where $r = |\mathbf{x}|$ and the convolution applies to the spatial variables. Since $f_\alpha = 0$ for $t < 0$, this implies that

$$\left(S(t)\, v_0\right)(\mathbf{x}) = \frac{1}{4\pi r} \left[f_0(t-r, ra, \gamma) + \sum_{n=1}^{N} a_n\, f_{\gamma_n}(t-r, ra, \gamma) \right] *_x\, v_0(\mathbf{x})$$

in agreement with the results of the previous section.

REMARK 1 The problem considered in the Corollary 1 can be formulated in terms of the Caputo fractional derivatives. In fact

$$K(t) = 2\sum_{n=1}^{N} a_n\, \chi_{1-\gamma_n}(t) + 2\sum_{n=1}^{N} \sum_{m=n+1}^{N} a_n\, a_m\, \chi_{1-\gamma_n-\gamma_m}(t) + \sum_{n=1}^{N} a_n^2\, \chi_{1-2\gamma_n}(t) \tag{47}$$

Using the definition of a Caputo fractional derivative

$$\mathrm{D}^\gamma f(t) = \int_0^t \tau^{M-\gamma-1}\, \partial^M f(t-\tau)/\partial t^M\, \mathrm{d}\tau/\Gamma(M-\gamma) \quad \text{for } M \in \mathbb{N},\ M-1 < \gamma \le M$$

as well as the identity $\mathrm{D}^\gamma\, \mathrm{D} = \mathrm{D}^{1+\gamma}$ [24], for K given by eq. (47), eq. (1) with the initial condition (2) is seen to be equivalent to the equation

$$u_{,tt} + 2\sum_{n=1}^{N} a_n\, \mathrm{D}^{1+\gamma_n}\, u + \sum_{n=1}^{N}\sum_{m=1}^{N} a_n\, a_m\, \mathrm{D}^{\gamma_n+\gamma_m}\, u - A\, u = 0 \tag{48}$$

A related equation was studied in [2] as a mechanical damping model.

REMARK 2 Theorem 6 can be applied to fractional powers of the Laplacian $A = -(-\overline{\Delta})^\nu$ on $Y = \mathcal{L}^2\left(\mathbb{R}^d\right)$, $d = 1, 2, 3, \dots$, $0 < \nu < 1$, since A generates a cosine family on Y (Hanyga, this volume). ($\overline{\Delta}$ denotes the closure of the Laplacian $\Delta = \nabla^2$ in Y.)

7. – Concluding remarks

In mechanical applications one has to consider systems of equations with several different memory operators (up to six different memory kernels appear in anisotropic viscoelasticity). Scalar equations with singular memory offer however the possibility of a better insight into the structure of the solutions. The basis functions are very helpful in this respect. Basis functions also appear in the construction of asymptotic solutions for systems of integro-differential equations with weakly singular convolution kernels [17, 19].

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An existence result for semilinear equations in viscoelasticity: the case of regular kernels

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1. – Introduction

This paper is concerned with the abstract semilinear integro-differential equation

$$\ddot{u}(t) + Au(t) + \int_0^t B(t-s)u(s)ds = F(u(t)) + f(t) \qquad t \geq 0, \tag{1}$$

that may be regarded as a model problem for some elastic systems with memory, like the ones considered in [2, 3, 6]. Here, A is a positive operator on a Hilbert space X with domain $D(A)$, $B(t)$ is a self-adjoint linear operator on X with domain $D(B(t)) \supset D(A)$, commuting with A, and F is a locally Lipschitz map defined on the domain of $\sqrt{A}$.

The linear version of (1) , that is for $F \equiv 0$, can be reduced to an integral form and then solved applying known existence results, see [5]. Such a procedure, however, requires smooth initial conditions and provides no maximal regularity results that are needed to study the nonlinear problem.

The purpose of the present paper is twofold. First, we shall complete the above mentioned procedure with the derivation of suitable maximal regularity estimates for the resolvent of the linear problem. Then, we shall apply the properties of such a resolvent to obtain a local existence result for (1) by standard fixed-point arguments.

An important aspect of our analysis is that we require no sign condition on $B(t)$. Instead, we assume that $B(\cdot)y$ is absolutely continuous in t for any $y \in D(A)$. We plan to study the case of singular kernels in a forthcoming paper.

The outline of this paper is the following. In section 2 we recall some preliminaries and prove our maximal regularity result for the resolvent of the linear problem. Section 3 is devoted to the solution of the Cauchy problem for linear equations, while, in section 4, we obtain local existence results for the nonlinear problem. Finally, in section 5, we describe a typical system in viscoelasticity that can be studied by our abstract approach.

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2. – Existence of resolvent

In this section, X will denote a real Hilbert space with scalar product $\langle\cdot\,,\,\cdot\rangle$ and norm $\|\cdot\|$.

For any $T>0$ we denote by $L^1(0,T;X)$ the usual space of measurable functions $u:[0,T]\to X$ such that

$$\|u\|_{1,T}:=\int_0^T\|u(t)\|\,dt<\infty.$$

For any $u\in L^1(0,T;X)$ and any $\psi\in L^1(0,T;\mathbb{R})$ the symbol $\psi*u$ stands for convolution from 0 to t, that is

$$\psi*u\,(t)=\int_0^t\psi(t-s)u(s)\,ds,\qquad t\in[0,T].$$

In the following we denote by $W^{1,1}(0,T;X)$ the Banach space of functions $u\in L^1(0,T;X)$ such that $\dot u\in L^1(0,T;X)$ and by $W^{1,1}_{\rm loc}(0,+\infty;X)$ the space of functions belonging to $W^{1,1}(0,T;X)$ for any $T>0$.

A and $B(t)$ are linear unbounded self-adjoint operators with domains $D(A)$ and $D(B(t))$ respectively, such that $D(A)\subset D(B(t))$ for any $t\geq 0$ and $D(A)$ is dense in X. Furthermore, we assume

$$\text{(2)}\qquad \langle Ay,y\rangle\geq a_0\|y\|^2\,,\qquad \text{for any } y\in D(A)\quad (a_0>0)\,;$$

$$\text{(3)}\qquad B(\cdot)y\in W^{1,1}_{\rm loc}(0,+\infty;X)\,,\qquad \text{for any } y\in D(A)\,;$$

$B(t)$ commutes with A, that is

$$\text{(4)}\qquad B(t)D(A^2)\subset D(A)\,\text{and}\;\; AB(t)y=B(t)Ay\,,\;\; y\in D(A^2)\,,\;\; t\geq 0.$$

In the sequel, $D(A)$ will be regarded as a Hilbert space with the norm $\|Ax\|$.

The notion of resolvent is recalled below.

DEFINITION **1** A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators in X is called a resolvent for the equation

$$\text{(5)}\qquad \ddot u(t)+Au(t)+\int_0^t B(t-s)u(s)\,ds=0$$

if the following conditions are satisfied:

(S1) $S(0)=I$ and $S(t)$ is strongly continuous on $[0,+\infty[$, that is, for all $x\in X$ $S(\cdot)x$ is continuous on $[0,+\infty[$;

(S2) $S(t)$ commutes with A, which means that $S(t)D(A)\subset D(A)$ and $AS(t)y=S(t)Ay$ for all $y\in D(A)$ and $t\geq 0$;

(S3) for any $y\in D(A)$ $S(\cdot)y$ is twice continuously differentiable in X on $[0,+\infty)$ and $\dot S(0)=0$;

(S4) for any $y \in D(A)$ and $t \geq 0$ the resolvent equation

$$\ddot{S}(t)y + AS(t)y + \int_0^t B(t-\tau)S(\tau)y\,d\tau = 0$$

holds.

REMARK 1 (i) The equation (5) is formally equivalent to the integral equation

$$(6)\ u(t) = u(0) + \dot{u}(0)t - \int_0^t (t-s)Au(s)ds - \int_0^t \left(\int_0^{t-s} (t-s-\tau)B(\tau)\,d\tau\right) u(s)ds.$$

(ii) The uniqueness of the resolvent follows from that of the resolvent for integral equations (see, e.g., [5, Proposition 6.1]).

The existence of resolvent follows from that for integral equations proved by Prüss in [5].

THEOREM 1 *Assume* (2)-(4). *Then, there exists the resolvent* $S(t)$ *for equation* (5).

PROOF. To simplify the notations, we introduce

$$\mathcal{A}(t) = -t\,A - \int_0^t (t-s)B(s)\,ds\,, \quad t \geq 0\,, \tag{7}$$

denoting the kernel of the integral equation (6).

We can apply [5, Corollaries 6.1 and 6.3] to (6): there exists a family $\{S(t)\}_{t\geq 0}$ of bounded linear operators in X such that

(i) $S(0) = I$ and $S(t)$ is strongly continuous on $[0, +\infty[$;

(ii) for any $y \in D(A)$, $S(\cdot)y \in D(A)$ a.e. and the function $AS(\cdot)y$ is locally bounded on $(0, +\infty)$;

(iii) for any $y \in D(A)$ and $t \geq 0$ the equations

$$S(t)y = y + \int_0^t \mathcal{A}(t-\tau)S(\tau)y\,d\tau\,, \qquad S(t)y = y + \int_0^t S(t-\tau)\mathcal{A}(\tau)y\,d\tau\,,$$

hold;

(iv) the function $\int_0^t (t-s)S(s)x\,ds$ belongs to $C([0, +\infty); D(A))$ for each $x \in X$;

(v) for all $y \in D(A)$ $S(\cdot)y$ is continuously differentiable in X and twice differentiable a.e. on $[0, +\infty)$.

First, we observe that for any $y \in D(A)$ and $t \geq 0$

$$\int_0^t S(t-\tau)\mathcal{A}(\tau)y\,d\tau = -\int_0^t (t-s)S(s)Ay\,ds - \int_0^t \left(\int_0^{t-\tau} (t-\tau-s)S(s)ds\right) B(\tau)y\,d\tau$$

and so, by (iv) and the second equation in (iii) it follows that $S(t)y \in D(A)$ and $AS(\cdot)y$ is continuous.

Now, we prove that $S(t)$ commutes with A. In fact, if $y \in D(A^2)$, then, thanks to (4), we have that

$$A^{-1}S(t)Ay = y + \int_0^t \mathcal{A}(t-\tau)A^{-1}S(\tau)Ay\,d\tau\,,$$

$$A^{-1}S(t)Ay = y + \int_0^t A^{-1}S(t-\tau)A\mathcal{A}(\tau)y\,d\tau\,;$$

by uniqueness of the resolvent it follows

$$S(t)y = A^{-1}S(t)Ay\,.$$

Moreover, by density we have

$$AS(t)y = S(t)Ay\,, \qquad \text{for any} \quad y \in D(A)\,.$$

Since $S(t)$ commutes with A, the equations in (iii) are the same. In addition, if we differentiate two times one of the equations in (iii), then we obtain that the resolvent equation in (S4) holds.

Finally, the family $\{S(t)\}_{t\geq 0}$ satisfies the properties (S1) - (S4), which characterize the resolvent of (5). ∎

In the following proposition we list some properties about the resolvent, which will be useful in the sequel to solve non-homogeneous equations.

