CHAPTER 5

Number Theory plays an important role in computer science, medical research, and satellite technology.

NUMBER THEORY AND THE REAL NUMBER SYSTEM

It is impossible to live in our modern world without encountering numbers on a regular basis. In addition to playing a role in our everyday lives, numbers are used to describe the natural world, to communicate vast quantities of information, and to model problems facing scientists and researchers. *Number theory,* the study of numbers and their properties, makes all these roles possible. Mathematicians and computer scientists use number theory extensively to improve the speed of computers in our homes, businesses, and schools. Research scientists use number theory along with another branch of mathematics known as ***knot theory*** to conduct DNA research, research new drugs, and study how infectious diseases spread. Engineers use number theory in satellite technology which is used in virtually every modern means of communication.

5.1 NUMBER THEORY

This chapter introduces ***number theory***, the study of numbers and their properties. The numbers we use to count are called the ***counting numbers*** or ***natural numbers***. Since we begin counting with the number 1, the set of natural numbers begins with 1. The set of natural numbers is frequently denoted by N:

$$N = \{1, 2, 3, 4, 5, \dots\}$$

Any natural number can be expressed as a product of two or more natural numbers. For example, $8 = 2 \times 4$, $16 = 4 \times 4$, and $19 = 1 \times 19$. The natural numbers that are multiplied together are called factors of the product. For example,

$$\underset{\uparrow}{2} \times \underset{\uparrow}{4} = 8$$

Factors

A natural number may have many factors. For example, what pairs of numbers have a product of 18?

$$1 \cdot 18 = 18$$
$$2 \cdot 9 = 18$$
$$3 \cdot 6 = 18$$

The numbers 1, 2, 3, 6, 9, and 18 are all factors of 18. Each of these numbers divides 18 without a remainder.

If a and b are natural numbers, we say that a is a ***divisor*** of b or a ***divides*** b, symbolized $a \mid b$, if the quotient of b divided by a has a remainder of 0. If a divides b, then b is ***divisible*** by a. For example, 4 divides 12, symbolized $4 \mid 12$, since the quotient of 12 divided by 4 has a remainder of 0. Note that 12 is divisible by 4. The notation $7 \nmid 12$ means that 7 does not divide 12. Note that every factor of a natural number is also a divisor of the natural number. *Caution:* Do not confuse the symbols $a \mid b$ and a / b; $a \mid b$ means "a divides b" and a / b means "a divided by b" $(a \div b)$. The symbols a / b and $a \div b$ indicate that the operation of division is to be performed, and b may or may not be a divisor of a.

Prime and Composite Numbers

Every natural number greater than 1 can be classified as either a prime number or a composite number.

> A **prime number** is a natural number greater than 1 that has exactly two factors (or divisors), itself and 1.

The number 5 is a prime number because it is divisible only by the factors 1 and 5. The first eight prime numbers are 2, 3, 5, 7, 11, 13, 17, and 19. The number 2 is the only even prime number. All other even numbers have at least three divisors: 1, 2, and the number itself.

> A **composite number** is a natural number that is divisible by a number other than itself and 1.

Any natural number greater than 1 that is not prime is composite. The first eight composite numbers are 4, 6, 8, 9, 10, 12, 14, and 15.

The number 1 is neither prime nor composite; it is called a ***unit***. The number 38 has at least three divisors, 1, 2, and 38, and hence is a composite number. In contrast, the number 23 is a prime number since its only divisors are 1 and 23.

More than 2000 years ago, the ancient Greeks developed a technique for determining which numbers are prime numbers and which are not. This technique is named the ***sieve of Eratosthenes***, for the Greek mathematician Eratosthenes of Cyrene who first used it.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~

Figure 5.1

To find the prime numbers less than or equal to any natural number, say, 50, using this method, list the first 50 counting numbers (Fig. 5.1). Cross out 1 since it is not a prime number. Circle 2, the first prime number. Then cross out all the multiples of 2: 4, 6, 8, . . . , 50. Circle the next prime number, 3. Cross out all multiples of 3 that are not already crossed out. Continue this process until you reach the prime number p, such that $p \cdot p$, or p^2, is greater than the last number listed, in this case 50. Therefore, we next circle 5 and cross out its multiples. Then circle 7 and cross out its multiples. The next prime number is 11, and $11 \cdot 11$, or 121, is greater than 50, so you are done. At this point, circle all the remaining numbers to obtain the prime numbers less than or equal to 50. The prime numbers less than or equal to 50 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47.

Now we turn our attention to composite numbers and their factors. The rules of divisibility given in the chart on page 210 are helpful in finding divisors (or factors) of composite numbers.

The test for divisibility by 6 is a particular case of the general statement that the product of two prime divisors of a number is a divisor of the number. Thus, for example, if both 3 and 7 divide a number, then 21 will also divide the number.

Note that the chart does not list rules of divisibility for the number 7. There is a rule for 7, but it is difficult to remember. The easiest way to check divisibility by 7 is just to perform the division.

PROFILE IN MATHEMATICS

ERATOSTHENES OF CYRENE

Eratosthenes of Cyrene (275–195 B.C.) was born in northern Africa near the present-day city of Shahhat, Libya. Eratosthenes is best known for being the first to estimate accurately the diameter of Earth. He is also credited for developing a method of finding prime numbers known as the **sieve of Eratosthenes**. Although he is most known for his work in mathematics, Eratosthenes also was influential in the fields of history, geography, and astronomy. In addition, Eratosthenes served for many years as the director of the famous library in Alexandria, Egypt. Although Eratosthenes was a highly regarded scholar throughout the ancient world, only fragments of his writing remain today. Eratosthenes was near 80 years old when, after going blind, he died from voluntary starvation.

DID YOU KNOW

No More Odd Days!

The date November 19, 1999, was a very special day. It was a rare "odd day." The numerical format is 11-19-1999 (or some may write it as 19-11-1999), which contains only odd digits. The next odd day will be January 1, 3111, which would be written 1-1-3111, and is over a thousand years away—a date we certainly will never see.

Days such as 8-27-2002 have both odd and even digits. Thus, it is neither an odd day nor an even day. The first of many even days in the year 2000 was 2-2-2000, the first one since 8-28-888.

So now you have a reason to celebrate, since you have seen your last odd day on Earth!

Rules of Divisibility

Divisible by	Test	Example
2	The number is even.	924 is divisible by 2 since 924 is even.
3	The sum of the digits of the number is divisible by 3.	924 is divisible by 3 since the sum of the digits, $9 + 2 + 4 = 15$, and 15 is divisible by 3.
4	The number formed by the last two digits of the number is divisible by 4.	924 is divisible by 4 since the number formed by the last two digits, 24, is divisible by 4.
5	The number ends in 0 or 5.	265 is divisible by 5 since the number ends in 5.
6	The number is divisible by both 2 and 3.	924 is divisible by 6 since it is divisible by both 2 and 3.
8	The number formed by the last three digits of the number is divisible by 8.	5824 is divisible by 8 since the number formed by the last three digits, 824, is divisible by 8.
9	The sum of the digits of the number is divisible by 9.	837 is divisible by 9 since the sum of the digits, 18, is divisible by 9.
10	The number ends in 0.	290 is divisible by 10 since the number ends in 0.

EXAMPLE 1 *Using the Divisibility Rules*

Determine whether 145,860 is divisible by

a) 2 b) 3 c) 4 d) 5 e) 6 f) 8 g) 9 h) 10

SOLUTION:

a) Since 145,860 is even, it is divisible by 2.

b) The sum of the digits of 145,860 is $1 + 4 + 5 + 8 + 6 + 0 = 24$. Since 24 is divisible by 3, the number 145,860 is divisible by 3.

c) The number formed by the last two digits is 60. Since 60 is divisible by 4, the number 145,860 is divisible by 4.

d) Since 145,860 has 0 as the last digit, 145,860 is divisible by 5.

e) Since 145,860 is divisible by both 2 and 3, 145,860 is divisible by 6.

f) The number formed by the last three digits is 860. Since 860 is not divisible by 8, the number 145,860 is not divisible by 8.

g) The sum of the digits of 145,860 is 24. Since 24 is not divisible by 9, the number 145,860 is not divisible by 9.

h) Since 145,860 has 0 as the last digit, 145,860 is divisible by 10.

Every composite number can be expressed as a product of prime numbers. The process of breaking a given number down into a product of prime numbers is called ***prime factorization***. The prime factorization of 18 is $3 \times 3 \times 2$. No other natural number listed as a product of primes will have the same prime factorization as 18. The *fundamental theorem of arithmetic* states this concept formally. (A ***theorem*** is a statement or proposition that can be proven true.)

The Fundamental Theorem of Arithmetic
Every composite number can be expressed as a *unique* product of prime numbers.

In writing the prime factorization of a number, the order of the factors is immaterial. For example, we may write the prime factors of 18 as $3 \times 3 \times 2$ or $2 \times 3 \times 3$ or $3 \times 2 \times 3$.

A number of techniques can be used to find the prime factorization of a number. Two methods are illustrated.

Method 1: Branching

To find the prime factorization of a number, select any two numbers whose product is the number to be factored. If the factors are not prime numbers, then continue factoring each composite number until all numbers are prime.

EXAMPLE 2 *Prime Factorization by Branching*

Write 2100 as a product of primes.

SOLUTION: Select any two numbers whose product is 2100. Among the many choices, two possibilities are $21 \cdot 100$ and $30 \cdot 70$. Let us consider $21 \cdot 100$. Since neither 21 nor 100 are prime numbers, find any two numbers whose product is 21 and any two numbers whose product is 100. Continue branching as shown in Fig. 5.2 until the numbers in the last row are all prime numbers. To determine the answer, write the product of all the prime factors. The branching diagram is sometimes called a *factor tree.*

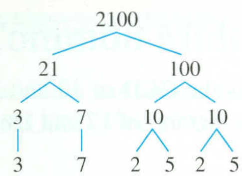


Figure 5.2

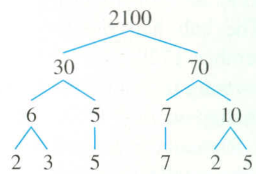


Figure 5.3

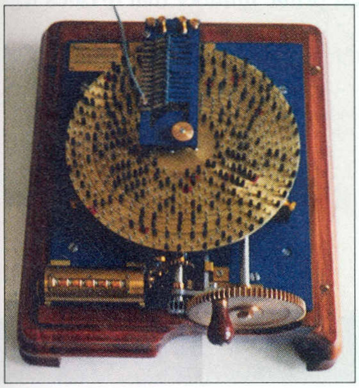
We see that the numbers in the last row of factors in Fig. 5.2 are all prime numbers. Thus, the prime factorization of 2100 is $3 \cdot 7 \cdot 2 \cdot 5 \cdot 2 \cdot 5 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 7 = 2^2 \cdot 3 \cdot 5^2 \cdot 7$. Note from Fig. 5.3 that had we chosen 30 and 70 as the first pair of factors, the prime factorization would still be $2^2 \cdot 3 \cdot 5^2 \cdot 7$. ▲

Method 2: Division

To obtain the prime factorization of a number by this method, divide the given number by the smallest prime number by which it is divisible. Place the quotient under the given number. Then divide the quotient by the smallest prime number by which it is divisible and again record the quotient. Repeat this process until the quotient is a prime number. The prime factorization is the product of all the prime divisors and the prime (or last) quotient. This procedure is illustrated in Example 3.

DID YOU KNOW

The Long-Lost Factoring Machine



In 1989, Jeffrey Shallit of the University of Waterloo came across an article in a 1920 French journal regarding a machine that was built in 1914 for factoring numbers. Eugene Oliver Carissan, an amateur mathematician who had invented the machine, wrote the article. After reading the article, Shallit wondered whatever became of the factoring machine.

After considerable searching, Shallit found the machine in a drawer of an astronomical observatory in Floirac, France. The machine was in good condition, and it still worked. By rotating the machine by a hand crank at two revolutions per minute, an operator could process 35 to 40 numbers per second. Carissan needed just 10 minutes to prove that 708,158,977 is a prime number, an amazing feat in precomputer times. The machine is now housed at the Conservatoire National des Arts et Métiers in Paris.

EXAMPLE 3 *Prime Factorization by Division*

Write 2100 as a product of prime numbers.

SOLUTION: Because 2100 is an even number, the smallest prime number that divides it is 2. Divide 2100 by 2. Place the quotient, 1050, below the 2100. Repeat this process of dividing each quotient by the smallest prime number that divides it.

2	2100
2	1050
3	525
5	175
5	35
	7

The final quotient, 7, is a prime number, so we stop. The prime factorization of 2100 is

$$2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 7 = 2^2 \cdot 3 \cdot 5^2 \cdot 7.$$

Note that, despite the different methods used in Examples 2 and 3, the answer is the same.