THEOREM 2 *In the same assumptions of theorem* 1, *the resolvent* $S(t)$ *for equation* (5) *verifies the following properties.*

(i) *The operators* $S(t)$ *are self-adjoint.*

(ii) $S(t)$ *commutes with* $\sqrt{A}$, *that is* $S(t)D(\sqrt{A}) \subset D(\sqrt{A})$ *and* $\sqrt{A}S(t)y = S(t)\sqrt{A}y$ *for all* $y \in D(\sqrt{A})$ *and* $t \geq 0$.

(iii) *For any* $x \in X$ *the function* $t \to \int_0^t S(\tau)x\,d\tau$ *belongs to* $C([0,+\infty[;D(\sqrt{A}))$ *and*

$$(8) \qquad \|S(t)x\| + \left\|\sqrt{A}\int_0^t S(\tau)x\,d\tau\right\| \leq C_T\,\|x\| \qquad \forall\, T>0\,,\quad t \in [0,T]\,.$$

(iv) *For any* $y \in D(\sqrt{A})$ *the function* $t \to \int_0^t S(r)y\,dr$ *belongs to* $C([0,+\infty[;D(A))$ *and for any* $T > 0$

$$(9) \qquad \left\|A\int_0^t S(\tau)y\,d\tau\right\| \leq C_T\,\|\sqrt{A}y\|\,, \qquad t \in [0,T]\,,$$

$$(10) \qquad \|\dot{S}(t)y\| \leq C_T\,(\|y\| + \|\sqrt{A}\,y\|)\,, \qquad t \in [0,T]\,,$$

$$(11) \qquad \dot{S}(t)y + A\int_0^t S(r)y\,dr + B*1*S(t)y = 0\,, \qquad t \geq 0\,.$$

(v) *For any $y \in D(A)$ the function $\dot S(\cdot)y$ belongs to $C([0,+\infty[;D(\sqrt{A}))$.*

Here C_T denotes a positive constant depending only on T.

PROOF. (i) Since the operators $\mathcal{A}(t)$ are self-adjoint, for any y, $z \in D(A)$ we have

$$\begin{aligned}\langle S(t)^* y, z\rangle &= \langle y, S(t)z\rangle = \langle y,z\rangle + \int_0^t \langle \mathcal{A}(t-\tau)y, S(\tau)z\rangle\, d\tau \\ &= \langle y + \int_0^t S(\tau)^*\mathcal{A}(t-\tau)y\, d\tau, z\rangle\,;\end{aligned}$$

by the density of $D(A)$ in X and uniqueness of the resolvent, it follows

$$S(t) = S(t)^*\,,$$

that is the operators $S(t)$ are self-adjoint.
(ii) The statement follows from the integral rappresentation formula for the operator $\sqrt{A}$ (see [4, Theorem 6.9]), taking into account that $S(t)$ commutes with A.
(iii) Fixed $T > 0$, we observe that by Banach-Steinhaus theorem it follows

$$(12) \qquad \|S(t)x\| \le C_T\,\|x\| \qquad \text{for any } t \in [0,T]\,, \quad x \in X\,.$$

To prove the other estimate in (8), set

$$U(t)x = \int_0^t S(r)x\, dr\,, \qquad x \in X\,,$$

by the resolvent equation we obtain

$$(13) \qquad AU(t)y = -\dot S(t)y - B * U(t)y\,, \qquad \text{for any} \quad y \in D(A)\,.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{A}U(t)y\|^2 = \langle AU(t)y, \dot U(t)y\rangle = -\frac{1}{2}\frac{d}{dt}\|S(t)y\|^2 - \langle B * U(t)y, S(t)y\rangle\,,$$

from which, integrating from 0 to t, we get

$$(14) \qquad \|\sqrt{A}U(t)y\|^2 + \|S(t)y\|^2 = \|y\|^2 - 2\int_0^t \langle B * U(\tau)y, S(\tau)y\rangle\, d\tau\,.$$

Now, to evaluate the term $B * U(t)y$ we note that

$$\begin{aligned}\int_0^t B(t-s)U(s)y\, ds &= \int_0^t B(0)A^{-1}AU(s)y\, ds + \int_0^t \left(\int_s^t \dot B(t-r)\, dr\right) U(s)y\, ds \\ &= B(0)A^{-1}1 * AU(t)y + \int_0^t \dot B(t-r)\left(\int_0^r U(s)y\, ds\right) dr \\ &= B(0)1 * U(t)y + \dot B * 1 * U(t)y\,.\end{aligned}$$

Since for $x \in X$ the function $A(1 * U(t)x) = A\int_0^t (t-s)S(s)x\, ds$ is continuous in t, in view of the previous equality the operator $B * U(t)$ can be extended to the whole space X and for any $x \in X$ $B * U(\cdot)x$ is continuous on $[0, +\infty)$. Therefore, again by Banach-Steinhaus theorem we have

(15) $$\|B * U(t)x\| \leq C_T\|x\|, \qquad \text{for any } t \in [0,T],\ x \in X;$$

thanks also to (12), by (14) it follows

$$\|\sqrt{A}U(t)y\| \leq C_T\|y\|, \qquad \text{for any } t \in [0,T], \quad y \in D(A).$$

So, using again the density of $D(A)$ in X (iii) is completely verified.
(iv) First, we observe that if $y \in D(\sqrt{A})$, then by (ii) we have

$$A\int_0^t S(\tau)y d\tau = \sqrt{A}\int_0^t S(\tau)\sqrt{A}y d\tau,$$

and hence, by (iii) the function $t \to A\int_0^t S(\tau)y d\tau$ is continuous and (9) holds.

Moreover, for $y \in D(A)$ by (13), (9) and (15) we get

$$\|\dot{S}(t)y\| \leq \left\|A\int_0^t S(r)y\, dr\right\| + \|B * 1 * S(t)y\| \leq C_T\left(\|\sqrt{A}\, y\| + \|y\|\right);$$

consequently, this inequality also holds for $y \in D(\sqrt{A})$, thanks to the density of $D(A)$ in $D(\sqrt{A})$. Finally, (11) follows again by (13) and the density of $D(A)$ in $D(\sqrt{A})$, taking into account (9) and (10).
(v) Fixed $y \in D(A)$, in view of (13) and (ii) we have

$$\sqrt{A}\dot{S}(t)y = -A\int_0^t S(r)\sqrt{A}y\, dr - B * 1 * S(t)\sqrt{A}y,$$

and hence the continuity of $\sqrt{A}\dot{S}(\cdot)y$ follows by (iv). ∎

REMARK **2** We observe that in the proof of theorem 2 (iii) we don't use the assumption that the operators $B(t)$ commute with A.

3. – Solvability of linear problems

The existence of the resolvent $S(t)$ allows us to solve the non-homogeneous equation

(16) $$\ddot{u}(t) + Au(t) + \int_0^t B(t-s)u(s)ds = f(t), \qquad t \geq 0.$$

We recall the notions of mild and strong solutions for equation (16).

Let $T > 0$ be given.

DEFINITION **2** Let $f \in C([0,T];X)$. We say that u is a strong solution of (16) on $[0,T]$ if $u \in C^2([0,T];X) \cap C([0,T];D(A))$ and u verifies (16) in $[0,T]$.

Let $f \in L^1(0,T;X)$ and u_0, $u_1 \in X$. The *mild solution* of (16) on $[0,T]$ with initial conditions

(17) $$u(0) = u_0, \quad \dot{u}(0) = u_1,$$

is the function $u \in C([0,T];X)$ defined by

(18) $$u(t) = S(t)u_0 + \int_0^t S(\tau)u_1 d\tau + \int_0^t 1 * S(t-\tau)f(\tau)d\tau, \quad t \in [0,T].$$

PROPOSITION 1 *A strong solution of the equation* (16) *is also a mild one.*

PROOF. A strong solution u of the equation (16) is also a strong solution of the integral equation

$$u(t) = u(0) + \dot{u}(0)t + 1 * 1 * f(t) + \int_0^t \mathcal{A}(t-s)u(s)ds\,,$$

where the operators $\mathcal{A}(t)$ are defined in (7). Therefore, by [5, Proposition 6.3 (i)] we have

$$\begin{aligned} u(t) &= \frac{d}{dt}\int_0^t S(r)[u(0) + \dot{u}(0)(t-r) + 1 * 1 * f(t-r)]\,dr \\ &= S(t)u(0) + \int_0^t S(r)\dot{u}(0)dr + \int_0^t S(r)1 * f(t-r)dr\,, \end{aligned}$$

which completes the proof. ∎

THEOREM 3 *Let* $f \in L^1(0,T;X)$, $u_0 \in D(\sqrt{A})$ *and* $u_1 \in X$. *Then, the mild solution of* (16)-(17) *belongs to* $C([0,T];D(\sqrt{A}))$.

PROOF. For $u_0 \in D(\sqrt{A})$, in view of theorem 2 (ii) and (iii) the mild solution is continuous with values in $D(\sqrt{A})$. ∎

THEOREM 4 *Let* $f \in W^{1,1}(0,T;X)$, $u_0 \in D(A)$ *and* $u_1 \in D(\sqrt{A})$. *Then, the mild solution of the Cauchy problem* (16)-(17) *is a strong solution and belongs to* $C^1([0,T];D(\sqrt{A}))$.

PROOF. First, the mild solution u of the Cauchy problem (16)-(17) is a strong solution thanks to theorem 2 (iv) and [5, Proposition 6.3 (iv)].

Concerning the regularity of $\dot{u}$, since

$$\dot{u}(t) = \dot{S}(t)u_0 + S(t)u_1 + \int_0^t S(t-\tau)f(\tau)d\tau\,, \quad t \in [0,T]\,,$$

we have that the functions $\sqrt{A}\dot{S}(t)u_0$, $\sqrt{A}S(t)u_1$ are continuous , thanks to (v) and (ii) of theorem 2 respectively. To complete the proof, we have to verify that the function $\sqrt{A}\int_0^t S(t-\tau)f(\tau)d\tau$ is continuous. In fact,

$$\sqrt{A}\int_0^t S(\tau)f(t-\tau)d\tau = \sqrt{A}\int_0^t S(\tau)f(0)d\tau + \int_0^t \sqrt{A}\left(\int_0^r S(\tau)d\tau\right)\dot{f}(t-r)\,dr\,,$$

and the statement follows again by theorem 2 (iii). ∎

4. – Semilinear problems

We consider a map $F : D(\sqrt{A}) \to X$ Lipschitz continuous from bounded subsets of $D(\sqrt{A})$ to X, that is for all $R > 0$ there exists a constant $L(R) > 0$ such that for any y_1, $y_2 \in D(\sqrt{A})$, $\|\sqrt{A}y_1\| \le R$, $\|\sqrt{A}y_2\| \le R$, we have

$$\|F(y_1) - F(y_2)\| \le L(R)\ \|\sqrt{A}y_1 - \sqrt{A}y_2\|\,. \tag{19}$$

We now state an existence result for the semilinear equation

$$\ddot{u}(t) + Au(t) + \int_0^t B(t-s)u(s)ds = F(u(t)) + f(t) \qquad t \ge 0\,, \tag{20}$$

Let $T > 0$ be given.

THEOREM 5 *Let $f \in L^1(0,T;X)$, $u_0 \in D(\sqrt{A}))$ and $u_1 \in X$. Then there exists $T_0 \in (0,T]$ and a unique solution $u \in C([0,T_0];D(\sqrt{A}))$ of*

$$u(t) = S(t)u_0 + \int_0^t S(\tau)u_1 d\tau + \int_0^t (1 * S)(t-\tau)[F(u(\tau)) + f(\tau)]d\tau \tag{21}$$

in $[0, T_0]$.

PROOF. Set

$$M = C_T(\|\sqrt{A}u_0\| + \|u_1\| + \|f\|_{1,T})\,, \qquad R = 2M + \|F(0)\|\,,$$

where $C_T > 0$ is the positive constant in (8).

We introduce the space

$$E = \{v \in C([0,T_0];D(\sqrt{A})),\ \|\sqrt{A}v(t)\| \le R \quad \forall t \in [0,T_0]\}\,,$$

where $T_0 \in (0,T]$ is to be found. If we equip E with the distance generated by the norm of $C([0,T_0];D(\sqrt{A}))$, that is

$$d(v_1,v_2) = \max_{0\le t\le T_0} \|\sqrt{A}v_1(t) - \sqrt{A}v_2(t)\|\,,$$

then (E,d) is a complete metric space. Define a nonlinear operator Γ on E by

$$(\Gamma v)(t) = S(t)u_0 + \int_0^t S(\tau)u_1 d\tau + \int_0^t (1 * S)(t-\tau)[F(v(\tau)) + f(\tau)]d\tau\,, \tag{22}$$

for any $v \in E$ and $t \in [0,T_0]$. Clearly, a function $v \in E$ is a solution of (21) in $[0,T_0]$ if and only if v is a fixed point of Γ.

We shall show that Γ maps E into itself and it is a contraction, provided T_0 is sufficiently small.

First, we note that for any $v \in E$ and $\tau \in [0,T_0]$ we have $F(v(\tau)) = F(0) + F(v(\tau)) - F(0)$; by (19) it follows

$$\|F(v(\tau))\| \le \|F(0)\| + RL(R) \le \frac{M + \|F(0)\|}{T_0 C_T}\,,$$

if we take

$$T_0 \leq \frac{M + \|F(0)\|}{C_T(\|F(0)\| + RL(R))}. \tag{23}$$

Then, by (8) we have

$$\|\sqrt{A}(\Gamma v)(t)\| \leq M + C_T \int_0^t \|F(v(\tau))\| \, d\tau \leq M + t\, C_T \frac{M + \|F(0)\|}{T_0 C_T} \leq R,$$

so Γ maps E into itself. Furthermore, again by (8) for any v_1, $v_2 \in E$ we have

$$\begin{aligned} \|\sqrt{A}(\Gamma v_1)(t) - \sqrt{A}(\Gamma v_2)(t)\| &\leq \int_0^t \|\sqrt{A}(1 * S)(t-\tau)[F(v_1(\tau)) - F(v_2(\tau))]\| d\tau \\ &\leq C_T \int_0^t \|F(v_1(\tau)) - F(v_2(\tau))\| d\tau \\ &\leq C_T L(R) \int_0^t \|\sqrt{A} v_1(\tau) - \sqrt{A} v_2(\tau)\| d\tau \\ &\leq T_0\, C_T L(R) d(v_1, v_2) \leq \frac{1}{2} d(v_1, v_2), \end{aligned} \tag{24}$$

if we also take

$$T_0 \leq \frac{1}{2 C_T L(R)}. \tag{25}$$

Therefore, if T_0 satisfies (23) and (25), then Γ is a contraction with Lipschitz constant $1/2$, and so Γ has a fixed point $u \in E$, which is a solution of (21) in $[0, T_0]$. So, the proof is complete. ∎

Theorem 6 *Let $f \in W^{1,1}(0,T;X)$, $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$. Then, a solution u of equation (21) in $[0,T_0]$, $T_0 \in (0,T]$, is a strong solution of equation (20), that is $u \in C^2([0,T_0];X) \cap C([0,T_0];D(A))$ and u verifies (20) in $[0,T_0]$. In addition $u \in C^1([0,T_0];D(\sqrt{A}))$.*

Proof. Let $0 < h < T_0$ and let $t \in [0, T_0 - h]$; taking into account that $\sqrt{A}$ commutes with $S(t)$ (see theorem 2 (ii)), it is easy to check that

$$\begin{aligned} \sqrt{A}u(t+h) - \sqrt{A}u(t) &= \int_t^{t+h} \dot{S}(\tau)\sqrt{A}u_0 \, d\tau + \int_t^{t+h} S(\tau)\sqrt{A}u_1 d\tau \\ &+ \int_0^t \sqrt{A}(1 * S)(\tau) \left(\int_0^h \dot{f}(t+s-\tau)\, ds \right) d\tau \\ &+ \int_0^h \sqrt{A}(1 * S)(t+h-\tau)[f(\tau) + F(u(\tau))] d\tau \\ &+ \int_0^t \sqrt{A}(1 * S)(\tau)[F(u(t+h-\tau)) - F(u(t-\tau))] d\tau. \end{aligned}$$

Set

$$R = \sup_{\tau \in [0,T_0]} \|\sqrt{A}u(\tau)\|,$$

in view of (10), (8) and (19) we have

$$\begin{aligned}\|\sqrt{A}u(t+h)-\sqrt{A}u(t)\| \;\leq\;& h\,C_T\left(\|u_0\|+\|Au_0\|+\|\sqrt{A}u_1\|+\|\dot{f}\|_{1,T}\right)\\ &+h\,C_T(\sup_{\tau\in[0,T]}\|f(\tau)\|+RL(R)+\|F(0)\|)\\ &+C_TL(R)\int_0^t\|\sqrt{A}u(\tau+h)-\sqrt{A}u(\tau)\|d\tau\,.\end{aligned}$$

Applying Gronwall's lemma we get

$$\|\sqrt{A}u(t+h)-\sqrt{A}u(t)\|\leq C\,h\,,\qquad \text{for any }\; 0\leq t<t+h\leq T_0\,,$$

where $C>0$ is a constant independent of t and h; hence, $u:[0,T_0]\to D(\sqrt{A})$ is Lipschitz continuous. It follows that $F(u)$ is also Lipschitz continuous, and consequently $F(u)$ has bounded derivative on $[0,T_0]$. Therefore, if we consider u as the mild solution of the linear equation (16) with non-homogeneous term given by $F(u)+f$, then we can conclude thanks to theorem 4. ∎

5. – An application to viscoelasticity

Let Ω be a bounded open domain in $\mathbb{R}^N$, $N\geq 1$, with smooth boundary $\partial\Omega$. In this section we will consider the following semilinear history value problem, arising in the study of viscoelasticity (see [2, 3, 6])

$$(26)\quad \begin{cases}\dfrac{\partial^2 u}{\partial t^2}(t,\xi)-A_{ij}\dfrac{\partial^2 u}{\partial\xi_i\partial\xi_j}(t,\xi)+\displaystyle\int_{-\infty}^t B_{ij}(t-s)\dfrac{\partial^2 u}{\partial\xi_i\partial\xi_j}(s,\xi)\,ds\\ \qquad\qquad =g(u(t,\xi))+h(t,\xi)\,,\quad t\geq 0\,,\ \xi\in\Omega,\\ u(t,\xi)=v(t,\xi),\quad t\leq 0,\quad \xi\in\Omega,\\ u(t,\xi)=0,\qquad t\in\mathbb{R},\quad \xi\in\partial\Omega.\end{cases}$$

Here $u(t,\xi)$ takes its values in $\mathbb{R}^N$. We employ the summation convention.

We shall assume that for any $i,j=1,2,\dots,N$, $A_{ij}=\{A_{ij}^{kl}\}$ and $B_{ij}(t)=\{B_{ij}^{kl}(t)\}$, $t\geq 0$, are $N\times N$ matrices, such that

$$A_{ij}^{kl}=A_{ji}^{lk}\,,\qquad B_{ij}^{kl}(t)=B_{ji}^{lk}(t)\,,\quad t\geq 0\,;$$

$$\sum_{i,j=1}^N\lambda_i\lambda_j(A_{ij}\eta|\eta)\geq\nu|\eta|^2|\lambda|^2\,,\qquad \text{for any }\;\lambda\,,\eta\in\mathbb{R}^N\qquad(\nu>0)\,;$$

$$B_{ij}^{kl}(\cdot)\in W^{1,1}_{\mathrm{loc}}(0,+\infty;\mathbb{R})\,.$$

Moreover, the functions h and v are given, and $g:\mathbb{R}^N\to\mathbb{R}^N$ is a function of class C^1 satisfying a suitable growth condition, to be specified later.

We can rewrite (26) as an abstract problem of the type (20). Let $X=L^2(\Omega;\mathbb{R}^N)$ be endowed with the usual norm and scalar product, and consider the operators A,

$B(t) : D(A) \subset X \to X$ defined by

$$D(A) = H^2(\Omega; \mathbb{R}^N) \cap H_0^1(\Omega; \mathbb{R}^N)$$

$$(Ay)(\xi) = -A_{ij} \frac{\partial^2 y}{\partial \xi_i \partial \xi_j}(\xi), \qquad \xi \in \Omega,\ y \in D(A),$$

$$(B(t)y)(\xi) = B_{ij}(t) \frac{\partial^2 y}{\partial \xi_i \partial \xi_j}(\xi), \qquad \xi \in \Omega,\ y \in D(A).$$

Notice that A commutes with $B(t)$ and verifies assumption (2) see, e.g., [1] . Moreover, the fractional power $\sqrt{A}$ of A is well defined and

$$D(\sqrt{A}) = H_0^1(\Omega; \mathbb{R}^N).$$

Next, if $N \geq 3$ we assume that g satisfies, for some constant $c_0 > 0$, the growth condition

$$|\nabla g(\eta)| \leq c_0 \left(1 + |\eta|^{\frac{2}{N-2}}\right), \qquad \text{for any } \eta \in \mathbb{R}^N. \tag{27}$$

We observe that no growth condition is required on ∇g in the case of $N = 1$. If $N = 2$, then the exponent $\frac{2}{N-2}$ in (27) can be replaced by any positive number. Then, standard arguments show that the composition operator

$$F(x)(\xi) = g(x(\xi)), \qquad \xi \in \Omega,\ x \in X$$

is well defined from $D(\sqrt{A})$ into X and F fulfils (19).

We note that if the term

$$\int_{-\infty}^{0} B_{ij}(t-s) \frac{\partial^2 v}{\partial \xi_i \partial \xi_j}(s, \xi)\, ds,$$

associated with the past history, is meaningful, then it can be absorbed into the forcing term h. Therefore, under suitable assumptions on v and h, theorems 5 and 6 can be applied to derive similar existence, uniqueness and regularity results for problem (26).

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Phase Transitions and Hysteresis in One-Dimensional Thermo-Plasticity

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The present analysis is concerned with the mathematical modeling of the thermo-mechanical evolution of a visco-plastic material which may undergo solid-solid phase transformations.

On the one hand, we are interested in how the phase transformations at the microscopic level (for instance the possible change of geometric configuration of the crystal lattice) may affect the global thermo-visco-plastic behavior of the material.