Greatest Common Divisor

The discussion in Section 5.3 of how to reduce fractions makes use of the greatest common divisor (GCD). One technique of finding the GCD is to use prime factorization.

The **greatest common divisor (GCD)** of a set of natural numbers is the largest natural number that divides (without remainder) every number in that set.

What is the GCD of 12 and 18? One way to determine it is to make a list of the divisors (or factors) of 12 and 18:

Divisors of 12	{**1, 2, 3**, 4, **6**, 12}
Divisors of 18	{**1, 2, 3, 6**, 9, 18}

The common divisors are 1, 2, 3, and 6. Therefore, the greatest common divisor is 6.

If the numbers are large, this method of finding the GCD is not practical. The GCD can be found more efficiently by using prime factorization.

To Find the Greatest Common Divisor of Two or More Numbers

1. Determine the prime factorization of each number.
2. Find each prime factor with the smallest exponent that appears in each of the prime factorizations.
3. Determine the product of the factors found in step 2.

PROFILE IN MATHEMATICS

SRINIVASA RAMANUJAN

One of the most interesting mathematicians of modern times is Srinivasa Ramanujan (1887–1920). Born to an impoverished middle-class family in India, he virtually taught himself higher mathematics. He went to England to study with the number theorist G. H. Hardy. Hardy tells the story of a taxicab ride he took to visit Ramanujan. The cab had the license plate number 1729, and he challenged the young Indian to find anything interesting in that. Without hesitating, Ramanujan pointed out that it was the smallest positive integer that could be represented in two different ways as the sum of two cubes: $1^3 + 12^3$ and $9^3 + 10^3$.

Example 4 illustrates this procedure.

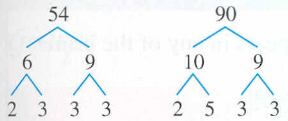


Figure 5.4

DID YOU KNOW

Friendly Numbers

The ancient Greeks often thought of numbers as having human qualities. For example, the numbers 220 and 284 were considered "friendly" or "amicable" numbers because each number was the sum of the other number's proper factors. (A proper factor is any factor of a number other than the number itself.) If you sum all the proper factors of 284 ($1 + 2 + 4 + 71 + 142$), you get the number 220, and if you sum all the proper factors of 220 ($1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110$), you get 284.

EXAMPLE 4 *Using Prime Factorization to Find the GCD*

Find the GCD of 54 and 90.

SOLUTION: The branching method of finding the prime factors of 54 and 90 is illustrated in Fig. 5.4.

a) The prime factorization of 54 is $2 \cdot 3^3$, and the prime factorization of 90 is $2 \cdot 3^2 \cdot 5$.

b) The prime factors with the smallest exponents that appear in each of the factorizations of 54 and 90 are 2 and 3^2.

c) The product of the factors found in step 2 is $2 \cdot 3^2 = 2 \cdot 9 = 18$. The GCD of 54 and 90 is 18. Eighteen is the largest number that divides both 54 and 90.

EXAMPLE 5 *Finding the GCD*

Find the GCD of 225 and 525.

SOLUTION:

a) The prime factorization of 225 is $3^2 \cdot 5^2$, and the prime factorization of 525 is $3 \cdot 5^2 \cdot 7$ (you should verify these using the branching method or the division method).

b) The prime factors with the smallest exponents that appear in each of the factorizations of 225 and 525 are 3 and 5^2.

c) The product of the factors found in step 2 is $3 \cdot 5^2 = 3 \cdot 25 = 75$. The GCD of 225 and 525 is 75.

Two numbers with a GCD of 1 are said to be ***relatively prime***. The numbers 9 and 14 are relatively prime, since the GCD is 1.

Least Common Multiple

To perform addition and subtraction of fractions (Section 5.3), we use the least common multiple (LCM). One technique of finding the LCM is to use prime factorization.

The **least common multiple (LCM)** of a set of natural numbers is the smallest natural number that is divisible (without remainder) by each element of the set.

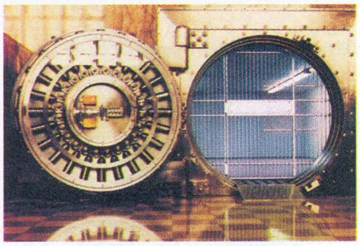
What is the least common multiple of 12 and 18? One way to determine the LCM is to make a list of the multiples of each number:

Multiples of 12	{12, 24, **36**, 48, 60, **72**, 84, 96, **108**, 120, 132, **144**, . . . }
Multiples of 18	{18, **36**, 54, **72**, 90, **108**, 126, **144**, 162, . . . }

Some common multiples of 12 and 18 are 36, 72, 108, and 144. The least common multiple, 36, is the smallest number that is divisible by both 12 and 18. Usually, the most efficient method of finding the LCM is to use prime factorization.

DID YOU KNOW

Safer Secrets



Prime numbers have for many years played an essential role in protecting the databases of government and business. From 1977 through 2000, data were *encrypted* to allow access only to legitimate users. The key to entering the database was a large composite number over 300 digits long. This number, n, was the product of two very large prime numbers, $p \cdot q$. The number n was publicly available, so information could be entered into the database. The numbers p and q, however, were given only to those users who had the right to access the database. The system was considered secure because computers of the day could not factor a number as large as n. However, with the rapid advances in computer science and mathematics research, larger and larger numbers had to be found to keep databases secure. By 1997, realizing a better protection system was needed, the National Institute for Standardization and Technology began a global contest to choose a better protection system. In October 2000, the Rijndael Block Cipher, developed by two Belgian cryptographers, was selected as the winner. The new system involves various areas of mathematics, including many that are discussed in this book such as permutations, modular arithmetic, polynomials, matrices, and group theory.

To Find the Least Common Multiple of Two or More Numbers

1. Determine the prime factorization of each number.
2. List each prime factor with the greatest exponent that appears in any of the prime factorizations.
3. Determine the product of the factors found in step 2.

Example 6 illustrates this procedure.

EXAMPLE 6 *Using Prime Factorization to Find the LCM*

Find the LCM of 54 and 90.

SOLUTION:

a) Find the prime factors of each number. In Example 4, we determined that

$$54 = 2 \cdot 3^3 \quad \text{and} \quad 90 = 2 \cdot 3^2 \cdot 5$$

b) List each prime factor with the greatest exponent that appears in either of the prime factorizations: $2, 3^3, 5$.

c) Determine the product of the factors found in step 2:

$$2 \cdot 3^3 \cdot 5 = 2 \cdot 27 \cdot 5 = 270$$

Thus, 270 is the LCM of 54 and 90. It is the smallest number that is divisible by both 54 and 90.

EXAMPLE 7 *Finding the LCM*

Find the LCM of 225 and 525.

SOLUTION:

a) Find the prime factors of each number. In Example 5, we determined that

$$225 = 3^2 \cdot 5^2 \quad \text{and} \quad 525 = 3 \cdot 5^2 \cdot 7$$

b) List each prime factor with the greatest exponent that appears in either of the prime factorizations: $3^2, 5^2, 7$.

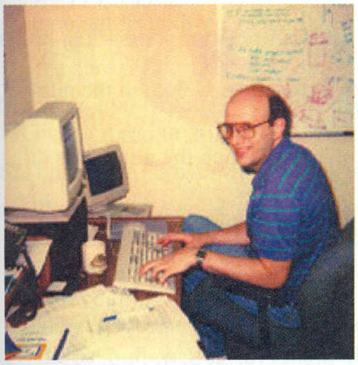
c) Determine the product of the factors found in step 2:

$$3^2 \cdot 5^2 \cdot 7 = 9 \cdot 25 \cdot 7 = 1575$$

Thus, 1575 is the least common multiple of 225 and 525. It is the smallest number divisible by both 225 and 525.

MATHEMATICS *Everywhere*

Gimps



George Woltman

The Great Internet Mersenne Prime Search (GIMPS) was started by George Woltman to provide free software and access to a large database to coordinate the efforts of thousands of people interested in seeking new Mersenne prime numbers. Scott Kurowski provided the technology and services needed to make it simple for anyone to join the project via the Internet. GIMPS allows participants to use their computers to search for prime numbers while their computers are idle. A computer communicates, via the specialized software, with the database. Currently, there are over 130,000 worldwide participants in GIMPS. As of July 2003, the largest Mersenne prime was found by the 20-year-old Michael Cameron on November 14, 2001 in conjunction with GIMPS. The number, $2^{13,466,917} - 1$, which is over 4 million digits long, is the 39th Mersenne prime ever found and the fifth discovered by GIMPS participants. Michael's home computer, an AMD T-Bird 880 MHz with 512 Megs of RAM, took 42 days of idle time to conclude that this number was prime.

The Search For Larger Prime Numbers

More than 2000 years ago, the Greek mathematician Euclid proved that there is no largest prime number. Mathematicians, however, continue to strive to find larger and larger prime numbers.

Marin Mersenne (1588–1648), a seventeenth-century monk, found that numbers of the form $2^n - 1$ are often prime numbers when n is a prime number. For example,

$$2^2 - 1 = 4 - 1 = 3 \qquad 2^3 - 1 = 8 - 1 = 7$$
$$2^5 - 1 = 32 - 1 = 31 \qquad 2^7 - 1 = 128 - 1 = 127$$

Numbers of the form $2^n - 1$ that are prime are referred to as ***Mersenne primes***. The first 10 Mersenne primes occur when $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89$. The first time the expression $2^n - 1$ does not generate a prime number, for prime number n, is when n is 11. The number $2^{11} - 1$ is a composite number (see Exercise 90).

Scientists frequently use Mersenne primes in their search for larger and larger primes. The largest prime number found to date was discovered on November 14, 2001, by 20-year-old Michael Cameron of Owen Sound, Ontario, Canada in conjunction with the Great Internet Mersenne Prime Search (GIMPS). The number is the Mersenne prime $2^{13,466,917} - 1$. This record prime is the 39th known Mersenne prime, and when written out, it is 4,053,946 digits long—over 11 miles long if written using the same size font as this textbook!

More About Prime Numbers

Another mathematician who studied prime numbers was Pierre de Fermat (1601–1665). A lawyer by profession, Fermat became interested in mathematics as a hobby. He became one of the finest mathematicians of the seventeenth century. Fermat conjectured that each number of the form $2^{2^n} + 1$, now referred to as a ***Fermat number***, was prime for each natural number n. Recall that a *conjecture* is a supposition that has not been proved nor disproved. In 1732, Leonhard Euler proved that for $n = 5$, $2^{32} + 1$ was a composite number, thus disproving Fermat's conjecture.

Since Euler's time, mathematicians have only been able to evaluate the sixth, seventh, eighth, ninth, tenth, and eleventh Fermat numbers to determine whether they are prime or composite. Each of these numbers has been shown to be composite. The eleventh Fermat number was factored by Richard Brent and François Morain in 1988. The sheer magnitude of the numbers involved makes it difficult to test these numbers, even with supercomputers.

In 1742, Christian Goldbach conjectured in a letter to Euler that every even number greater than or equal to 4 can be represented as the sum of two (not necessarily distinct) prime numbers (for example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 5 + 5$, $12 = 5 + 7$). This conjecture became known as ***Goldbach's conjecture***, and it remains unproven to this day. The ***twin prime conjecture*** is another famous long-standing conjecture. ***Twin primes*** are primes of the form p and $p + 2$ (for example, 3 and 5, 5 and 7, 11 and 13). This conjecture states that there are an infinite number of pairs of twin primes. At the time of this writing the largest twin primes are of the form $665,551,035 \cdot 2^{80,025}$ plus or minus 1, which were found by David Underbakke and Phil Carmody on November 28, 2000.

SECTION 5.1 EXERCISES

Concept/Writing Exercises

1. What is number theory?
2. What does "a and b are factors of c" mean?
3. a) What does "a divides b" mean?
   b) What does "a is divisible by b" mean?
4. What is a prime number?
5. What is a composite number?
6. What does the fundamental theorem of arithmetic state?
7. a) What is the least common multiple of a set of natural numbers?
   b) In your own words, explain how to find the LCM of a set of natural numbers by using prime factorization.
   c) Find the LCM of 16 and 40 by using the procedure given in part (b).
8. a) What is the greatest common divisor of a set of natural numbers?
   b) In your own words, explain how to find the GCD of a set of natural numbers by using prime factorization.
   c) Find the GCD of 16 and 40 by using the procedure given in part (b).
9. What are Mersenne primes?
10. What is a conjecture?
11. What is Goldbach's conjecture?
12. What are twin primes?

Practice the Skills

13. Use the sieve of Eratosthenes to find the prime numbers up to 100.
14. Use the sieve of Eratosthenes to find the prime numbers up to 150.

In Exercises 15–26, determine whether the statement is true or false. Modify each false statement to make it a true statement.

15. 9 is a factor of 54.
16. $4|36$.
17. 7 is a multiple of 21.
18. 35 is a divisor of 5.
19. 8 is divisible by 56.
20. 15 is a factor of 45.
21. If a number is not divisible by 5, then it is not divisible by 10.
22. If a number is not divisible by 10, then it is not divisible by 5.
23. If a number is divisible by 3, then every digit of the number is divisible by 3.
24. If every digit of a number is divisible by 3, then the number itself is divisible by 3.
25. If a number is divisible by 2 and 3, then the number is divisible by 6.
26. If a number is divisible by 3 and 4, then the number is divisible by 12.

In Exercises 27–32, determine whether the number is divisible by each of the following numbers: 2, 3, 4, 5, 6, 8, 9, and 10.

27. 10,368
28. 19,200
29. 2,763,105
30. 3,126,120
31. 1,882,320
32. 3,941,221
33. Determine a number that is divisible by 2, 3, 4, 5, and 6.
34. Determine a number that is divisible by 3, 4, 5, 9, and 10.

In Exercises 35–46, find the prime factorization of the number.

35. 45
36. 52
37. 196
38. 198
39. 303
40. 400
41. 513
42. 663
43. 1336
44. 1313
45. 2001
46. 3190

In Exercises 47–56, find (a) the greatest common divisor (GCD) and (b) the least common multiple (LCM).

47. 6 and 15
48. 20 and 36
49. 48 and 54
50. 22 and 231
51. 40 and 900
52. 120 and 240
53. 96 and 212
54. 240 and 285
55. 24, 48, and 128
56. 18, 78, and 198

Problem Solving

57. Find the next two sets of twin primes that follow the set 11, 13.
58. The primes 2 and 3 are consecutive natural numbers. Is there another pair of consecutive natural numbers both of which are prime? Explain.

59. For each pair of numbers, determine whether the numbers are relatively prime. Write yes or no as your answer.
a) 14, 15
b) 21, 30
c) 24, 25
d) 119, 143

60. Find the first three Fermat numbers and determine whether they are prime or composite.

61. Show that Goldbach's conjecture is true for the even numbers 4 through 20.

62. Find the first five Mersenne prime numbers.

63. *Barbie and Ken* Mary Lois King collects Barbie dolls and Ken dolls. She has 350 Barbie dolls and 140 Ken dolls. Mary Lois wishes to display the dolls in groups so that the same number of dolls are in each group and that each doll belongs to one group. If each group is to consist only of Barbie dolls or only of Ken dolls, what is the largest number of dolls Mary Lois can have in each group?

64. *Toy Car Collection* Martha Goshaw collects Matchbox® and HotWheels® toy cars. She has 288 red cars and 192 blue cars. She wants to line up her cars in groups so that each group has the same number of cars and each group contains only red cars or only blue cars. What is the largest number of cars she can have in a group?

65. *Stacking Trading Cards* Desmond Freeman collects trading cards. He has 432 baseball cards and 360 football cards. He wants to make stacks of cards on a table so that each stack contains the same number of cards and each card belongs to one stack. If the baseball and football cards must not be mixed in the stacks, what is the largest number of cards that he can have in a stack?

66. *Tree Rows* Elizabeth Dwyer is the manager at Queen Palm Nursery and is in charge of displaying potted trees in rows. Elizabeth has 150 citrus trees and 180 palm trees. She wants to make rows of trees so that each row has the same number of trees and each tree is in a row. If the citrus trees and the palm trees must not be mixed in the rows, what is the largest number of trees that she can have in a row?

67. *Airport Activity* O'Hare International Airport in Chicago has a flight leaving for New York City every 45 minutes and a flight leaving for Atlanta every 60 minutes. If a flight to New York City and a flight to Atlanta leave at the same time, how many minutes will it be before a flight to New York City and a flight to Atlanta again leave at the same time?

O'Hare International Airport

68. *Car Maintenance* For many sport utility vehicles, it is recommended that the oil be changed every 3500 miles and that the tires be rotated every 6000 miles. If Carmella Gonzalez just had the oil changed and tires rotated on her SUV during the same visit to her mechanic, how many miles will she drive before she has the oil changed and tires rotated again during the same visit?

69. *Work Schedules* Sara Pappas and Harry Kinnan both work the 3:00 P.M. to 11:00 P.M. shift. Sara has every fifth night off and Harry has every sixth night off. If they both have tonight off, how many days will pass before they have the same night off again?

70. *Restaurant Service* Peter Theodus runs a professional accounting service for restaurants. Peter goes to Arturo's Family Restaurant every 15 days, and he goes to Xang's Great Wall Restaurant every 18 days. If on October 1 Peter visits both restaurants, how many days would it be before he visited both restaurants on the same day again?

71. *U.S. Senate Committees* The U.S. Senate consists of 100 members. Senate committees are to be formed so that each of the committees contains the same number of senators and each senator is a member of exactly one committee. The committees are to have more than 2 members but fewer than 50 members. There are various ways that these committees can be formed.
a) What size committees are possible?
b) How many committees are there for each size?

72. *Prime Numbers* Consider the first eight prime numbers greater than 3. The numbers are 5, 7, 11, 13, 17, 19, 23, and 29.
a) Determine which of these prime numbers differs by 1 from a multiple of the number 6.
b) Use inductive reasoning and the results obtained in part (a) to make a conjecture regarding prime numbers.
c) Select a few more prime numbers and determine whether your conjecture appears to be correct.

73. State a procedure that defines a divisibility test for 15.

74. State a procedure that defines a divisibility test for 22.

Euclidean Algorithm *Another method that can be used to find the greatest common divisor is known as the Euclidean algorithm. We illustrate this procedure by finding the GCD of 60 and 220.*

First divide 220 by 60 as shown below. Disregard the quotient 3 and then divide 60 by the remainder 40. Continue this process of dividing the divisors by the remainders until you obtain a remainder of 0. The divisor in the last division, in which the remainder is 0, is the GCD.

$$\begin{array}{r} 3 \\ 60\overline{)220} \\ \underline{180} \\ 40 \end{array} \qquad \begin{array}{r} 1 \\ 40\overline{)60} \\ \underline{40} \\ 20 \end{array} \qquad \begin{array}{r} 2 \\ 20\overline{)40} \\ \underline{40} \\ 0 \end{array}$$

Since 40/20 had a remainder of 0, the GCD is 20.

In Exercises 75–80, use the Euclidean algorithm to find the GCD.

75. 15, 35

76. 16, 28

77. 36, 108

78. 76, 240

79. 150, 180

80. 210, 560

Perfect Numbers *A number whose **proper factors** (factors other than the number itself) add up to the number is called a **perfect number**. For example, 6 is a perfect number because its proper factors are 1, 2, and 3, and $1 + 2 + 3 = 6$. Determine which, if any, of the following numbers are perfect.*

81. 12

82. 28

83. 496

84. 48

Challenge Problems/Group Activities

85. ***Number of Factors*** The following procedure can be used to determine the *number of factors* (or *divisors*) of a composite number. Write the number in prime factorization form. Examine the exponents on the prime numbers in the prime factorization. Add 1 to each exponent and then find the product of these numbers. This product gives the number of positive divisors of the composite number.
    a) Use this procedure to determine the number of divisors of 60.
    b) To check your answer, list all the divisors of 60. You should obtain the same number of divisors found in part (a).

86. Recall that if a number is divisible by both 2 and 3, then the number is divisible by 6. If a number is divisible by both 2 and 4, is the number necessarily divisible by 8? Explain your answer.

87. The product of any three consecutive natural numbers is divisible by 6. Explain why.

88. A number in which each digit except 0 appears exactly three times is divisible by 3. For example, 888,444,555 and 714,714,714 are both divisible by 3. Explain why this outcome must be true.

89. Use the fact that if $a|b$ and $a|c$, then $a|(b + c)$ to determine whether 36,018 is divisible by 18. (*Hint:* Write 36,018 as $36{,}000 + 18$.)

90. Show that the $2^n - 1$ is a (Mersenne) prime for $n = 2, 3, 5$, and 7 but composite for $n = 11$.

91. Goldbach also conjectured in his letter to Euler that *every* integer greater than 5 is the sum of three prime numbers. For example, $6 = 2 + 2 + 2$ and $7 = 2 + 2 + 3$. Show that this conjecture is true for integers 8 through 20.

Recreational Mathematics

92. ***Country, Animal, Fruit*** Select a number from 1 to 10. Multiply your selected number by 9. Add the digits of the product together (if the product has more than one digit). Now subtract 5. Determine which letter of the alphabet corresponds to the number you ended with (for example, $1 = A$, $2 = B$, $3 = C$, and so on). Think of a country whose name begins with that letter. Remember the last letter of the name of that country. Think of an animal whose name begins with that letter. Remember the last letter in the name of that animal. Think of a fruit whose name begins with that letter. a) What country, animal, and fruit did you select? Turn to the answer section to see if your response matches the responses of over 90% of the people who attempt this activity. b) Can you explain why most people select the given answer?

Research Activities

93. Do research and explain what *deficient numbers* and *abundant numbers* are. Give an example of each type of number. References include history of mathematics books, encyclopedias, and the Internet.

94. Conduct an Internet search on the GIMPS project. Write a report describing the history and development of the project. Include a current update of the project's findings.

5.2 THE INTEGERS

In Section 5.1, we introduced the natural or counting numbers:

$$N = \{1, 2, 3, 4, \dots\}$$

Another important set of numbers, the ***whole numbers***, help to answer the question "How many?"

$$\text{Whole numbers} = \{0, 1, 2, 3, 4, \dots\}$$

Note that the set of whole numbers contains the number 0 but that the set of counting numbers does not. If a farmer were asked how many chickens were in a coop, the answer would be a whole number. If the farmer had no chickens, he or she would answer zero. Although we use the number 0 daily and take it for granted, the number 0 as we know it was not used and accepted until the sixteenth century.

If the temperature is 12°F and drops 20°, the resulting temperature is −8°F. This type of problem shows the need for negative numbers. The set of ***integers*** consists of the negative integers, 0, and the positive integers.

$$\text{Integers} = \{\underbrace{\dots, -4, -3, -2, -1}_{\text{Negative integers}}, 0, \underbrace{1, 2, 3, \dots}_{\text{Positive integers}}\}$$

The term *positive integers* is yet another name for the natural numbers or counting numbers.

An understanding of addition, subtraction, multiplication, and division of the integers is essential in understanding algebra (Chapter 6). To aid in our explanation of addition and subtraction of integers, we introduce the real number line (Fig. 5.5). To construct the real number line, arbitrarily select a point for zero to serve as the starting point. Place the positive integers to the right of 0, equally spaced from one another. Place the negative integers to the left of 0, using the same spacing. The real number line contains the integers and all the other real numbers that are not integers. Some examples of real numbers that are not integers are indicated in Fig. 5.5, namely $-\frac{5}{2}$, $\frac{1}{2}$, $\sqrt{2}$, and π. We discuss real numbers that are not integers in the next two sections.

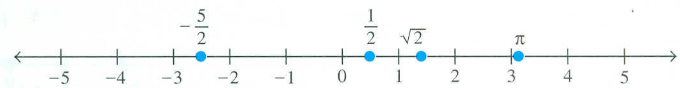


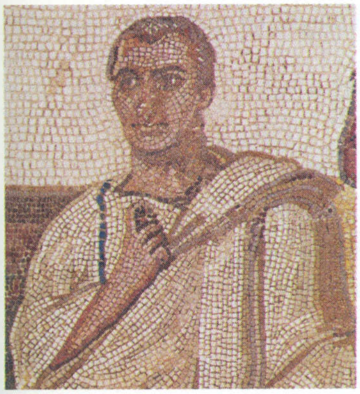
Figure 5.5

The arrows at the ends of the real number line indicate that the line continues indefinitely in both directions. Note that for any natural number, n, on the number line, the *opposite of* that number, $-n$, is also on the number line. This real number line was drawn horizontally, but it could just as well have been drawn vertically. In fact, in the next chapter, we show that the axes of a graph are the union of two number lines, one horizontal and the other vertical.

The number line can be used to determine the greater (or lesser) of two integers. Two *inequality symbols* that we will use in this chapter are $>$ and $<$. The symbol $>$ is read "is greater than," and the symbol $<$ is read "is less than." Expressions that

DID YOU KNOW

Virgil's Party



The poet Virgil was born in 70 B.C. In 1930, some highly respected scholars decided to celebrate his 2000th birthday. The only problem was that the accounting of time does not include a year 0 between B.C. and A.D. So Virgil would have turned 70 in our year 1 A.D. and, had he lived so long, 2000 in 1931. This fact was pointed out to the celebrants, but only after the party was well under way.

contain an inequality symbol are called *inequalities.* On the number line, the numbers increase from left to right. The number 3 is greater than 2, written $3 > 2$. Observe that 3 is to the right of 2. Similarly, we can see that $0 > -1$ by observing that 0 is to the right of -1 on the number line.