On the other hand, we want to describe the effect on some solid-solid phase transition driven by micro- and mesoscopic stresses.

As for an example of an interesting class of materials which strongly exhibit the above described relations between phase changes and stresses one should consider the so called *shape memory materials*. The latter are metallic alloys that can be mechanically deformed (avoiding fractures) and then be forced to recover their original shape just by thermal means. The motivation for such a surprising behavior relies indeed in a structural thermo-stress-driven transformation in the metallic lattices. In particular, the geometric configuration of the crystal lattice of the alloy changes between hi-regularity variants and low-regularity ones. Let us stress that this kind of phenomena are actually present in a large variety of ordinary materials (including steels) although to a smaller extent.

Let us sketch a classical approach to this kind of problems in a one-dimensional setting. Henceforth let ϑ, u, σ, w, and ψ denote absolute temperature, displacement, stress, phase variable (a so called *generalized freezing index* [10]) and the thermodynamic force the drives the phase transition, respectively. Moreover, let $\varepsilon := u_x$ denote the linearized strain.

Hence, we start by introducing a *local free energy* function of the form

$$F(\varepsilon, w, \vartheta) \;:=\; C_V\vartheta(1-\ln\vartheta) \;+\; F_1(\varepsilon, w) \;+\; \vartheta F_2(\varepsilon, w),$$

where $C_V > 0$ stands for the *specific heat* (and will be taken to be 1 without loss of generality) while F_1, F_2 are suitable coupling terms. We remark that the complement of the purely caloric part of F, i.e. $F_1 + \vartheta F_2$, is linear with respect to the temperature. Then, we perform the standard choices

$$(1) \qquad \sigma \;=\; \frac{\partial F}{\partial \varepsilon}(\varepsilon, w, \vartheta), \qquad \psi = \frac{\partial F}{\partial w}(\varepsilon, w, \vartheta),$$

$$(2) \qquad S(\varepsilon, w, \vartheta) \;:=\; -\frac{\partial F}{\partial \vartheta}(\varepsilon, w, \vartheta) \qquad \text{(entropy)},$$

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$$U(\varepsilon, w, \vartheta) \;:=\; (F + \vartheta S)(\varepsilon, w, \vartheta) \qquad \text{(internal energy)}, \tag{3}$$

and we address the coupling of the conservation law for the *momentum*

$$u_{tt} - \tilde{\sigma}_x = f \qquad (\tilde{\sigma} = \sigma + \text{viscous stress } = \sigma + \varepsilon_t), \tag{4}$$

the *balance of internal energy* (where the Fourier law for the heat flux is assumed)

$$U_t - \vartheta_{xx} = \tilde{\sigma} u_{xt} + g, \tag{5}$$

and the *evolution equation* for the phase

$$w_t = -\psi, \tag{6}$$

where f and g are a load and an heat source density, respectively.

Let us come to the second important issue of our modeling, i.e. the possibility of describing hysteresis effects in the evolution of the material. Indeed, it is well known that a strong temperature-dependent hysteretic behavior in the macroscopic stress-strain relation can be observed for a wide class of ordinary materials. On the other hand, phase transition phenomena may be accompanied by strong macroscopic hysteretic effects often related to the thermal and mechanical stresses acting on the micro- and mesoscales.

Unfortunately relations like (1) are not suitable of describing hysteresis. Indeed, even when $F(\cdot, w, \vartheta)$ and $F(\varepsilon, \cdot, \vartheta)$ are *non-convex*, the latter relations may not ensure the occurrence of hysteresis. Moreover, position (1) obviously cannot take into the correct account the inherent memory structures that are responsible for the complicated loopings in the interior of the external hysteresis loops that are observed in the experiments.

In order to avoid this difficulties, Krejčí and Sprekels [8, 9, 10] have recently proposed to incorporate hysteresis directly in the model by replacing (1) by

$$\sigma = \mathcal{H}_1[\varepsilon, w] + \vartheta \mathcal{H}_2[\varepsilon, w], \tag{7}$$
$$\psi = \mathcal{H}_3[\varepsilon, w] + \vartheta \mathcal{H}_4[\varepsilon, w]. \tag{8}$$

If one chooses

$$\mathcal{H}_1[\varepsilon, w] = \frac{\partial F_1}{\partial \varepsilon}(\varepsilon, w), \qquad \mathcal{H}_2[\varepsilon, w] = \frac{\partial F_2}{\partial \varepsilon}(\varepsilon, w), \tag{9}$$
$$\mathcal{H}_3[\varepsilon, w] = \frac{\partial F_1}{\partial w}(\varepsilon, w), \qquad \mathcal{H}_4[\varepsilon, w] = \frac{\partial F_2}{\partial w}(\varepsilon, w), \tag{10}$$

then (7)-(8) turn out to be exactly equivalent to (1). On the contrary, we are not going to assume (9)-(10) and we actually choose $\mathcal{H}_i$ for $i = 1, \ldots, 4$ to be no longer real valued *functions* but *hysteresis operators* acting on functions. In this regard, the use of square brackets will refer to a possible *functional* dependence on the arguments (see below).

The idea of inserting hysteresis directly in the model has been successfully applied to both one-dimensional thermo-plasticity without phase change [6, 7] and multidimensional phase change without mechanical effects [3, 9, 8, 10, 11, 12, 15]. We now address the full coupling of the system.

We shall now briefly recall some basic facts on the notion of *hysteresis operator*, referring indeed to the monographs [2, 4, 5, 16, 17] for the details. Let $T > 0$ denote some reference time. A mapping $\mathcal{H}$ from the set $\mathrm{Map}[0,T] := \{w : [0,T] \to \mathbb{R}\}$ into itself is called a *hysteresis operator* if it is *causal*, that is, if for all $w_1, w_2 \in \mathrm{Map}[0,T]$ and $t \in [0,T]$ we have the implication

$$w_1(\tau) = w_2(\tau) \quad \forall\, \tau \in [0,t] \;\Rightarrow\; \mathcal{H}[w_1](t) = \mathcal{H}[w_2](t),$$

and if it is *rate-independent*, that is, if for every $w \in \mathrm{Map}[0,T]$ and every continuous increasing mapping α of $[0,T]$ onto $[0,T]$ we have

$$\mathcal{H}[w \circ \alpha](t) = \mathcal{H}[w](\alpha(t)) \quad \forall\, t \in [0,T].$$

In the case of partial differential equations, when the input functions not only depend on a time variable $t \in [0,T]$ but also on a space variable x, it is necessary to extend the above notion by simply putting

$$\hat{\mathcal{H}}[w](x,t) := \mathcal{H}[w(x,\cdot)](t)$$

and identify the operators $\mathcal{H}$ and $\hat{\mathcal{H}}$. The hysteresis operators in (7)-(8) have to be understood in this way.

The mathematical treatment of such kind of nonlinearities is often very complicated. Indeed, the input-output behavior of hysteretic nonlinearities often cannot be explicitly described. Moreover, we shall observe that, due to rate independence, an hysteresis operator cannot be expressed in term of a convolution-type integral, i.e. the memory contained in an hysteretic process is not of *fading type*. Finally, hysteresis operators usually show only very restricted smoothness properties and the *chain rule equality* has to be replaced by suitable inequalities.

In particular, for the purposes of our analysis we have to assume the existence of further hysteresis operators $\mathcal{F}_1$ and $\mathcal{F}_2$ such that, for all $(\varepsilon, w) \in (W^{1,1}(0,T))^2$ it holds, for almost every $t \in (0,T)$,

$$\frac{d}{dt}\mathcal{F}_1[\varepsilon,w](t) \;\le\; \mathcal{H}_1[\varepsilon,w](t)\varepsilon_t(t) + \mathcal{H}_3[\varepsilon,w](t)w_t(t), \tag{11}$$

$$\frac{d}{dt}\mathcal{F}_2[\varepsilon,w](t) \;\le\; \mathcal{H}_2[\varepsilon,w](t)\varepsilon_t(t) + \mathcal{H}_4[\varepsilon,w](t)w_t(t). \tag{12}$$

Then, we are led to define the free energy, entropy, and internal energy as the hysteresis operators

$$\mathcal{F}[\varepsilon,w,\vartheta] \;:=\; \vartheta(1-\ln\vartheta) + \mathcal{F}_1[\varepsilon,w] + \vartheta\mathcal{F}_2[\varepsilon,w], \tag{13}$$

$$\mathcal{S}[\varepsilon,w,\vartheta] \;:=\; \ln\vartheta - \mathcal{F}_2[\varepsilon,w], \tag{14}$$

$$\mathcal{U}[\varepsilon,w,\vartheta] \;:=\; \vartheta + \mathcal{F}_1[\varepsilon,w], \tag{15}$$

define σ and ψ as in (7)-(8) and make use of the constitutive equations (4)-(6) in order to obtain the system

$$u_{tt} - u_{xxt} = \sigma_x + f, \tag{16}$$

$$\sigma = \mathcal{H}_1[u_x, w] + \vartheta \mathcal{H}_2[u_x, w], \tag{17}$$

$$(\vartheta + \mathcal{F}_1[u_x, w])_t - \vartheta_{xx} = u_{xt}^2 + \sigma u_{xt} + g(\cdot, \vartheta), \tag{18}$$

$$w_t + \mathcal{H}_3[u_x, w] + \vartheta \mathcal{H}_4[u_x, w] = 0, \tag{19}$$

to be fulfilled almost everywhere in the domain $I \times (0,T)$ where $I := (0,1)$ and $T > 0$.

We are now in the position of complementing the latter system with initial and boundary conditions as

$$u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1,\ \vartheta(\cdot,0) = \vartheta_0,\ w(\cdot,0) = w_0 \quad \text{a.e. in } I, \tag{20}$$

$$u(0,\cdot) = 0,\ u_{xt}(1,\cdot) + \sigma(1,\cdot) = 0,\ \vartheta_x(0,\cdot) = \vartheta_x(1,\cdot) = 0 \quad \text{a.e. in } (0,T), \tag{21}$$

namely our wire is thermally insulated, fixed in 0 and stress-free in 1.

Let us now state our assumptions on data and operators:

(H1) $u_0 \in H^2(I)$, $u_1 \in H^1(I)$, $\vartheta_0 \in H^1(I)$, $w_0 \in H^1(I)$, it holds $\vartheta_0(x) \geq \delta > 0$ for all $x \in \bar{I}$, and the compatibility condition $u_0(0) = u_1(0) = 0$ is satisfied.

(H2) It holds $f \in H^1(0,T; L^2(I))$.

(H3) We assume that $g : I \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$\exists\, g_0 \in L^\infty(I \times (0,T)) : \quad \theta \leq 0 \Rightarrow g(x,t,\theta) = g_0(x,t),$$

$$\exists\, K_1 > 0 : \quad \left|\frac{\partial g}{\partial \theta}\right| \leq K_1 \quad \text{a.e. in } I \times (0,T) \times \mathbb{R},$$

$$g_0(x,t) \geq 0 \quad \text{a.e. in } I \times (0,T).$$

(H4) The operators $\mathcal{H}_j$, $1 \leq j \leq 4$, and $\mathcal{F}_1$ are causal and map $C[0,T] \times C[0,T]$ into $C[0,T]$ and $W^{1,1}(0,T) \times W^{1,1}(0,T)$ into $W^{1,1}(0,T)$. Besides, the following conditions are satisfied:

(i) $\exists\, K_2 > 0 : \forall \varepsilon, w \in C[0,T]$ it holds

$$\max_{j \in \{2,4\}} \|\mathcal{H}_j[\varepsilon, w]\|_\infty \leq K_2, \quad \mathcal{F}_1[\varepsilon, w](t) \geq -K_2 \quad \forall t \in [0,T].$$

(ii) $\exists\, K_3 > 0 : \forall \varepsilon, w \in W^{1,1}(0,T)$ it holds, for a.e. $t \in (0,T)$,

$$\max_{1 \leq j \leq 4} |\mathcal{H}_j[\varepsilon, w]_t(t)| + |\mathcal{F}_1[\varepsilon, w]_t(t)| \leq K_3 \left(|\varepsilon_t(t)| + |w_t(t)|\right).$$

(iii) $\exists\, K_4 > 0 : \forall \varepsilon_1, w_1, \varepsilon_2, w_2 \in C[0,T]$ it holds, for every $t \in [0,T]$,

$$\max_{1 \leq j \leq 4} |\mathcal{H}_j[\varepsilon_1, w_1](t) - \mathcal{H}_j[\varepsilon_2, w_2](t)|$$
$$\leq K_4 \max_{0 \leq r \leq t} \left(|\varepsilon_1(r) - \varepsilon_2(r)| + |w_1(r) - w_2(r)|\right),$$

$$|\mathcal{F}_1[\varepsilon_1, w_1](t) - \mathcal{F}_1[\varepsilon_2, w_2](t)| \leq K_4 \Big[|\varepsilon_1(0) - \varepsilon_2(0)| + |w_1(0) - w_2(0)|$$
$$+ \int_0^t \left(|\varepsilon_{1,t}(r) - \varepsilon_{2,t}(r)| + |w_{1,t}(r) - w_{2,t}(r)|\right) dr\Big].$$

(H5) There exist causal operators $\mathcal{F}_2 : W^{1,1}(0,T)\times W^{1,1}(0,T) \to W^{1,1}(0,T)$, $\mathcal{G} : W^{1,1}(0,T) \to W^{1,1}(0,T)$, and a constant $K_5 > 0$, such that the following conditions are satisfied:

(i) For every $\varepsilon, w \in W^{1,1}(0,T)$ it holds

$$\begin{aligned} \mathcal{F}_1[\varepsilon, w]_t &\leq \varepsilon_t \mathcal{H}_1[\varepsilon, w] + \mathcal{G}[w]_t \mathcal{H}_3[\varepsilon, w] \quad \text{a.e. in } (0,T), \\ \mathcal{F}_2[v, w]_t &\leq \varepsilon_t \mathcal{H}_2[\varepsilon, w] + \mathcal{G}[w]_t \mathcal{H}_4[\varepsilon, w] \quad \text{a.e. in } (0,T). \end{aligned}$$

(ii) For every $w \in W^{1,1}(0,T)$ it holds

$$|\mathcal{G}[w]_t(t)|^2 \leq K_5\, w_t(t)\, \mathcal{G}[w]_t(t) \quad \text{for a.e. } t \in (0,T).$$

A typical example where **(H4)**, **(H5)** are fulfilled is given by *Prandtl-Ishlinskii operators* of the form

$$\begin{aligned} \mathcal{H}_j[\varepsilon, w] &:= \int_0^\infty \varphi_j(r)\, s_r\left[\sigma_r^{0,j}, \varepsilon\right] dr, \quad j = 1, 2, \\ \mathcal{H}_j[\varepsilon, w] &:= \int_0^\infty \varphi_j(r)\, s_r\left[\sigma_r^{0,j}, w\right] dr, \quad j = 3, 4, \end{aligned}$$

where $\sigma_r^{0,j} \in [-r, +r]$, $1 \leq j \leq 4$, are suitable initial values and the weight functions φ_j are non-negative on $[0, +\infty)$ and satisfy

$$\max_{1\leq j\leq 4} \int_0^\infty (1 + r^2)\, \varphi_j(r)\, dr < +\infty.$$

The above operator s_r stands for the so called *stop operator* or *Prandtl's normalized elastic-perfectly plastic element* i.e., for all $\varepsilon \in W^{1,1}(0,T)$ and all $\sigma_r^0 \in [-r, r]$, we define $s_r[\sigma_r^0, \varepsilon] := \sigma_r$ as the solution of the variational inequality

$$\begin{aligned} &\sigma_r(t) \in [-r, r] \quad \text{a.e. in } (0,T), \quad \sigma_r(0) = \sigma_r^0, \\ &(\varepsilon_t(t) - \sigma_{r,t}(t))(\sigma_r(t) - \eta) \geq 0 \quad \forall \eta \in [-r, r], \quad \text{a.e. in } (0,T). \end{aligned}$$

Indeed, defining the (energy) operators

$$\begin{aligned} \mathcal{F}_1[\varepsilon, w] &:= \frac{1}{2} \int_0^\infty \left(\varphi_1(r)\, s_r^2\left[\sigma_r^{0,1}, \varepsilon\right] + \varphi_3(r)\, s_r^2\left[\sigma_r^{0,3}, w\right]\right) dr, \\ \mathcal{F}_2[\varepsilon, w] &:= \frac{1}{2} \int_0^\infty \left(\varphi_2(r)\, s_r^2\left[\sigma_r^{0,2}, \varepsilon\right] + \varphi_4(r)\, s_r^2\left[\sigma_r^{0,4}, w\right]\right) dr, \end{aligned}$$

choosing $\mathcal{G}[w] = w$, we easily verify the validity of **(H4)**, **(H5)**. In particular relations (11)-(12) are exactly **(H5.i)**. For examples where the $\mathcal{H}_j$ are not Prandtl-Ishlinskii operators and $\mathcal{G}$ differs from the identity operator we refer to [10, 11].

We can now formulate a well-posedness result for the problem (16)-(19), (20)-(21).

THEOREM 1 *Suppose that the hypotheses* **(H1)** - **(H5)** *are satisfied. Then the problem (16)-(19), (20)-(21) admits a unique strong solution* (u, θ, w) *such that*

$$\begin{aligned} &u \in H^2(0,T; L^2(I)) \cap H^1(0,T; H^2(I)), \\ &w \in H^2(0,T; L^2(I)) \cap H^1(0,T; H^1(I)), \\ &\theta \in H^1(0,T; L^2(I)) \cap L^2(0,T; H^2(I)). \end{aligned}$$

Besides, with the finite norms $\beta_1 := \|u_{xt}\|_{L^1(0,T;L^\infty(I))}$ *and* $\beta_2 := \|w_t\|_{C(\bar I\times[0,T])}$ *it holds*

$$\theta(x,t) \geq \delta\, e^{-((K_1+K_2K_5\beta_2)t+K_2\beta_1)} \quad \textit{for all } (x,t)\in \bar I\times[0,T]. \tag{22}$$

Of course we will not give the proof of this result here and we refer the interested reader to [13] for the details. The idea of this proof relies indeed in a *local existence – global a priori estimates – passage to the limit* procedure and the key point is the careful use of a variable transformation due to Andrews [1]. The latter transformation forces us to consider the stress-free boundary condition of (21). In particular, let us stress that our analysis cannot extend to the case of zero boundary conditions.

A second well-posedness result may be achieved by means of similar techniques for another model of one-dimensional thermo-visco-plasticity with phase change obtained replacing (16) with

$$u_{tt} - u_{xxt} + u_{xxxx} = \sigma_x + f, \tag{23}$$

namely including an additional curvature term and prescribing the boundary conditions for u

$$u(0,\cdot) = u(1,\cdot) = u_{xx}(0,\cdot) = u_{xx}(1,\cdot) = 0 \quad \text{a.e. in } (0,T).$$

Let us stress that, in this situation, zero boundary conditions can be accepted as well other choices. Omitting the details about regularity and compatibility requirements to the ingredients of the problem, we just stress that an analogous to the latter Theorem holds for the system (17)-(19), (23). The reader is referred to the paper [14]) for the details.

Let us conclude this discussion by remarking the crucial role of **(H5.i)-(H5.ii)** in connection with the thermodynamic consistency of our models. In particular, it is worth noting that, owing to (22), the temperature stays positive for all times. We now aim to prove the consistency with the *Second Principle of Thermodynamics* by obtaining the *Clausius-Duhem inequality* which is in the form (see (5))

$$\vartheta\,\frac{d}{dt}\mathcal{S}[\varepsilon,w,\vartheta] - \frac{d}{dt}\mathcal{U}[\varepsilon,w,\vartheta] \geq -\tilde\sigma\varepsilon_t \quad \text{a.e. in } I\times(0,T),$$

where $\tilde\sigma = \sigma + \varepsilon_t$ is again the total stress. Indeed, we simply compute from (5), (13)-(15), and **(H5.i)-(H5.ii)**

$$\begin{aligned}
\theta\,\mathcal{S}[\varepsilon,w,\theta]_t - \mathcal{U}[\varepsilon,w,\theta]_t + \tilde\sigma\,\varepsilon_t &= -\theta\,\mathcal{F}_2[\varepsilon,w]_t - \mathcal{F}_1[\varepsilon,w]_t + \sigma\,\varepsilon_t + \varepsilon_t^2\\
\geq -\Big(\mathcal{H}_1[\varepsilon,w] + \theta\,\mathcal{H}_2[\varepsilon,w]\Big)\,\varepsilon_t &- \Big(\mathcal{H}_3[\varepsilon,w] + \theta\,\mathcal{H}_4[\varepsilon,w]\Big)\,\mathcal{G}[w]_t + \sigma\,\varepsilon_t + \varepsilon_t^2\\
&\geq \varepsilon_t^2 + w_t\,\mathcal{G}[w]_t \geq 0 \quad \text{a.e. in } I\times(0,T).
\end{aligned}$$

Hence, the Clausius-Duhem inequality is fulfilled. Let us remark that this thermodynamic consistency proof is still valid when referred to the second model containing the additional fourth order curvature term, i.e. relation (23) instead of (16).

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Longterm dynamics of a conserved phase-field system with memory

F. M. Vegni *

1. – Introduction

In the framework of phase-field theories (cf. Brokate-Sprekels [2], and its references), materials which exhibit phase transitions due to temperature variations are usually described by two state variables: the (absolute) temperature field θ and the so-called order parameter (or phase variable) χ accounting for phase changes.

Here we introduce and analyze a phase transition model for a material with thermal memory. We recall that memory (or relaxation) effects in the heat propagation are related, e.g., to certain liquids of high viscosity (see Jäckle [11]).

Our model basically consists of an integrodifferential evolution equation ruling the temperature coupled with a nonlinear fourth-order evolution equation governing the order parameter. This system is phenomenologically derived combining the theory of heat conduction in materials with memory with the Ginzburg-Landau theory for phase transitions. We then associate with the evolution system a set of initial and boundary conditions. The well-posedness of the corresponding initial and boundary value problem as well as the longtime behavior analysis is investigated.