Instead of stating that 3 is greater than 2, we could state that 2 is less than 3, written $2 < 3$. Note that 2 is to the left of 3 on the number line. We can also see that $-1 < 0$ by observing that -1 is to the left of 0. The inequality symbol always points to the smaller of the two numbers when the inequality is true.

EXAMPLE 1 *Writing an Inequality*

Insert either $>$ or $<$ in the shaded area between the paired numbers to make the statement correct.

a) -3 ■ 1 b) -3 ■ -5 c) -6 ■ -4 d) 0 ■ -7

SOLUTION:

a) $-3 < 1$ since -3 is to the left of 1 on the number line.

b) $-3 > -5$ since -3 is to the right of -5 on the number line.

c) $-6 < -4$ since -6 is to the left of -4 on the number line.

d) $0 > -7$ since 0 is to the right of -7 on the number line.

Addition of Integers

Addition of integers can be represented geometrically with a number line. To do so, begin at 0 on the number line. Represent the first addend (the first number to be added) by an arrow starting at 0. Draw the arrow to the right if the addend is positive. If the addend is negative, draw the arrow to the left. From the tip of the first arrow, draw a second arrow to represent the second addend. Draw the second arrow to the right or left, as just explained. The sum of the two integers is found at the tip of the second arrow.

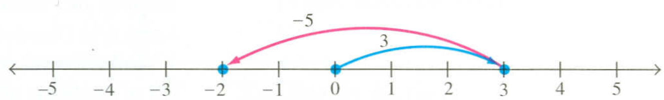
EXAMPLE 2 *Adding Integers*

Evaluate the following using the number line.

a) $3 + (-5)$ b) $-1 + (-4)$ c) $-6 + 4$ d) $3 + (-3)$

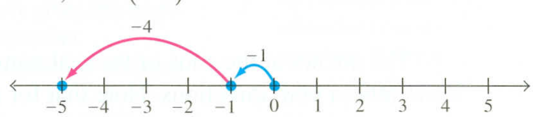
SOLUTION:

a)



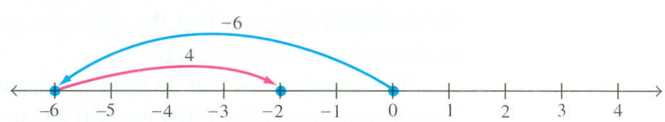
Thus, $3 + (-5) = -2$

b)



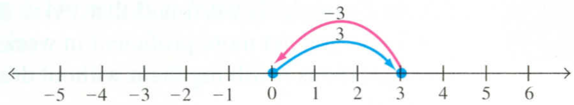
Thus, $-1 + (-4) = -5$

c)



Thus, $-6 + 4 = -2$

d)



Thus, $3 + (-3) = 0$.

In Example 2(d), the number -3 is said to be the additive inverse of 3 and 3 is the additive inverse of -3, because their sum is 0. In general, the ***additive inverse*** of the number n is $-n$, since $n + (-n) = 0$. Inverses are discussed more formally in Chapter 10.

Subtraction of Integers

Any subtraction problem can be rewritten as an addition problem. To do so, we use the following definition of subtraction.

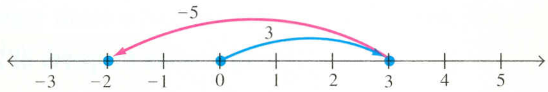
Subtraction

$$a - b = a + (-b)$$

The rule for subtraction indicates that to subtract b from a, *add* the additive inverse of b to a. For example,

$$\underset{\substack{\uparrow\\ \text{Subtraction}}}{3 - 5} = \underset{\substack{\uparrow\\ \text{Addition}}}{3 +} \underset{\substack{\uparrow\\ \text{Additive inverse of 5}}}{(-5)}$$

Now we can determine the value of $3 + (-5)$.

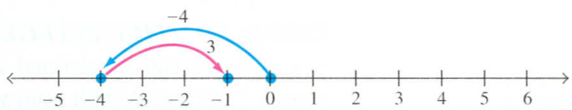


Thus, $3 - 5 = 3 + (-5) = -2$.

EXAMPLE 3 *Subtracting Integers*

Evaluate $-4 - (-3)$ using the number line.

SOLUTION: We are subtracting -3 from -4. The additive inverse of -3 is 3; therefore, we add 3 to -4. We now add $-4 + 3$ on the number line to obtain the answer -1.



Thus, $-4 - (-3) = -4 + 3 = -1$.

DID YOU KNOW

Sparsely Populated

More people have made it into outer space than have stood on the top of Nepal's 29,035-foot-high Mount Everest. Fewer people still have visited the Pacific Ocean's 36,198-foot-deep Mariana Trench. In 1960, Jacques Piccard and Don Walsh were the first to make the journey in a vessel designed to withstand the immense pressure: 7 tons per square inch. The diving time was $8\frac{1}{2}$ hours, of which only 20 minutes were spent on the ocean floor.

In Example 3, we found that $-4 - (-3) = -4 + 3$. In general, $a - (-b) = a + b$. As you get more proficient in working with integers, you should be able to answer questions involving them without drawing a number line.

EXAMPLE 4 *Subtracting: Adding the Inverse*

Evaluate

a) $-5 - 2$ b) $-5 - (-2)$ c) $5 - (-2)$ d) $5 - 2$

SOLUTION:

a) $-5 - 2 = -5 + (-2) = -7$ b) $-5 - (-2) = -5 + 2 = -3$

c) $5 - (-2) = 5 + 2 = 7$ d) $5 - 2 = 5 + (-2) = 3$

EXAMPLE 5 *Elevation Difference*

The highest point on Earth is Mount Everest, in the Himalayas, at a height of 29,035 ft above sea level. The lowest point on Earth is the Mariana Trench, in the Pacific Ocean, at a depth of 36,198 ft below sea level (−36,198 ft). Find the vertical height difference between Mount Everest and the Mariana Trench.

SOLUTION: We obtain the vertical difference by subtracting the lower elevation from the higher elevation.

$$29{,}035 - (-36{,}198) = 29{,}035 + 36{,}198 = 65{,}233$$

The vertical difference is 65,233 ft.

Multiplication of Integers

The multiplication property of zero is important in our discussion of multiplication of integers. It indicates that the product of 0 and any number is 0.

Multiplication Property of Zero

$$a \cdot 0 = 0 \cdot a = 0$$

We will develop the rules for multiplication of integers using number patterns. The four possible cases are

1. positive integer × positive integer,
2. positive integer × negative integer,
3. negative integer × positive integer, and
4. negative integer × negative integer.

CASE 1: ***POSITIVE INTEGER × POSITIVE INTEGER*** The product of two positive integers can be defined as repeated addition of a positive integer. Thus, $3 \cdot 2$ means $2 + 2 + 2$. This sum will always be positive. Thus, a positive integer times a positive integer is a positive integer.

CASE 2: ***POSITIVE INTEGER × NEGATIVE INTEGER*** Consider the following patterns:

$$3(3) = 9$$
$$3(2) = 6$$
$$3(1) = 3$$

Note that each time the second factor is reduced by 1, the product is reduced by 3. Continuing the process gives

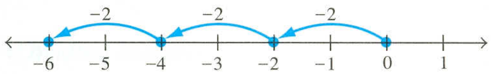
$$3(0) = 0$$

What comes next?

$$3(-1) = -3$$
$$3(-2) = -6$$

The pattern indicates that a positive integer times a negative integer is a negative integer.

We can confirm this result by using the number line. The expression $3(-2)$ means $(-2) + (-2) + (-2)$. Adding $(-2) + (-2) + (-2)$ on the number line, we obtain a sum of -6.



CASE 3: ***NEGATIVE INTEGER × POSITIVE INTEGER*** A procedure similar to that used in case 2 will indicate that a negative integer times a positive integer is a negative integer.

CASE 4: ***NEGATIVE INTEGER × NEGATIVE INTEGER*** We have illustrated that a positive integer times a negative integer is a negative integer. We make use of this fact in the following pattern:

$$4(-4) = -16$$
$$3(-4) = -12$$
$$2(-4) = -8$$
$$1(-4) = -4$$

In this pattern, each time the first term is decreased by 1, the product is increased by 4. Continuing this process gives

$$0(-4) = 0$$
$$(-1)(-4) = 4$$
$$(-2)(-4) = 8$$

This pattern illustrates that a negative integer times a negative integer is a positive integer.

The examples were restricted to integers. The rules for multiplication, however, can be used for any numbers. We summarize them as follows.

Rules for Multiplication

1. The product of two numbers with *like signs* (positive $\times$ positive or negative $\times$ negative) is a *positive number.*
2. The product of two numbers with *unlike signs* (positive $\times$ negative or negative $\times$ positive) is a *negative number.*

EXAMPLE 6 ***Multiplying Integers***

Evaluate

a) $5 \cdot 9$ b) $5 \cdot (-9)$ c) $(-5) \cdot 9$ d) $(-5)(-9)$

SOLUTION:

a) $5 \cdot 9 = 45$ b) $5 \cdot (-9) = -45$ c) $(-5) \cdot 9 = -45$
d) $(-5)(-9) = 45$

Division of Integers

You may already realize that a relationship exists between multiplication and division.

$$6 \div 2 = 3 \quad \text{means that} \quad 3 \cdot 2 = 6$$

$$\frac{20}{10} = 2 \quad \text{means that} \quad 2 \cdot 10 = 20$$

These examples demonstrate that division is the reverse process of multiplication.

Division

For any a, b, and c where $b \neq 0$, $\frac{a}{b} = c$ means that $c \cdot b = a$.

We discuss the four possible cases for division, which are similar to those for multiplication.

CASE 1: ***POSITIVE INTEGER ÷ POSITIVE INTEGER*** A positive integer divided by a positive integer is positive.

$$\frac{6}{2} = 3 \quad \text{since} \quad 3(2) = 6$$

CASE 2: ***POSITIVE INTEGER ÷ NEGATIVE INTEGER*** A positive integer divided by a negative integer is negative.

$$\frac{6}{-2} = -3 \quad \text{since} \quad (-3)(-2) = 6$$

CASE 3: ***NEGATIVE INTEGER ÷ POSITIVE INTEGER*** A negative integer divided by a positive integer is negative.

$$\frac{-6}{2} = -3 \qquad \text{since} \qquad (-3)(2) = -6$$

CASE 4: ***NEGATIVE INTEGER ÷ NEGATIVE INTEGER*** A negative integer divided by a negative integer is positive.

$$\frac{-6}{-2} = 3 \qquad \text{since} \qquad 3(-2) = -6$$

The examples were restricted to integers. The rules for division, however, can be used for any numbers. You should realize that division of integers does not always result in an integer. The rules for division are summarized as follows.

Rules for Division

1. The quotient of two numbers with *like signs* (positive ÷ positive or negative ÷ negative) is a *positive number.*
2. The quotient of two numbers with *unlike signs* (positive ÷ negative or negative ÷ positive) is a *negative number.*

EXAMPLE 7 *Dividing Integers*

Evaluate

a) $\frac{56}{8}$ b) $\frac{-56}{8}$ c) $\frac{56}{-8}$ d) $\frac{-56}{-8}$

SOLUTION:

a) $\frac{56}{8} = 7$ b) $\frac{-56}{8} = -7$ c) $\frac{56}{-8} = -7$ d) $\frac{-56}{-8} = 7$ ▲

In the definition of division, we stated that the denominator could not be 0. Why not? Suppose we are trying to find the quotient $\frac{5}{0}$. Let's say that this quotient is equal to some number x. Then we would have $\frac{5}{0} = x$. If true, this would mean that $5 = x \cdot 0$. The right side of the equation is $x \cdot 0$, which is equal to 0 for any real value of x. This leads us to conclude that $5 = 0$, which is false. Thus, there is no number that can replace x that makes the equation $\frac{5}{0} = x$ true. Therefore, in mathematics, division by 0 is not allowed and we say that a quotient of any number divided by zero is *undefined.*

SECTION 5.2 EXERCISES

Concept/Writing Exercises

1. Explain how to add numbers using a number line.
2. What is the additive inverse of a number n?
3. Explain how to rewrite a subtraction problem as an addition problem.
4. Explain the rule for multiplication of real numbers.
5. Explain the rule for division of real numbers.

6. Explain why the quotient of a number divided by 0 is undefined.

Practice the Skills

In Exercises 7–16, evaluate the expression.