Let $\Omega \subset \mathbb{R}^3$ be a fixed, bounded domain occupied by an isotropic, rigid and homogeneous heat conductor. We consider only small variations of the absolute temperature and its gradient, and we suppose that at each point $x \in \Omega$ and at each time $t \in (\tau, \infty)$, τ being a fixed initial time, the state of the material is described by the triplet $(\vartheta, \chi, \vartheta^t)$. Here, ϑ is the *temperature variation field*, $\vartheta = (\theta - \theta_c)/\theta_c$, where θ_c is the reference temperature at which transition occurs. Moreover, we recall that the *phase variable* χ describes the macroscopic solid-liquid transition, and $\vartheta^t(s) = \vartheta(t-s)$ is the *past history* of ϑ up to time t.

The evolution of the temperature dependent phenomenon is governed by the *energy balance* equation

$$\partial_t e + \nabla \cdot \mathbf{q} = f \tag{1}$$

where e is the *internal energy*, $\mathbf{q}$ is the *heat flux vector*, and f is the external heat supply. Taking into consideration a linearized version of the Coleman-Gurtin theory, we assume that e and $\mathbf{q}$ are described by the following constitutive equations

$$e(x,t) = e_c + c_v\theta_c\vartheta(x,t) + \int_0^\infty a(\sigma)\vartheta^t(x,\sigma)d\sigma + \theta_c\lambda(\chi(x,t)) \tag{2}$$

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$$\mathbf{q}(x,t) \quad = \quad -k\nabla\vartheta(x,t) - \int_0^\infty b(\sigma)\nabla\vartheta^t(x,\sigma)d\sigma \tag{3}$$

for $(x,t) \in \Omega \times \mathbb{R}$. Here a, b are smooth positive functions such that a', a'', b, b' are summable and $a(0) > 0$. The positive constants e_c, c_v, and k are the internal energy at equilibrium, the specific heat and the *instantaneous conductivity*, respectively. As a consequence, the Fourier heat conduction law is recovered when the memory term in (3) is neglected.

Using (2)-(3), the energy balance (1) reads

$$\begin{aligned} &c_v\theta_c\partial_t\vartheta(t) + a(0)\vartheta(t) + \int_0^\infty a'(\sigma)\vartheta(t-\sigma)d\sigma + \theta_c\lambda'(\chi(t))\partial_t\chi(t) \\ &-k\Delta\vartheta(t) - \int_0^\infty b(\sigma)\Delta\vartheta(t-\sigma)d\sigma = f(t) \end{aligned} \tag{4}$$

We couple this equation with a Cahn-Hilliard type equation which rules the phase evolution (see, e.g., Brokate-Sprekels [2], and Novick–Cohen [13] for its justification)

$$\partial_t\chi - \Delta w = 0 \tag{5}$$

where

$$w = -\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda'(\chi)\vartheta$$

represents the so-called chemical potential, being γ a smooth function.

Concerning initial conditions, given at time τ, besides the values of ϑ and χ, it will be necessary to know the past history of ϑ as well

$$\begin{array}{rcll} \vartheta(\tau) & = & \vartheta_0 & \text{in } \Omega \\ \chi(\tau) & = & \chi_0 & \text{in } \Omega \\ \vartheta(\tau - s) & = & \vartheta_0(s) & \text{in } \Omega \times (0,\infty) \end{array}$$

Provided that χ may represent the density of some substance (e.g., a component in an alloy), the total amount of the substance does not change in time, provided that the mass flux through the boundary is null. Then, the most natural boundary condition associated with (5) is the homogeneous Neumann condition for both χ and the chemical potential w

$$\begin{array}{rcll} \partial_{\mathbf{n}}\chi & = & 0 & \text{on } \partial\Omega \times (\tau,\infty) \\ \partial_{\mathbf{n}}w & = & 0 & \text{on } \partial\Omega \times (\tau,\infty) \end{array}$$

Therefore, a formal spatial integration of (5) yields the conservation of $\int_\Omega \chi d\Omega$. Phase-field models displaying this feature are commonly called *conserved models.*

As far as ϑ is concerned, we suppose that the Dirichlet homogeneous boundary condition is satisfied on $\Gamma_0 \subseteq \partial\Omega$, while the adiabatic boundary condition holds on $\Gamma_1 = \partial\Omega \setminus \Gamma_0$

$$\begin{array}{rcll} \vartheta & = & 0 & \text{on } \Gamma_0 \times (\tau,\infty) \\ \mathbf{q}\cdot\mathbf{n} & = & 0 & \text{on } \Gamma_1 \times (\tau,\infty) \end{array}$$

Notice that the latter condition depends on the explicit expression of the heat flux vector, i.e., on the heat conduction law under consideration.

The longterm behavior of the standard (homogeneous) conserved phase-field system (4)-(5) without memory effects has been analyzed in Brochet-Hilhorst-Novick–Cohen [1]. In the hyperbolic case with $a \equiv 0$, well-posedness results with more general nonlinearities are in Colli-Gilardi-Grasselli-Schimperna [4].

Nevertheless, in all the previous papers on models with memory effects, the problem is formulated by incorporating the given past history of the temperature in the heat source term. This approach does not allow to interpret the problem as a dynamical system.

In order to construct a dynamical system associated with our initial and boundary value problem, we need to formulate the problem in a history space setting. Following an idea of Dafermos [5] (cf. also Giorgi-Grasselli-Pata [7]), we introduce an additional variable, the *summed past history* of ϑ

$$\eta^t(x,s) = \int_0^s \vartheta^t(x,y)dy = \int_{t-s}^t \vartheta(x,y)dy$$

and we easily check that η satisfies the first order linear evolution equation

$$\partial_t \eta^t(s) + \partial_s \eta^t(s) = \vartheta(t) \qquad \text{in } \Omega, \quad (t,s) \in (\tau,\infty) \times (0,\infty)$$

along with the initial and boundary conditions

$$\begin{aligned} \eta^\tau &= \eta_0 && \text{in } \Omega \times (0,\infty) \\ \eta^t(0) &= 0 && \text{on } \Omega \times (\tau,\infty) \end{aligned}$$

where

$$\eta_0(x,s) = \int_0^s \vartheta_0(x,y)dy$$

is the *initial summed past history* of ϑ, and the homogeneous boundary condition is a direct consequence of the definition of η.

On account of (4), and making physically reasonable assumptions on the past history and the memory kernels, we observe that a formal integration by parts yields

$$\begin{aligned} \int_0^\infty b(\sigma)\nabla\vartheta^t(\sigma)d\sigma &= -\int_0^\infty b'(\sigma)\nabla\eta^t(\sigma)d\sigma && \text{in } \Omega \times (\tau,\infty) \\ \int_0^\infty a'(\sigma)\vartheta^t(\sigma)d\sigma &= -\int_0^\infty a''(\sigma)\eta^t(\sigma)d\sigma && \text{in } \Omega \times (\tau,\infty) \end{aligned}$$

Then we set

$$\begin{aligned} \mu(s) &= -b'(s) \\ \nu_0\nu(s) &= -a''(s) \end{aligned}$$

for every $s > 0$, where $\nu_0 = 1$ or $\nu_0 = -1$ if a is bounded, nondecreasing, and concave or if a is summable, nonincreasing, and convex, respectively. Both cases are thermodynamically consistent (see Giorgi-Grasselli-Pata [7], and references therein).

For the sake of simplicity, we set all the constants appearing in equation (4), except k, equal to 1. In particular we have $a(0) = 1$, even though it will be clear that one can take $a \equiv 0$, provided that Γ_0 has positive measure. We observe that the introduction of the summed past history as a third state variable, and the related formulation in a history space setting, seems unavoidable if one is interested in a structural stability analysis with respect to the initial data and the heat supply (see [8]).

Then we can formulate the original initial and boundary value problem as follows.
Problem P. Find (ϑ, χ, η) solution to the system

$$\partial_t(\vartheta + \lambda(\chi)) + \vartheta - k\Delta\vartheta + \nu_0 \int_0^\infty \nu(\sigma)\eta(\sigma)d\sigma - \int_0^\infty \mu(\sigma)\Delta\eta(\sigma)d\sigma = f$$

$$\partial_t\chi - \Delta w = 0$$

$$w = -\Delta\chi + \chi^3 + \gamma'(\chi) - \lambda'(\chi)\vartheta$$

$$\partial_t\eta + \partial_s\eta = \vartheta$$

in Ω, for any $t > \tau$, and any $s > 0$, which satisfies the initial conditions

$$\begin{array}{rclll} \vartheta(\tau) & = & \vartheta_0 & \text{in } \Omega \\ \chi(\tau) & = & \chi_0 & \text{in } \Omega & \qquad (6)\\ \eta^\tau & = & \eta_0 & \text{in } \Omega \times (0,\infty) \end{array}$$

and the boundary conditions

$$\begin{array}{rcll} \vartheta & = & 0 & \text{on } \Gamma_0 \times (\tau,\infty) \\ \partial_{\mathbf{n}}\chi & = & 0 & \text{on } \partial\Omega \times (\tau,\infty) \\ \partial_{\mathbf{n}} w & = & 0 & \text{on } \partial\Omega \times (\tau,\infty) \\ \eta & = & 0 & \text{on } \Gamma_0 \times (\tau,\infty) \times (0,\infty) \\ \eta^t(0) & = & 0 & \text{on } \Omega \times (\tau,\infty) \\ k\partial_{\mathbf{n}}\vartheta + \int_0^\infty \mu(\sigma)\partial_{\mathbf{n}}\eta(\sigma)d\sigma & = & 0 & \text{on } \Gamma_1 \times (\tau,\infty) \times (0,\infty) \end{array}$$

Observe that **P** includes two distinct models according to the possible choices of ν_0.

Our goal is to analyze the well-posedness of problem **P** and the longtime behavior of its solutions. More precisely, we prove some continuous dependence and uniqueness result, which allow to interpret **P** as a dynamical system. Then, we show the dissipativity of the dynamical system by constructing an absorbing set which is uniform with respect to f. Finally, we prove the existence of a uniform attractor of finite fractal dimension.

This analysis extends to the so-called conserved case characterized by (5) the results obtained in Giorgi-Grasselli-Pata [7] for the nonconserved case, where (5) is substituted with

$$\partial_t\chi + w = 0$$

2. – The existence and uniqueness

We refer to [9]. With standard notation, we consider two functions ν and μ (memory kernels), which satisfy the assumptions:

(k1) $\quad \nu(\sigma),\ \mu(\sigma) \in C^1(0,\infty) \cap L^1(0,\infty)$

(k2) $\quad \nu(\sigma) \geq 0,\ \mu(\sigma) \geq 0 \qquad \forall\, \sigma \in (0,\infty)$

(k3) $\quad \nu'(\sigma) \leq 0,\ \mu'(\sigma) \leq 0 \qquad \forall\, \sigma \in (0,\infty)$

About initial data, we consider

$$\begin{aligned} \vartheta_0 \in H &\equiv L^2(\Omega) \\ \chi_0 \in V &\equiv H^1_{\Gamma_0}(\Omega) \\ \eta_0 \in \mathcal{M} &\equiv L^2_\nu(0,\infty; L^2(\Omega)) \cap L^2_\mu(0,\infty; H^1_{\Gamma_0}(\Omega)) \end{aligned}$$

The nonlinearities involved in the problem are taken

$$\gamma \in C^2(\mathbb{R}) \text{ with } \gamma',\ \gamma'' \in L^\infty(\mathbb{R}) \tag{7}$$

and

$$\lambda \in C^2(\mathbb{R}) \text{ with } \lambda'' \in L^\infty(\mathbb{R}) \tag{8}$$

$$f \in L^1_{\text{loc}}(\mathbb{R}; H) + L^2_{\text{loc}}(\mathbb{R}; V^*) \tag{9}$$

Then, problem **P** has a solution which is valued in the space $H \times V \times \mathcal{M}$ where initial data are set. We make use of a Faedo-Galerkin approximating scheme to project problem **P** on a finite dimensional space, where we can locally solve a Cauchy problem for a system of ordinary differential equations, having local solution, with standard techniques. Then, we provide some a priori estimates which guarantee that any local solution to the problem is actually global. Finally we prove that the approximating solutions converges to the actual solution to problem **P**, exploiting weak convergence properties in Banach spaces.