7. $-6 + 9$
8. $4 + (-5)$
9. $(-7) + 9$
10. $(-3) + (-3)$
11. $[6 + (-11)] + 0$
12. $(2 + 5) + (-4)$
13. $[(-3) + (-4)] + 9$
14. $[8 + (-3)] + (-2)$
15. $[(-23) + (-9)] + 11$
16. $[5 + (-13)] + 18$

In Exercises 17–26, evaluate the expression.

17. $3 - 6$
18. $-3 - 7$
19. $-4 - 6$
20. $7 - (-1)$
21. $-5 - (-3)$
22. $-4 - 4$
23. $14 - 20$
24. $8 - (-3)$
25. $[5 + (-3)] - 4$
26. $6 - (8 + 6)$

In Exercises 27–36, evaluate the expression.

27. $-4 \cdot 5$
28. $4(-3)$
29. $(-12)(-12)$
30. $5(-5)$
31. $[(-8)(-2)] \cdot 6$
32. $4(-5)(-6)$
33. $(5 \cdot 6)(-2)$
34. $(-9)(-1)(-2)$
35. $[(-3)(-6)] \cdot [(-5)(8)]$
36. $[(-8 \cdot 4) \cdot 5](-2)$

In Exercises 37–46, evaluate the expression.

37. $-26 \div (-13)$
38. $-56 \div 8$
39. $23 \div (-23)$
40. $-64 \div 16$
41. $56/-8$
42. $-75/15$
43. $-210/14$
44. $186/-6$
45. $144 \div (-3)$
46. $(-900) \div (-4)$

In Exercises 47–56, determine whether the statement is true or false. Modify each false statement to make it a true statement.

47. Every whole number is an integer.
48. Every integer is a whole number.
49. The difference of any two negative integers is a negative integer.
50. The sum of any two negative integers is a negative integer.
51. The product of any two positive integers is a positive integer.
52. The difference of a positive integer and a negative integer is always a negative integer.
53. The quotient of a negative integer and a positive integer is always a negative number.
54. The quotient of any two negative integers is a negative number.
55. The sum of a positive integer and a negative integer is always a positive integer.
56. The product of a positive integer and a negative integer is always a positive integer.

In Exercises 57–66, evaluate the expression.

57. $(5 + 7) \div 2$
58. $(-4) \div [14 \div (-7)]$
59. $[6(-2)] - 5$
60. $[(-5)(-6)] - 3$
61. $(4 - 8)(3)$
62. $[18 \div (-2)](-3)$
63. $[2 + (-17)] \div 3$
64. $(5 - 9) \div (-4)$
65. $[(-22)(-3)] \div (2 - 13)$
66. $[15(-4)] \div (-6)$

In Exercises 67–70, write the numbers in increasing order from left to right.

67. $0, -5, -10, 10, 5, -15$
68. $-20, 30, -40, 10, 0, -10$
69. $-5, -2, -3, -1, -4, -6$
70. $106, 33, -47, -108, 72, -76$

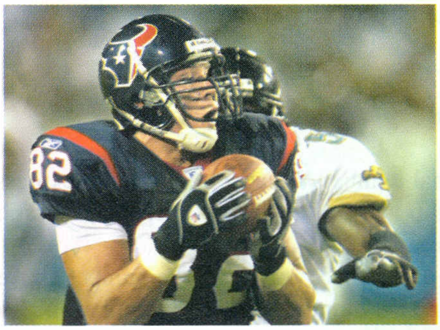
Problem Solving

71. *Extreme Temperatures* The hottest temperature ever recorded in the United States was 134°F, which occurred at Greenland Ranch, California, in Death Valley on July 10, 1913. The coldest temperature ever recorded in the United States was −79.8°F, which occurred at Prospect Creek Camp, Alaska, in the Endicott Mountains on January 23, 1971. Determine the difference between these two temperatures.
72. *NASDAQ Average* On August 28, 2002, the NASDAQ composite average opened at 1347 points. During that day it lost 33 points. On August 29 it gained 22 points, and on August 30 it lost 21 points. What was the closing NASDAQ composite average on August 30, 2002?
73. *Pit Score* While playing the game of Pit, John Pearse began with a score of zero points. He then gained 100 points, lost 40 points, gained 90 points, lost 20 points, and gained 80 points on his next five rounds. What is John's score after five rounds?
74. *Elevation Difference* Mount Whitney, in the Sierra Nevada mountains of California, is the highest point in the

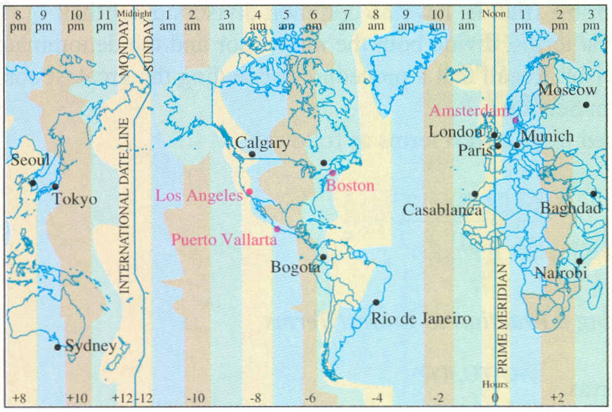
contiguous United States. It is 14,495 ft above sea level. Death Valley, in California and Nevada, is the lowest point in the United States, 282 ft below sea level. Find the vertical height difference between Mount Whitney and Death Valley.

75. *Vertical Distance Traveled* A helicopter drops a package from a height of 842 ft above sea level. The package lands in the ocean and settles at a point 927 ft below sea level. What was the vertical distance the package traveled?

76. *Football Yardage* In the first four plays of the game, the Texans gained 8 yd, lost 5 yd, gained 3 yd, and gained 4 yd. What is the total number of yards gained in the first four plays? Did the Texans make a first down? (Ten yards are needed for a first down.)



77. *Time Zone Calculations* Part of a World Standard Time Zones chart used by airlines and the United States Navy is shown. The scale along the bottom is just like a number line with the integers $-12, -11, \ldots, 11, 12$ on it.
   a) Find the difference in time between Amsterdam (zone $+1$) and Los Angeles (zone -8).
   b) Find the difference in time between Boston (zone -5) and Puerto Vallarta (zone -7).



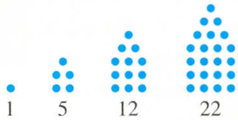
78. Explain why $\dfrac{a}{b} = \dfrac{-a}{-b}$.

Challenge Problems/Group Activities

79. Find the quotient:

$$\frac{-1 + 2 - 3 + 4 - 5 + \cdots - 99 + 100}{1 - 2 + 3 - 4 + 5 - \cdots + 99 - 100}$$

80. *Pentagonal Numbers* Triangular numbers and square numbers were introduced in the Section 1.1 Exercises. There are also **pentagonal numbers**, which were also studied by the Greeks. Four pentagonal numbers are 1, 5, 12, and 22.



   a) Determine the next three pentagonal numbers.
   b) Describe a procedure to determine the next five pentagonal numbers without drawing the figures.
   c) Is 72 a pentagonal number? Explain how you determined your answer.

81. Place the appropriate plus or minus signs between each digit so that the total will equal 1.

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 = 1$$

Recreational Mathematics

82. *Four 4's* The game of *Four 4's* is a challenging way to learn about some of the operations on integers. In this game you must use exactly four 4's along with one or more of the operations of addition, subtraction, multiplication, and division* to write and evaluate expressions. The object of the game is to write expressions that when evaluated will give each whole number from 0 through 9. You may use as many grouping symbols as you wish, but you must use exactly four 4's. For example, one way to obtain 1 is as follows: $(4 + 4) \div (4 + 4) = 8 \div 8 = 1$. One way to obtain 2 is as follows: $(4 \div 4) + (4 \div 4) = 1 + 1 = 2$.
   a) Use the rules as defined above to obtain each whole number 0 through 9.
   b) Use the rules as defined above to obtain the following whole numbers: 12, 15, 16, 17, 20.
   c) We will now change our rules to allow the number 44 to count as two of the four 4's. Use the number 44 and two other fours to obtain the whole number 10.

Research Activity

83. Do research and write a report on the history of the number 0 in the Hindu–Arabic numeration system.

*This game will be expanded in future exercise sets to include other operations such as exponents and square roots.

5.3 THE RATIONAL NUMBERS

"When you can measure what you are talking about and express it in numbers, you know something about it."
Lord Kelvin

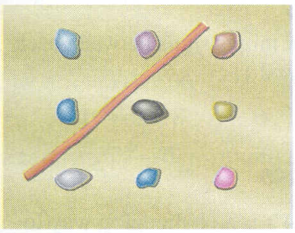
We introduced the number line in Section 5.1 and discussed the integers in Section 5.2. The numbers that fall between the integers on the number line are either rational or irrational numbers. In this section, we discuss the rational numbers, and in Section 5.4, we discuss the irrational numbers.

Any number that can be expressed as a quotient of two integers (denominator not 0) is a rational number.

The set of **rational numbers**, denoted by Q, is the set of all numbers of the form p/q, where p and q are integers and $q \neq 0$.

DID YOU KNOW

Triangular Numbers



One very simple reason that the ancient Greek mathematicians thought of mathematics in terms of whole numbers and their ratios (the rational numbers) was that they were still working with numbers that were represented by objects, such as squares and triangles, not number symbols. Still, that did not prevent them from drawing some conclusions about number theory. Consider two consecutive triangular numbers, for instance 3 and 6. You can see from the diagram that the sum of the two triangular numbers is equal to the square number 9, the square of a side of the larger triangle.

The following numbers are examples of rational numbers:

$$\frac{1}{3}, \quad \frac{3}{4}, \quad -\frac{7}{8}, \quad 1\frac{2}{3}, \quad 2, \quad 0, \quad \frac{15}{7}$$

The integers 2 and 0 are rational numbers because each can be expressed as the quotient of two integers: $2 = \frac{2}{1}$ and $0 = \frac{0}{1}$. In fact, every integer n is a rational number, since it can be written in the form of $\frac{n}{1}$.

Numbers such as $\frac{1}{3}$ and $-\frac{7}{8}$ are also called ***fractions***. The number above the fraction line is called the ***numerator***, and the number below the fraction line is called the ***denominator***.

Reducing Fractions

Sometimes the numerator and denominator in a fraction have a common divisor (or common factor). For example, both the numerator and denominator of the fraction $\frac{6}{10}$ have the common divisor 2. When a numerator and denominator have a common divisor, we can *reduce the fraction to its lowest terms.*

A fraction is said to be in its lowest terms (or reduced) when the numerator and denominator are relatively prime (that is, have no common divisors other than 1). To reduce a fraction to its lowest terms, divide both the numerator and the denominator by the greatest common divisor. Recall that a procedure for finding the greatest common divisor was discussed in Section 5.1.

The fraction $\frac{6}{10}$ is reduced to its lowest terms as follows.

$$\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$$

EXAMPLE 1 *Reducing a Fraction to Lowest Terms*

Reduce $\frac{54}{90}$ to its lowest terms.

SOLUTION: On page 213 in Example 4 of Section 5.1, we determined that the GCD of 54 and 90 is 18. Divide the numerator and the denominator by GCD, 18.

$$\frac{54}{90} = \frac{54 \div 18}{90 \div 18} = \frac{3}{5}$$

Since there are no common divisors of 3 and 5 other than 1, this fraction is in its lowest terms.

Mixed Numbers and Improper Fractions

Consider the number $2\frac{3}{4}$. It is an example of a ***mixed number***. It is called a mixed number because it consists of an integer, 2, and a fraction, $\frac{3}{4}$. The mixed number $2\frac{3}{4}$ means $2 + \frac{3}{4}$. The mixed number $-4\frac{1}{4}$ means $-(4 + \frac{1}{4})$. Rational numbers greater than 1 or less than -1 that are not integers may be represented as mixed numbers, or as ***improper fractions***. An improper fraction is a fraction whose numerator is greater than its denominator. An example of an improper fraction is $\frac{8}{5}$. Figure 5.6 shows both mixed numbers and improper fractions indicated on a number line. In this section, we show how to convert mixed numbers to improper fractions and vice versa.

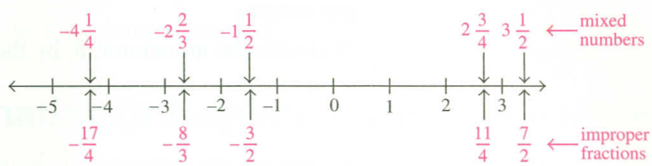


Figure 5.6

We begin by limiting our discussion to positive mixed numbers and positive improper fractions.