We prove continuous dependence estimates for solutions to problems **P** (uniqueness straightforwardly follows from these results). Some restrictions on λ and γ are needed. More precisely, when (8) is replaced by

$$\lambda \in C^2(\mathbb{R}) \quad \text{with} \quad \lambda', \lambda'' \in L^\infty(\mathbb{R}) \tag{10}$$

we can obtain a first continuous dependence results. A stronger one holds, when we take

$$\lambda(r) = \lambda_0 r \qquad \forall\, r \in \mathbb{R} \tag{11}$$

$$\gamma \in C^3(\mathbb{R}) \quad \text{with} \quad \gamma',\ \gamma'',\ \gamma''' \in L^\infty(\mathbb{R}) \tag{12}$$

in place of (10) and (7), respectively. In this case the solution is such that

$$\theta \in C(\tau, \infty; H) \quad \chi \in C(\tau, \infty; V) \quad \eta \in C(\tau, \infty; \mathcal{M})$$

The proof is given exploiting the completeness properties of Banach spaces, after having constructed some a priori estimates on the difference of two solutions with different initial data.

3. – The longtime behavior

The dissipative nature of a system is evident if the set of its solutions, depending on initial data, possesses an absorbing set, i.e. a set into which all the orbits corresponding to different initial data eventually enter. Referring to [10, 14] for a deeper insight on properties of dynamical systems, we recall that, given a metric space X, a two-parameter family of operators

$$W(t,\tau) : X \longrightarrow X \qquad \tau \in \mathbb{R},\ t \geq \tau$$

is a (*strongly continuous*) *process*, or a *dynamical system*, on X if the following properties are satisfied:

$$\begin{array}{lll} \text{(i)} & W(\tau,\tau) = \mathbb{I} & \forall\, \tau \in \mathbb{R} \\ \text{(ii)} & W(t,\tau) = W(t,s)W(s,\tau) & \forall\, \tau \in \mathbb{R}, \quad t \geq s \geq \tau \\ \text{(iii)} & W(t,\tau)z_0 \in C([\tau,\infty); X) & \forall\, \tau \in \mathbb{R}, \quad \forall\, z_0 \in X \\ \text{(iv)} & W(t,\tau) \in C(X;X) & \forall\, \tau \in \mathbb{R}, \quad t \geq \tau \end{array}$$

The process is the natural extension of the concept of semigroup.

It is useful to highlight the dependence on a functional symbol $\varphi \in \mathcal{F}$, where the *symbol space* $\mathcal{F}$ is a complete metric space, and consider a family of processes

$$\{W_\varphi(t,\tau), \varphi \in \mathcal{F}\}$$

rather than a single process. The symbol commonly models an external force. A set $\mathcal{B}_0 \subset X$ is a *uniform absorbing set* (with respect to $\varphi \in \mathcal{F}$) for the family $\{W_\varphi(t,\tau),\ \varphi \in \mathcal{F}\}$ if for any bounded set $\mathcal{B} \subset X$ there exists $t_{\mathcal{B}} \geq 0$ such that

$$\bigcup_{\varphi \in \mathcal{F}} W_\varphi(t,\tau)\mathcal{B} \subset \mathcal{B}_0$$

for every $\tau \in \mathbb{R}$ and every $t \geq \tau + t_{\mathcal{B}}$.

3.1. – *The process associated with problem P.* – We introduce the product Hilbert space

$$\mathcal{H} = H \times V \times \mathcal{M}$$

For every $\alpha \geq 0$, we also introduce the subset $\mathcal{H}_\alpha$ of $\mathcal{H}$ defined by

$$\mathcal{H}_\alpha = \left\{ (\vartheta, \chi, \eta) \in \mathcal{H} : \left| \int_\Omega \chi d\Omega \right| \leq \alpha \right\}$$

Notice that $\mathcal{H}_\alpha$ is a complete metric space with respect to the metric induced by the norm of $\mathcal{H}$. We agree to denote by

$$U_f(t,\tau)(\vartheta_0, \chi_0, \eta_0) = (\vartheta(t), \chi(t), \eta^t)$$

the solution to **P** at time t, with initial data $(\vartheta_0, \chi_0, \eta_0)$ given at time $\tau \leq t$, and forcing term f. Since we are working with a phase-field system which conserves the phase variable, $(\vartheta_0, \chi_0, \eta_0) \in \mathcal{H}_\alpha$ implies $U_f(t,\tau)(\vartheta_0, \chi_0, \eta_0) \in \mathcal{H}_\alpha$.

If we let (k1)-(k3), (7), (9) and (10) hold, then the two-parameter family $U_f(t,\tau)$, acting either on $\mathcal{H}_\alpha$ or on $\mathcal{H}$, fulfills (i) and (ii). When λ is linear, and γ'' is lipschitz, we can also ensure that the continuity properties (iii) and (iv) are satisfied. Then, if we let (k1)-(k3), (9), (11) and (12) hold, the two-parameter family $U_f(t,\tau)$ defines a process, acting either on $\mathcal{H}_\alpha$ or on $\mathcal{H}$.

3.2. – *The absorbing set.* – The asymptotic behavior of solutions to problem **P** strongly depends on the decay properties of the memory kernels, and, in particular, on the choice of the parameter ν_0, i.e., on a (cf. Introduction of [7]). To prove the existence of an absorbing set, we consider first the case $\nu_0 = 1$ and we require that the memory kernels ν and μ satisfy, besides (k1)-(k3), the following *exponential decay* condition

$$\text{(k4)} \qquad \nu'(\sigma) + \delta\nu(\sigma) \leq 0 \qquad \mu'(\sigma) + \delta\mu(\sigma) \leq 0 \qquad \forall\, \sigma \in \mathbb{R}^+$$

for some $\delta > 0$. We point out that the exponential decay of the memory kernels is a standard assumption, even in the stability analysis of linear integrodifferential systems (see, e.g., [12]). To deal with the case $\nu_0 = -1$, we need to assume $|\Gamma_0| > 0$ in order to take advantage of Poincaré inequality. Furthermore, besides a decay condition for μ, we require ν to be suitably dominated by μ, that is,

$$\text{(k5)} \qquad \nu(\sigma) \leq \frac{\delta_0}{8a_0 C_P^2}\mu(\sigma) \qquad \mu'(\sigma) + \delta\mu(\sigma) \leq 0 \qquad \forall\, \sigma \in \mathbb{R}^+$$

for some $0 < \delta_0 < \delta$. We remark that, if (k5) holds, we can take $\mathcal{M} \equiv L^2_\mu(\mathbb{R}^+; V)$.

We need to introduce the space of L^p-translation bounded functions, valued in the Banach space X. We denote this space $\mathcal{T}^p(X)$, where we use the norm

$$\|f\|_{\mathcal{T}^p(X)} = \sup_{r\in\mathbb{R}} \left(\int_r^{r+1} \|f(\xi)\|_X^p d\xi \right)^{1/p}$$

for every $f \in L^p_{\mathrm{loc}}(R; X)$.

We assume (7), (10), (k1)-(k3) and either (k4) [if $\nu_0 = 1$] or (k5) [if $|\Gamma_0| > 0$ and $\nu_0 = -1$]. Let $\mathcal{F} \subset \mathcal{T}^1(H) + \mathcal{T}^2(V^*)$ be a bounded set, and denote

$$F = \sup_{f\in\mathcal{F}} \|f\|_{\mathcal{T}^1(H)+\mathcal{T}^2(V^*)}$$

Then, for any fixed $\alpha \geq 0$, there exists $R_0 = R_0(F) > 0$ such that, given any $R > 0$ and initial data

$$\text{(13)} \qquad (\vartheta_0, \chi_0, \eta_0) \in \mathcal{H}_\alpha$$

satisfying $\|(\vartheta_0, \chi_0, \eta_0)\|_{\mathcal{H}} \leq R$, there is $t_R = t_R(F) \geq 0$ such that

$$\text{(14)} \qquad \sup_{f\in\mathcal{F}} \|U_f(t,\tau)(\vartheta_0, \chi_0, \eta_0)\|_{\mathcal{H}} \leq R_0$$

for every $\tau \in \mathbb{R}$ and almost every $t \geq \tau + t_R$. If (11)-(12) are also assumed, then any ball of $\mathcal{H}_\alpha$ centered at zero of radius strictly greater than R_0 is a uniform absorbing set for the family of processes $\{U_f(t,\tau),\ f \in \mathcal{F}\}$ acting on $\mathcal{H}_\alpha$.

We prove the existence of the absorbing set by constructing some sharp and refined inequality concerning the behavior of the energy of the system. A Gronwall type lemma leads to (14). We point out that (13) is crucial, since the conservativity of the system.

3.3. – *The attracting set.* – Longtime behavior of a single solution to a well-posed initial value problem is often described by the so-called ω-limit set, i.e., a set to which the orbit converges as $t \to \infty$. We show that such sets exist for $\mathbf{P}$, and that they are part of a bigger compact invariant set $\mathcal{A}$ which attracts *all* orbits.

A set $\mathcal{K} \subset X$ is said to be *uniformly attracting* for $\{W_\varphi(t,\tau), \varphi \in \mathcal{F}\}$ if, for any $\tau \in \mathbb{R}$ and any bounded set $\mathcal{B}$ in X,

$$\lim_{t\to\infty}\left[\sup_{\varphi\in\mathcal{F}} \delta_X(W_\varphi(t,\tau)\mathcal{B},\mathcal{K})\right] = 0$$

where

$$\delta_X(\mathcal{B}_1,\mathcal{B}_2) = \sup_{z_1\in\mathcal{B}_1}\inf_{z_2\in\mathcal{B}_2} \operatorname{dist}_X(z_1,z_2)$$

is the Hausdorff semidistance of two sets $\mathcal{B}_1$ and $\mathcal{B}_2$ in X.

A family of processes that possesses a uniformly attracting compact set is said to be *uniformly asymptotically compact.*

A closed set $\mathcal{A} \subset X$ is said to be a *uniform attractor* for the family $\{W_\varphi(t,\tau), \varphi \in \mathcal{F}\}$ if it is at the same time uniformly attracting and contained in every closed uniformly attracting set.

We recall also that a curve $z(t) \in X$, $t \in \mathbb{R}$, is a *complete trajectory* of a process $W(t,\tau)$ if the identity

$$W(t,\tau)z(\tau) = z(t)$$

holds for every $\tau \in \mathbb{R}$ and every $t \geq \tau$ and that we say that the *kernel* $\mathrm{Ker}[W]$ of a process $W(t,\tau)$ consists of all bounded complete trajectories of the process. The set

$$\mathrm{Ker}[W;s] = \Big\{z(s) : z(\cdot) \in \mathrm{Ker}[W]\Big\}$$

is the *kernel section* at time $t = s$, $s \in \mathbb{R}$.

We exploit the following fundamental abstract result, which gives sufficient conditions to existence of an attracting set and its characterization. It appears as Theorem 3.2 in [3].

- *Theorem.* - Let $\mathcal{F}$ be compact, $\{W_\varphi(t,\tau), \varphi \in \mathcal{F}\}$ be uniformly asymptotically compact, and $W_\bullet(t,\tau)$ be continuous as a map from $X \times \mathcal{F}$ to X for every $\tau \in \mathbb{R}$ and every $t \geq \tau$. In addition, assume that there exists a semigroup $T(t)$ on $\mathcal{F}$ that satisfies the translation identity

$$W_\varphi(t+s,\tau+s) = W_{T(s)\varphi}(t,\tau) \tag{15}$$

for every $\varphi \in \mathcal{F}$, $\tau \in \mathbb{R}$, $s \geq 0$. Then the semigroup $\Sigma(t)$ on $X \times \mathcal{F}$, defined as

$$\Sigma(t)(z,\phi) = (W_\varphi(t,0), T(t)\varphi) \qquad (z,\varphi) \in X \times \mathcal{F}$$

possesses a compact global attractor $\tilde{\mathcal{A}}$, which is fully invariant with respect to $\Sigma(t)$, that is,

$$\Sigma(t)\tilde{\mathcal{A}} = \tilde{\mathcal{A}}$$

for every $t \geq 0$. Moreover, the following properties hold:

1. The projection on the first component $\Pi_1\tilde{\mathcal{A}} = \mathcal{A}$ is the uniform attractor of $\{W_\varphi(t,\tau), \varphi \in \mathcal{F}\}$ on X;

2. The projection on the second component $\Pi_2\tilde{\mathcal{A}} = \mathcal{A}_{\mathcal{F}}$ is the global attractor of $T(t)$ on $\mathcal{F}$;

3. $\tilde{\mathcal{A}} = \bigcup_{\varphi\in\mathcal{A}_{\mathcal{F}}} \Big(\mathrm{Ker}[W_\varphi;0] \times \{\varphi\}\Big)$. □

In order to exploit this result, we consider a translation compact function

$$g \in L^1_{\mathrm{loc}}(\mathbb{R}; H) \tag{16}$$

We recall that a function $g \in L^p_{\mathrm{loc}}(\mathbb{R}; X)$ is said to be *translation compact* in $L^p_{\mathrm{loc}}(\mathbb{R}; X)$ if the set $\{g^r\}_{r\in\mathbb{R}}$ of translates of g, where $g^r(\cdot) = g(\cdot + r)$, is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}; X)$. The set

$$\mathrm{H}(g) = \overline{\{g^r, r \in \mathbb{R}\}}$$

(the closure in $L^p_{\mathrm{loc}}(\mathbb{R}; X)$) is called the *hull* of g. Then, the hull $\mathrm{H}(g)$ is a compact metric space, on which it is naturally defined a strongly continuous semigroup, namely the translation semigroup $T(t)$

$$T(t)f = f^t \qquad \forall\, f \in \mathrm{H}(g)$$

This semigroup satisfies the translation equality (15). Notice that the space $\mathrm{H}(g)$ is compact by construction. In particular, the global attractor of $\mathrm{H}(g)$ coincide with $\mathrm{H}(g)$ itself.

In the sequel, we assume (11), (12), (k1)-(k3) and either (k4) [if $\nu_0 = 1$] or (k5) [if $|\Gamma_0| > 0$ and $\nu_0 = -1$]. For any given $\alpha \geq 0$, we study the asymptotic behavior of the family of processes

$$\{U_f(t,\tau), f \in \mathrm{H}(g)\}$$

acting on $\mathcal{H}_\alpha$, generated by the solutions to problem **P**.

To exploit theorem above, and to prove that $\{U_f(t,\tau), f \in \mathrm{H}(g)\}$ is uniformly asymptotically compact, we need to decompose the solution to **P** in three parts: the first vanishing at infinity, and the other two belonging to a compact set.

We introduce the triplets

$$\begin{aligned} z_0 &= (\vartheta_0, \chi_0, \eta_0) \\ z(t) &= (\vartheta(t), \chi(t), \eta(t)) = U_f(t,\tau) z_0 \end{aligned}$$

with

$$z_0 \in \mathcal{B}_0 \subset \mathcal{H}_\alpha$$

where $\mathcal{B}_0$ is a bounded, uniform absorbing set, whose existence has been proved. Observe that $z(t) \in \mathcal{H}_0$ whenever $z_0 \in \mathcal{H}_0$. We now decompose the solution to problem **P** as

$$z = z_L + z_E + z_N$$

where z_L is the solutions to the system

$$\begin{aligned}
\partial_t \vartheta_L &= -\vartheta_L - \lambda_0 \partial_t \chi_L + k\Delta\vartheta_L - \nu_0 \int_0^\infty \nu(\sigma)\eta_L(\sigma)d\sigma + \int_0^\infty \mu(\sigma)\Delta\eta_L(\sigma)d\sigma \\
\partial_t \chi_L &= \Delta(-\Delta\chi_L - \lambda_0\vartheta_L) \\
\partial_t \eta_L &= -\partial_s \eta_L + \vartheta_L \\
z_L(\tau) &= (\vartheta_0, \chi_0 - m_{\chi_0}, \eta_0)
\end{aligned}$$

The linearity of this system allows to conclude that $z_L(t)$ vanishes at infinity in $\mathcal{H}$. Instead, $z_E(t)$ is the solution to the system

$$\begin{aligned}
\partial_t \vartheta_E &= -\vartheta_E - \lambda_0 \partial_t \chi_E + k\Delta\vartheta_E - \nu_0 \int_0^\infty \nu(\sigma)\eta_E(\sigma)d\sigma + \int_0^\infty \mu(\sigma)\Delta\eta_E(\sigma)d\sigma + f \\
\partial_t \chi_E &= \Delta(-\Delta\chi_E - \lambda_0\vartheta_E) \\
\partial_t \eta_E &= -\partial_s \eta_E + \vartheta_E \\
z_E(\tau) &= (0,0,0)
\end{aligned}$$

and we can show that there exists a relatively compact set $\mathcal{K}_E \subset \mathcal{H}_0$ such that

$$\bigcup_{f\in \mathrm{H}(g)} \bigcup_{z_0\in\mathcal{B}_0} \bigcup_{\tau\in\mathbb{R}} \bigcup_{t\geq\tau} z_E(t) \subset \mathcal{K}_E$$

Finally, z_N is the solution to the system

$$\begin{aligned}
\partial_t \vartheta_N &= -\vartheta_N - \lambda_0 \partial_t \chi_N + k\Delta\vartheta_N - \nu_0 \int_0^\infty \nu(\sigma)\eta_N(\sigma)d\sigma + \int_0^\infty \mu(\sigma)\Delta\eta_N(\sigma)d\sigma \\
\partial_t \chi_N &= \Delta(-\Delta\chi_N + \chi^3 + \gamma'(\chi) - \lambda_0\vartheta_N) \\
\partial_t \eta_N &= -\partial_s \eta_N + \vartheta_N \\
z_N(\tau) &= (0, m_{\chi_0}, 0)
\end{aligned}$$

Along the lines of Lemma 7.7 in [6], we prove that $z_N(t)$ belongs to a compact, bounded set $\mathcal{K}_N$ in $\mathcal{H}$, depending only on $\sup_{z_0\in\mathcal{B}_0}\|z_0\|_{\mathcal{H}}$ and on g.

Existence and uniqueness theorems ensure that the map

$$U_\bullet(t,\tau) : \mathcal{H}_\alpha \times \mathrm{H}(g) \longrightarrow \mathcal{H}_\alpha$$

is continuous. Therefore, as a consequence of all the previous results, for every $\alpha \geq 0$ the family of processes $\{U_f(t,\tau), f \in \mathrm{H}(g)\}$ on $\mathcal{H}_\alpha$ has a compact (and connected) uniform attractor given by

$$\mathcal{A} = \mathcal{A}(\alpha) = \bigcup_{f\in\mathrm{H}(g)} \mathrm{Ker}[U_f; 0]$$

If we consider the more restrictive condition

$$g \in L^2_{\mathrm{loc}}(\mathbb{R}^+; H)$$

in place of (16), we can proceed with a double decomposition of the solution $z = z_L + z_N$, where the external source is now included in the nonlinear part, and the related results are obtained as well. It follows that the boundedness properties of $\mathcal{K}_N$ are reflected on $\mathcal{A}$.

3.4. – *The finite dimension of the attractor.* – To prove that the uniform attractor of the family of processes $\{U_f(t,\tau)\}$ has finite fractal and Hausdorff dimensions is of some importance for numerical simulations. This fact implies that the longterm system dynamics can be described by a finite number of parameters; in fact (we refer to [14]), if a subset $\mathcal{X}$ of the phase space has Hausdorff dimension less than or equal to N, then almost all projections of dimension $2N+1$ are injective on $\mathcal{X}$. If we assume that $\mathcal{X}$ be a subset of a metric space X, the *Hausdorff dimension* of $\mathcal{X}$ is

$$\dim_{\mathrm{H}}\mathcal{X} = \sup\Big\{\delta > 0 \ : \ \sup_{\varepsilon>0}\inf_{C_\varepsilon}\sum_{i\in J} r_i^\delta < \infty\Big\}$$

where $C_\varepsilon = \{\mathcal{B}_i(r_i)\}_{i\in J}$ is a covering of $\mathcal{X}$ of balls of radii $r_i < \varepsilon$.

The *fractal dimension* of $\mathcal{X}$ is

$$\dim_{\mathrm{F}}\mathcal{X} = \sup\Big\{\delta > 0 \ : \ \limsup_{\varepsilon\to0^+}\varepsilon^\delta n_{\mathcal{X}}(\varepsilon) < \infty\Big\}$$

where $n_{\mathcal{X}}(\varepsilon)$ is the minimum number of balls of radii ε which is necessary to cover $\mathcal{X}$.

It is straightforward from the definitions that

$$\dim_{\mathrm{H}}\mathcal{X} \le \dim_{\mathrm{F}}\mathcal{X}$$

We suppose now $f \in H$, i.e. constant in time. Thus, we are dealing with an autonomous dynamical system, and $U_f(t,0)$ reduces to a strongly continuous semigroup acting either on $\mathcal{H}_\alpha$ or on $\mathcal{H}$. We can prove he attractor $\mathcal{A}$ of the semigroup $U_f(t,0)$ has finite fractal (and Hausdorff) dimension.

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