Converting a Positive Mixed Number to an Improper Fraction

1. Multiply the denominator of the fraction in the mixed number by the integer preceding it.
2. Add the product obtained in step 1 to the numerator of the fraction in the mixed number. This sum is the numerator of the improper fraction we are seeking. The denominator of the improper fraction we are seeking is the same as the denominator of the fraction in the mixed number.

EXAMPLE 2 *From Mixed Number to Improper Fraction*

Convert the following mixed numbers to improper fractions.

a) $1\frac{7}{8}$ b) $3\frac{5}{6}$

SOLUTION:

a) $1\frac{7}{8} = \frac{8 \cdot 1 + 7}{8} = \frac{8 + 7}{8} = \frac{15}{8}$

b) $3\frac{5}{6} = \frac{6 \cdot 3 + 5}{6} = \frac{18 + 5}{6} = \frac{23}{6}$

Notice that both $\frac{15}{8}$ and $\frac{23}{6}$ have numerators larger than their denominators and that both are improper fractions.

Converting a Positive Improper Fraction to a Mixed Number

1. Divide the numerator by the denominator. Identify the quotient and the remainder.
2. The quotient obtained in step 1 is the integer part of the mixed number. The remainder is the numerator of the fraction in the mixed number. The denominator in the fraction of the mixed number will be the same as the denominator in the original fraction.

EXAMPLE 3 *From Improper Fraction to Mixed Number*

Convert the following improper fractions to mixed numbers.

a) $\frac{8}{5}$ b) $\frac{225}{8}$

SOLUTION:

a) Divide the numerator, 8, by the denominator, 5.

$$\text{Divisor} \rightarrow \begin{array}{r} 1 \\ 5\overline{)8} \\ \underline{5} \\ 3 \end{array} \begin{array}{l} \leftarrow \text{Quotient} \\ \leftarrow \text{Dividend} \\ \\ \leftarrow \text{Remainder} \end{array}$$

Therefore,

$$\frac{8}{5} = \overset{\substack{\text{Quotient} \\ \downarrow}}{1}\frac{3}{5} \begin{array}{l} \leftarrow \text{Remainder} \\ \leftarrow \text{Divisor} \end{array}$$

The mixed number is $1\frac{3}{5}$.

b) Divide the numerator, 225, by the denominator, 8.

$$\text{Divisor} \rightarrow \begin{array}{r} 28 \\ 8\overline{)225} \\ \underline{16} \\ 65 \\ \underline{64} \\ 1 \end{array} \begin{array}{l} \leftarrow \text{Quotient} \\ \leftarrow \text{Dividend} \\ \\ \\ \\ \leftarrow \text{Remainder} \end{array}$$

Therefore,

$$\frac{225}{8} = \overset{\substack{\text{Quotient} \\ \downarrow}}{28}\frac{1}{8} \begin{array}{l} \leftarrow \text{Remainder} \\ \leftarrow \text{Divisor} \end{array}$$

The mixed number is $28\frac{1}{8}$.

Up to this point, we have only worked with positive mixed numbers and positive improper fractions. When converting a negative mixed number to an improper fraction, or a negative improper fraction to a mixed number, it is best to ignore the negative sign temporarily. Perform the calculation as described earlier and then reattach the negative sign.

EXAMPLE 4 *Negative Mixed Numbers and Improper Fractions*

a) Convert $-1\frac{7}{8}$ to an improper fraction.

b) Convert $-\frac{8}{5}$ to a mixed number.

SOLUTION:

a) First, ignore the negative sign and examine $1\frac{7}{8}$. We learned in Example 2(a) that $1\frac{7}{8} = \frac{15}{8}$. Now to convert $-1\frac{7}{8}$ to an improper fraction, we reattach the negative sign. Thus, $-1\frac{7}{8} = -\frac{15}{8}$.

b) We learned in Example 3(a) that $\frac{8}{5} = 1\frac{3}{5}$. Therefore, $-\frac{8}{5} = -1\frac{3}{5}$. ▲

Terminating or Repeating Decimal Numbers

Note the following important property of the rational numbers.

Every *rational number* when expressed as a decimal number will be either a terminating or a repeating decimal number.

Examples of terminating decimal numbers are 0.5, 0.75, and 4.65. Examples of repeating decimal numbers are 0.333. . . , 0.2323. . . , and 8.13456456. . . . One way to indicate that a number or group of numbers repeat is to place a bar above the number or group of numbers that repeat. Thus, 0.333. . . may be written $0.\overline{3}$, 0.2323. . . may be written $0.\overline{23}$, and 8.13456456. . . may be written $8.13\overline{456}$.

EXAMPLE 5 *Terminating Decimal Numbers*

Show that the following rational numbers are terminating decimal numbers.

a) $\dfrac{2}{5}$ b) $-\dfrac{7}{8}$ c) $\dfrac{17}{16}$

SOLUTION: To express the rational number in decimal form, divide the numerator by the denominator. If you use a calculator, or use long division, you will obtain the following results.

a) $\dfrac{2}{5} = 0.4$ b) $-\dfrac{7}{8} = -0.875$ c) $\dfrac{17}{16} = 1.0625$ ▲

EXAMPLE 6 *Repeating Decimal Numbers*

Show that the following rational numbers are repeating decimal numbers.

a) $\dfrac{2}{3}$ b) $\dfrac{14}{99}$ c) $1\dfrac{4}{33}$

SOLUTION: If you use a calculator, or use long division, you will see that each fraction results in a repeating decimal number.

a) $2 \div 3 = 0.6666\ldots$ or $0.\overline{6}$

b) $14 \div 99 = 0.141414\ldots$ or $0.\overline{14}$

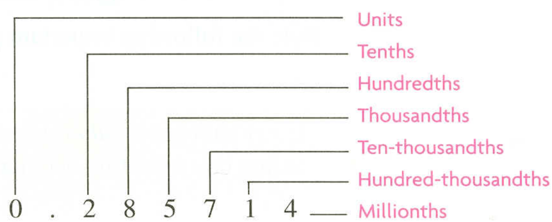
c) $1\dfrac{4}{33} = \dfrac{37}{33} = 1.121212\ldots$ or $1.\overline{12}$ ▲

Note that in each part of Example 6, the quotient has no final digit and continues indefinitely. Each number is a repeating decimal number.

When a fraction is converted to a decimal number, the maximum number of digits that can repeat is $n - 1$, where n is the denominator of the fraction. For example, when $\frac{2}{7}$ is converted to a decimal number, the maximum number of digits that can repeat is $7 - 1$, or 6.

Converting Decimal Numbers to Fractions

We can convert a terminating or repeating decimal number into a quotient of integers. The explanation of the procedure will refer to the positional values to the right of the decimal point, as illustrated here:



Example 7 demonstrates how to convert from a decimal number to a fraction.

EXAMPLE 7 *Converting a Decimal Number into a Fraction*

Convert the following terminating decimal numbers to a quotient of integers.

a) 0.4 b) 0.62 c) 0.062 d) 1.37

SOLUTION: When converting a terminating decimal number to a quotient of integers, we observe the last digit to the right of the decimal point. The position of this digit will indicate the denominator of the quotient of integers.

a) $0.4 = \frac{4}{10}$ because the 4 is in the tenths position.

b) $0.62 = \frac{62}{100}$ because the last digit on the right, 2, is in the hundredths position.

c) $0.062 = \frac{62}{1000}$ because the last digit on the right, 2, is in the thousandths position.

d) $1.37 = \frac{137}{100}$ because the last digit on the right, 7, is in the ten-thousandths position. Notice the numerator is the decimal number with the decimal point removed. When the decimal number is larger than 1, the numerator of the resulting fraction will be greater than the denominator or an improper fraction. ▲

Converting a repeating decimal number to a quotient of integers is more difficult. To do so, we must "create" another repeating decimal number with the same repeating digits so that when one repeating decimal number is subtracted from the other repeating decimal number, the difference will be a whole number. To create a number with the same repeating digits, multiply the original repeating decimal number by 10 if one digit repeats, by 100 if two digits repeat, by 1000 if three digits repeat, and so on. Examples 8 through 10 demonstrate this procedure.

DID YOU KNOW

Where Do These Fractions Come From?

Until recently, new investors in U.S. stock markets were often puzzled by the practice of quoting share prices in fractions of a dollar instead of in dollars and cents. For example, a stock quote of $20\frac{1}{8}$ meant \$20.125.

In the late 1700s, most stock traders were merchants who were involved in foreign trade and currency exchange. The Spanish dollar coin, or piece of eight, was a widely held and stable currency used for trading both stocks and goods. With a hammer and chisel, these dollars could be divided into halves, quarters, and eighths. Eighths were also known as bits. It is from this practice that the synonym of "two bits" for 25 cents comes.

Up until recently, trading in eighths remained part of the securities industry culture. By 2002, all the stock exchanges in the United States had converted from fractional to decimal representations for share prices.

EXAMPLE 8 *Converting a Repeating Decimal Number into a Fraction*

Convert $0.\overline{3}$ to a quotient of integers.

SOLUTION: $0.\overline{3} = 0.3\overline{3} = 0.33\overline{3}$, and so on.

Let the original repeating decimal number be n; thus, $n = 0.\overline{3}$. Because one digit repeats, we multiply both sides of the equation by 10, which gives $10n = 3.\overline{3}$. Then we subtract.

$$\begin{array}{r} 10n = 3.\overline{3} \\ -\quad n = 0.\overline{3} \\ \hline 9n = 3.0 \end{array}$$

Note that $10n - n = 9n$ and $3.\overline{3} - 0.\overline{3} = 3.0$.

Next, we solve for n by dividing both sides of the equation by 9.

$$\frac{9n}{9} = \frac{3.0}{9}$$

$$n = \frac{3}{9} = \frac{1}{3}$$

Therefore, $0.\overline{3} = \frac{1}{3}$. Evaluate $1 \div 3$ on a calculator now and see what value you get.

EXAMPLE 9 *Converting a Repeating Decimal Number into a Fraction*

Convert $0.\overline{35}$ to a quotient of integers.

SOLUTION: Let $n = 0.\overline{35}$. Since two digits repeat, multiply both sides of the equation by 100. Thus, $100n = 35.\overline{35}$. Now we subtract n from $100n$.

$$\begin{array}{r} 100n = 35.\overline{35} \\ -\quad n = 0.\overline{35} \\ \hline 99n = 35 \end{array}$$

Finally, we divide both sides of the equation by 99.

$$\frac{99n}{99} = \frac{35}{99}$$

$$n = \frac{35}{99}$$

Therefore, $0.\overline{35} = \frac{35}{99}$. Evaluate $35 \div 99$ on a calculator now and see what value you get.

EXAMPLE 10 *Converting a Repeating Decimal Number into a Fraction*

Convert $12.14\overline{2}$ to a quotient of integers.

SOLUTION: This problem is different from the two preceding examples in that the repeating digit, 2, is not directly to the right of the decimal point. When this situation arises, move the decimal point to the right until the repeating terms are directly to its right. For each place the decimal point is moved, the number is multiplied by 10. In this example, the decimal point must be moved two places to the right. Thus, the number must be multiplied by 100.

$$n = 12.14\overline{2}$$
$$100n = 100 \times 12.14\overline{2} = 1214.\overline{2}$$

Now proceed as in the previous two examples. Since one digit repeats, multiply both sides by 10.

$$\begin{aligned} 100n &= 1214.\overline{2} \\ 10 \times 100n &= 10 \times 1214.\overline{2} \\ 1000n &= 12142.\overline{2} \end{aligned}$$

Now subtract $100n$ from $1000n$ so that the repeating part will drop out.

$$\begin{aligned} 1000n &= 12142.\overline{2} \\ -\ 100n &= \ \ 1214.\overline{2} \\ \hline 900n &= 10928 \\ n &= \frac{10{,}928}{900} = \frac{2732}{225} \end{aligned}$$

Therefore, $12.14\overline{2} = \frac{2732}{225}$. Evaluate $2732 \div 225$ on a calculator now and see what value you get. ▲

Multiplication and Division of Fractions

The product of two fractions is found by multiplying the numerators together and multiplying the denominators together.

Multiplication of Fractions

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}, \quad b \neq 0, \quad d \neq 0$$

EXAMPLE 11 *Multiplying Fractions*

Evaluate the following.

a) $\frac{3}{5} \cdot \frac{7}{8}$ b) $\left(\frac{-2}{3}\right)\left(\frac{-4}{9}\right)$ c) $\left(1\frac{7}{8}\right)\left(2\frac{1}{4}\right)$

SOLUTION:

a) $\frac{3}{5} \cdot \frac{7}{8} = \frac{3 \cdot 7}{5 \cdot 8} = \frac{21}{40}$

b) $\left(\frac{-2}{3}\right)\left(\frac{-4}{9}\right) = \frac{(-2)(-4)}{(3)(9)} = \frac{8}{27}$

c) $\left(1\frac{7}{8}\right)\left(2\frac{1}{4}\right) = \frac{15}{8} \cdot \frac{9}{4} = \frac{135}{32} = 4\frac{7}{32}$ ▲

The ***reciprocal*** of any number is 1 divided by that number. The product of a number and its reciprocal must equal 1. Examples of some numbers and their reciprocals follow.

Number		Reciprocal		Product
3	$\cdot$	$\frac{1}{3}$	$=$	1
$\frac{3}{5}$	$\cdot$	$\frac{5}{3}$	$=$	1
-6	$\cdot$	$-\frac{1}{6}$	$=$	1

To find the quotient of two fractions, multiply the first fraction by the reciprocal of the second fraction.

Division of Fractions

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, \quad b \neq 0, \quad d \neq 0, \quad c \neq 0$$

EXAMPLE 12 *Dividing Fractions*

Evaluate the following.

a) $\frac{2}{3} \div \frac{5}{7}$ b) $\frac{-3}{5} \div \frac{5}{7}$

SOLUTION:

a) $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{2 \cdot 7}{3 \cdot 5} = \frac{14}{15}$

b) $\frac{-3}{5} \div \frac{5}{7} = \frac{-3}{5} \cdot \frac{7}{5} = \frac{-3 \cdot 7}{5 \cdot 5} = \frac{-21}{25}$ or $-\frac{21}{25}$

Addition and Subtraction of Fractions

Before we can add or subtract fractions, the fractions must have a common denominator. A common denominator is another name for a common multiple of the denominators. The ***lowest common denominator (LCD)*** is the least common multiple of the denominators.

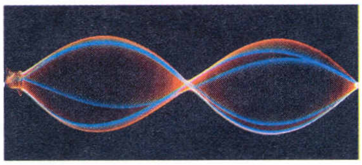
To add or subtract two fractions with a common denominator, we add or subtract their numerators and retain the common denominator.

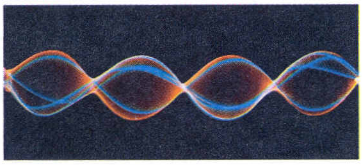
Addition and Subtraction of Fractions

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \quad c \neq 0; \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}, \quad c \neq 0$$

DID YOU KNOW

Mathematical Music





When the string of a musical instrument is plucked or bowed, it moves in a wave-like pattern like the strings shown here. The vibration this creates in the surrounding air is what your eardrums detect as sound.

The ancient Greeks believed that in nature, all harmony and everything of beauty could be explained with rational numbers. This belief was reinforced by the discovery that the sound of plucked strings could be quite pleasing if the strings plucked were in the ratio of 1 to 2 (an octave), 2 to 3 (a fifth), 3 to 4 (a fourth), and so on. Thus, the secret of harmony lies in the rational numbers such as $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$.

The theory of vibrating strings has applications today that go well beyond music. How materials vibrate, and hence the stress they can absorb, is a vital matter in the construction of rockets, buildings, and bridges.

EXAMPLE 13 *Adding and Subtracting Fractions with a Common Denominator*

Evaluate the following.

a) $\frac{3}{8} + \frac{2}{8}$

b) $\frac{15}{16} - \frac{7}{16}$

SOLUTION:

a) $\frac{3}{8} + \frac{2}{8} = \frac{3+2}{8} = \frac{5}{8}$

b) $\frac{15}{16} - \frac{7}{16} = \frac{15-7}{16} = \frac{8}{16} = \frac{1}{2}$

Note that in Example 13, the denominators of the fractions being added or subtracted were the same; that is, they have a common denominator. *When adding or subtracting two fractions with unlike denominators, first rewrite each fraction with a common denominator. Then add or subtract the fractions.*

Writing fractions with a common denominator is accomplished with the *fundamental law of rational numbers.*

Fundamental Law of Rational Numbers

If a, b, and c are integers, with $b \neq 0$ and $c \neq 0$, then

$$\frac{a}{b} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a \cdot c}{b \cdot c}$$

The terms $\frac{a}{b}$ and $\frac{a \cdot c}{b \cdot c}$ are called ***equivalent fractions***. For example, since $\frac{5}{12} = \frac{5 \cdot 5}{12 \cdot 5} = \frac{25}{60}$, the fractions $\frac{5}{12}$ and $\frac{25}{60}$ are equivalent fractions. We will see the importance of equivalent fractions in the next two examples.

EXAMPLE 14 *Subtracting Fractions with Unlike Denominators*

Evaluate $\frac{5}{12} - \frac{3}{10}$.

SOLUTION: Using prime factorization (Section 5.1), we find that the LCM of 12 and 10 is 60. We will therefore express each fraction as an equivalent fraction with a denominator of 60. Sixty divided by 12 is 5. Therefore, the denominator, 12, must be multiplied by 5 to get 60. If the denominator is multiplied by 5, the numerator must also be multiplied by 5 so that the value of the fraction remains unchanged. Multiplying both numerator and denominator by 5 is the same as multiplying by 1.

We follow the same procedure for the other fraction, $\frac{3}{10}$. Sixty divided by 10 is 6. Therefore, we multiply both the denominator, 10, and the numerator, 3, by 6 to obtain an equivalent fraction with a denominator of 60.

$$\begin{aligned}\frac{5}{12} - \frac{3}{10} &= \left(\frac{5}{12}\cdot\frac{5}{5}\right) - \left(\frac{3}{10}\cdot\frac{6}{6}\right)\\ &= \frac{25}{60} - \frac{18}{60}\\ &= \frac{7}{60}\end{aligned}$$

EXAMPLE 15 *Adding Fractions with Unlike Denominators*

Evaluate $\frac{1}{54} + \frac{1}{90}$.

SOLUTION: On page 214, in Example 6 of Section 5.1, we determined that the LCM of 54 and 90 is 270. Rewrite each fraction as an equivalent fraction using the LCM as the common denominator.

$$\begin{aligned}\frac{1}{54} + \frac{1}{90} &= \left(\frac{1}{54}\cdot\frac{5}{5}\right) + \left(\frac{1}{90}\cdot\frac{3}{3}\right)\\ &= \frac{5}{270} + \frac{3}{270}\\ &= \frac{8}{270}\end{aligned}$$

Now we reduce $\frac{8}{270}$ by dividing both 8 and 270 by 2, their greatest common factor.

$$\frac{8}{270} = \frac{8 \div 2}{270 \div 2} = \frac{4}{135}$$

EXAMPLE 16 *Rice Preparation*

Following are the instructions given on a box of Minute Rice. Determine the amount of (a) rice and water, (b) salt, and (c) butter or margarine needed to make 3 servings of rice.

Directions

1. Bring water, salt, and butter (or margarine) to a boil.
2. Stir in rice. Cover; remove from heat. Let stand 5 minutes. Fluff with fork.

To Make	Rice & Water (Equal Measures)	Salt	Butter or Margarine (If Desired)
2 servings	$\frac{2}{3}$ cup	$\frac{1}{4}$ tsp	1 tsp
4 servings	$1\frac{1}{3}$ cups	$\frac{1}{2}$ tsp	2 tsp

SOLUTION: Since 3 is halfway between 2 and 4, we can find the amount of each ingredient by finding the average of the amount for 2 and 4 servings. To do so, we add the amounts for 2 servings and 4 servings and divide the sum by 2.

a) Rice and water: $\dfrac{\frac{2}{3} + 1\frac{1}{3}}{2} = \dfrac{\frac{2}{3} + \frac{4}{3}}{2} = \dfrac{\frac{6}{3}}{2} = \dfrac{2}{2} = 1$ cup

b) Salt: $\dfrac{\frac{1}{4} + \frac{1}{2}}{2} = \dfrac{\frac{1}{4} + \frac{2}{4}}{2} = \dfrac{\frac{3}{4}}{2} = \dfrac{3}{4} \cdot \dfrac{1}{2} = \dfrac{3}{8}$ tsp

c) Butter or margarine: $\dfrac{1 + 2}{2} = \dfrac{3}{2}$, or $1\frac{1}{2}$ tsp

The solution to Example 16 can be found in other ways. Suggest two other procedures for solving the same problem.

SECTION 5.3 EXERCISES

Concept/Writing Exercises

1. Describe the set of rational numbers.
2. a) Explain how to write a terminating decimal number as a fraction.
   b) Write 0.397 as a fraction.
3. a) Explain how to reduce a fraction to lowest terms.
   b) Reduce $\frac{15}{27}$ to lowest terms by using the procedure in part (a).
4. Explain how to convert an improper fraction into a mixed number.
5. Explain how to convert a mixed number into an improper fraction.
6. a) Explain how to multiply two fractions.
   b) Multiply $\frac{15}{16} \cdot \frac{24}{25}$ by using the procedure in part (a).
7. a) Explain how to determine the reciprocal of a number.
   b) Using the procedure in part (a), determine the reciprocal of -2.
8. a) Explain how to divide two fractions.
   b) Divide $\frac{4}{15} \div \frac{16}{55}$ by using the procedure in part (a).
9. a) Explain how to add or subtract two fractions having a common denominator.
   b) Add $\frac{11}{36} + \frac{13}{36}$ by using the procedure in part (a).
   c) Subtract $\frac{37}{48} - \frac{13}{48}$ using the procedure in part (a).
10. a) Explain how to add or subtract two fractions having unlike denominators.
    b) Using the procedure in part (a), add $\frac{5}{12} + \frac{4}{9}$.
    c) Subtract $\frac{5}{6} - \frac{2}{15}$ using the procedure in part (a).
11. In your own words, state the fundamental law of rational numbers.
12. Are $\frac{4}{7}$ and $\frac{20}{35}$ equivalent fractions? Explain your answer.

Practice the Skills

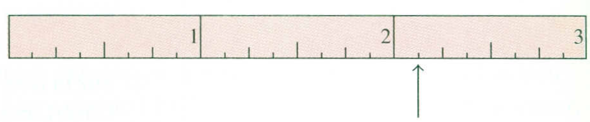
In Exercises 13–22, reduce each fraction to lowest terms.

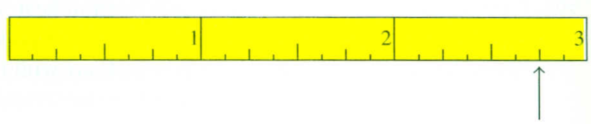
13. $\dfrac{4}{6}$
14. $\dfrac{21}{35}$
15. $\dfrac{26}{91}$
16. $\dfrac{36}{56}$
17. $\dfrac{525}{800}$
18. $\dfrac{13}{221}$
19. $\dfrac{112}{176}$
20. $\dfrac{120}{135}$
21. $\dfrac{45}{495}$
22. $\dfrac{124}{148}$

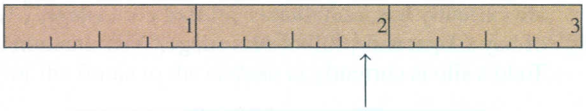
In Exercises 23–28, convert each mixed number into an improper fraction.

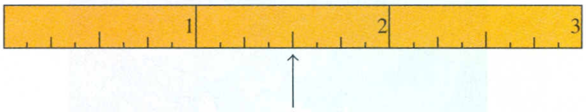
23. $3\dfrac{4}{7}$
24. $4\dfrac{5}{6}$
25. $-1\dfrac{15}{16}$
26. $-7\dfrac{1}{5}$
27. $-4\dfrac{15}{16}$
28. $11\dfrac{9}{16}$

In Exercises 29–32, write the number of inches indicated by the arrows as an improper fraction.

29.


30.


31. 

32. 

In Exercises 33–38, convert each improper fraction into a mixed number.

33. $\frac{11}{8}$ 34. $\frac{23}{4}$ 35. $-\frac{73}{6}$

36. $-\frac{457}{11}$ 37. $-\frac{878}{15}$ 38. $\frac{1028}{21}$

In Exercises 39–48, express each rational number as terminating or repeating decimal number.

39. $\frac{3}{5}$ 40. $\frac{15}{16}$ 41. $\frac{2}{9}$

42. $\frac{5}{6}$ 43. $\frac{3}{8}$ 44. $\frac{23}{7}$

45. $\frac{13}{3}$ 46. $\frac{115}{15}$ 47. $\frac{85}{15}$

48. $\frac{1002}{11}$

In Exercises 49–58, express each terminating decimal number as a quotient of two integers.

49. 0.25 50. 0.29 51. 0.045

52. 0.0125 53. 0.2 54. 0.251

55. 0.452 56. 0.2345 57. 0.0001

58. 0.2535

In Exercises 59–68, express each repeating decimal number as a quotient of two integers.

59. $0.\overline{6}$ 60. $0.\overline{5}$ 61. $1.\overline{9}$

62. $0.\overline{51}$ 63. $1.\overline{36}$ 64. $0.\overline{135}$

65. $1.0\overline{2}$ 66. $2.4\overline{9}$ 67. $3.4\overline{78}$

68. $5.2\overline{39}$

In Exercises 69–78, perform the indicated operation and reduce your answer to lowest terms.

69. $\frac{4}{11}\cdot\frac{3}{8}$ 70. $\frac{3}{5}\div\frac{6}{7}$

71. $\left(\frac{-3}{8}\right)\left(\frac{-16}{15}\right)$ 72. $\left(-\frac{3}{5}\right)\div\frac{10}{21}$

73. $\frac{7}{8}\div\frac{8}{7}$ 74. $\frac{3}{7}\div\frac{3}{7}$

75. $\left(\frac{3}{5}\cdot\frac{4}{7}\right)\div\frac{1}{3}$ 76. $\left(\frac{4}{7}\div\frac{4}{5}\right)\cdot\frac{1}{7}$

77. $\left[\left(\frac{-3}{4}\right)\left(\frac{-2}{7}\right)\right]\div\frac{3}{5}$ 78. $\left(\frac{3}{8}\cdot\frac{5}{9}\right)\cdot\left(\frac{4}{7}\div\frac{5}{8}\right)$

In Exercises 79–88, perform the indicated operation and reduce your answer to lowest terms.

79. $\frac{2}{3}+\frac{1}{5}$ 80. $\frac{5}{6}-\frac{1}{8}$

81. $\frac{5}{13}+\frac{11}{26}$ 82. $\frac{5}{12}+\frac{7}{36}$

83. $\frac{5}{9}-\frac{7}{54}$ 84. $\frac{13}{30}-\frac{17}{120}$

85. $\frac{1}{12}+\frac{1}{48}+\frac{1}{72}$ 86. $\frac{3}{5}+\frac{7}{15}+\frac{9}{75}$

87. $\frac{1}{30}-\frac{3}{40}-\frac{7}{50}$ 88. $\frac{4}{25}-\frac{9}{100}-\frac{7}{40}$

Problem Solving

Alternative methods for adding and subtracting two fractions are shown. These methods may not result in a solution in its lowest terms.

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

In Exercises 89–94, use one of the two formulas to evaluate the expression.

89. $\frac{2}{5}+\frac{7}{8}$ 90. $\frac{3}{4}+\frac{2}{9}$

91. $\frac{5}{6}-\frac{7}{8}$ 92. $\frac{7}{3}-\frac{5}{12}$

93. $\frac{3}{8}+\frac{5}{12}$ 94. $\left(\frac{2}{3}+\frac{1}{4}\right)-\frac{3}{5}$

In Exercises 95–100, evaluate each expression.

95. $\left(\frac{2}{3}\cdot\frac{9}{10}\right)+\frac{2}{5}$ 96. $\left(\frac{7}{6}\div\frac{4}{3}\right)-\frac{11}{12}$

97. $\left(\frac{1}{2}+\frac{3}{10}\right)\div\left(\frac{1}{5}+2\right)$

98. $\left(\frac{1}{9}\cdot\frac{3}{5}\right)+\left(\frac{2}{3}\cdot\frac{1}{5}\right)$

99. $\left(3 - \frac{4}{9}\right) \div \left(4 + \frac{2}{3}\right)$

100. $\left(\frac{2}{5} \div \frac{4}{9}\right)\left(\frac{3}{5} \cdot 6\right)$

In Exercises 101–114, write an expression that will solve the problem and then evaluate the expression.

101. ***Thistles*** Diane Helbing has four different varieties of thistles invading her pasture. She estimates that of these thistles, $\frac{1}{2}$ are Canada thistles, $\frac{1}{4}$ are bull thistles, $\frac{1}{6}$ are plumeless thistles, and the rest are musk thistles. What fraction of the thistles are musk thistles?

Bull Thistle Flower

102. ***Height Increase*** When David Conway finished his freshman year of high school, his height was $69\frac{7}{8}$ inches. When he returned to school after the summer, David's height was $71\frac{5}{8}$ inches. How much did David's height increase over the summer?

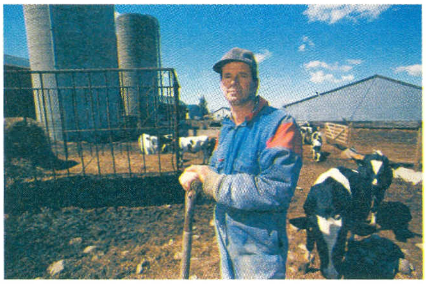
103. ***Stairway Height*** A stairway consists of 14 stairs, each $8\frac{5}{8}$ inches high. What is the vertical height of the stairway?

104. ***Math Team*** Julie Cholet is hosting the math team after school and wants to share a $67\frac{5}{8}$ oz bottle of soda. How much soda should she pour into each of six glasses so that each glass contains the same amount of soda?

105. ***Alphabet Soup*** Margaret Cannata's recipe for alphabet soup calls for (among other items) $\frac{1}{4}$ cup snipped parsley, $\frac{1}{8}$ teaspoon pepper, and $\frac{1}{2}$ cup sliced carrots. Margaret is expecting company and needs to multiply the amounts of the ingredients by $1\frac{1}{2}$ times. Determine the amount of (a) snipped parsley, (b) pepper, and (c) sliced carrots she needs for the soup.

106. ***Sprinkler System*** To repair his sprinkler system, Tony Gambino needs a total of $20\frac{5}{16}$ inches of PVC pipe. He has on hand pieces that measure $2\frac{1}{4}$ inches, $3\frac{7}{8}$ inches, and $4\frac{1}{4}$ inches in length. If he can combine these pieces and use them in the repair, how long of a piece of PVC pipe will Tony need to purchase to repair his sprinkler system?

107. ***Crop Storage*** Todd Schroeder has a silo on his farm in which he can store silage made from his various crops. He currently has a silo that is $\frac{1}{4}$ full of corn silage, $\frac{2}{5}$ full of hay silage, and $\frac{1}{3}$ full of oats silage. What fraction of Todd's silo is currently in use?

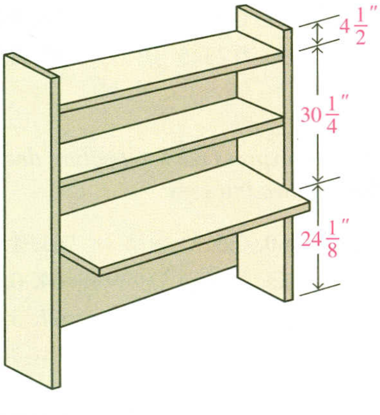


108. ***Department Budget*** Jaime Bailey is chair of the humanities department at Santa Fe Community College. Jaime has a budget in which $\frac{1}{2}$ of the money is for photocopying, $\frac{2}{5}$ of the money is for computer-related expenses, and the rest of the money is for student tutors in the foreign languages lab. What fraction of Jaime's budget is for student tutors?

109. ***Proofreading a Textbook*** To help proofread her new textbook, Chris Mishke assigns three students to proofread $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{2}$ of the book, respectively. She decides to proofread the rest of the book herself. If the book has 540 pages, how many pages must Chris proofread herself?

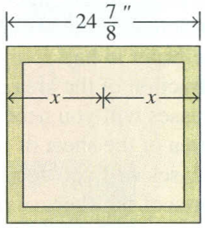
110. ***Art Supplies*** Denise Viale teaches kindergarten and is buying supplies for her class to make papier-mâché piggy banks. Each piggy bank to be made requires $1\frac{1}{4}$ cups of flour. If Denise has 15 students who are going to make piggy banks, how much flour does Denise need to purchase?

111. ***Height of a Computer Stand*** The instructions for assembling a computer stand include a diagram illustrating its dimensions. Find the total height of the stand.



112. ***Cutting Lumber*** A piece of wood measures $15\frac{3}{8}$ in.
   a) How far from one end should you cut the wood if you want to cut the length in half?
   b) What is the length of each piece after the cut? You must allow $\frac{1}{8}$ in. for the saw cut.

113. ***Width of a Picture*** The width of a picture is $24\frac{7}{8}$ in., as shown in the diagram. Find x, the distance from the edge of the frame to the center.



114. ***Floor Molding*** Rafela Weiss wants to place $\frac{1}{2}$ in. molding along the floor around the perimeter of her room (excluding door openings). She finds that she needs lengths of $26\frac{1}{2}$ in., $105\frac{1}{4}$ in., $53\frac{1}{4}$ in., and $106\frac{5}{16}$ in. How much molding will she need?

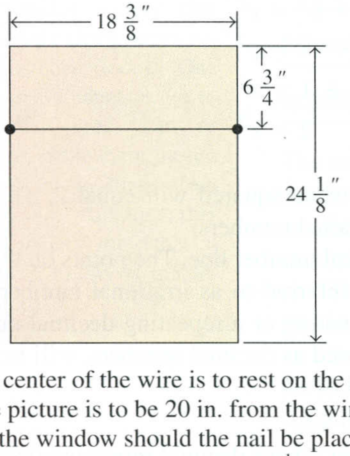
Challenge Problems/Group Activities

115. ***Cutting Lumber*** If a piece of wood $8\frac{3}{4}$ ft long is to be cut into four equal pieces, find the length of each piece. (Allow $\frac{1}{8}$ in. for each saw cut.)

116. ***Increasing a Book Size*** The dimensions of the cover of a book have been increased from $8\frac{1}{2}$ in. by $9\frac{1}{4}$ in. to $8\frac{1}{2}$ in. by $10\frac{1}{4}$ in. By how many square inches has the surface area increased? Use area = length $\times$ width.

117. ***Dimensions of a Room*** A rectangular room measures 8 ft 3 in. by 10 ft 8 in. by 9 ft 2 in. high.
   a) Determine the perimeter of the room in feet.
   b) Calculate the area of the floor of the room in square feet.
   c) Calculate the volume of the room in cubic feet.

118. ***Hanging a Picture*** The back of a framed picture that is to be hung is shown. A nail is to be hammered into the wall, and the picture will be hung by the wire on the nail.



   a) If the center of the wire is to rest on the nail and a side of the picture is to be 20 in. from the window, how far from the window should the nail be placed?
   b) If the top of the frame is to be $26\frac{1}{4}$ in. from the ceiling, how far from the ceiling should the nail be placed? (Assume the wire will not stretch.)
   c) Repeat part (b) if the wire will stretch $\frac{1}{4}$ in. when the picture is hung.

Dense Set of Numbers *A set of numbers is said to be a* **dense set** *if between any two distinct members of the set there exists a third distinct member of the set. The set of integers is not dense, since between any two consecutive integers, there is not another integer. For example, between 1 and 2 there are no other integers. The set of rational numbers is dense because between any two distinct rational numbers there exists a third distinct rational number. For example, we can find a rational number between 0.243 and 0.244. The number 0.243 can be written as 0.2430, and 0.244 can be written as 0.2440. There are many numbers between these two. Some of them are 0.2431, 0.2435, and 0.243912. In Exercises 119–126, find a rational number between the two numbers in each pair.*

119. 0.10 and 0.11
120. 5.03 and 5.003
121. -2.176 and -2.175
122. 1.3457 and 1.34571
123. 3.12345 and 3.123451
124. 0.4105 and 0.4106
125. 4.872 and 4.873
126. -3.7896 and -3.7895

Halfway Between Two Numbers *To find a rational number halfway between any two rational numbers given in fraction form, add the two numbers together and divide their sum by 2. In Exercises 127–134, find a rational number halfway between the two fractions in each pair.*

127. $\frac{1}{3}$ and $\frac{2}{3}$
128. $\frac{2}{7}$ and $\frac{3}{7}$
129. $\frac{1}{100}$ and $\frac{1}{10}$
130. $\frac{7}{13}$ and $\frac{8}{13}$
131. $\frac{1}{4}$ and $\frac{1}{5}$
132. $\frac{1}{3}$ and $\frac{2}{3}$
133. $\frac{1}{10}$ and $\frac{1}{100}$
134. $\frac{1}{2}$ and $\frac{2}{3}$

135. ***Cooking Oatmeal*** Following are the instructions given on a box of oatmeal. Determine the amount of water (or milk) and oats needed to make $1\frac{1}{2}$ servings by:
   a) Adding the amount of each ingredient needed for 1 serving to the amount needed for 2 servings and dividing by 2.
   b) Adding the amount of each ingredient needed for 1 serving to half the amount needed for 1 serving.

Directions

1. Boil water or milk and salt (if desired).
2. Stir in oats.
3. Stirring occasionally, cook over medium heat for 5 minutes.

Servings	1	2
Water (or milk)	1 cup	$1\frac{3}{4}$ cup
Oats	$\frac{1}{2}$ cup	1 cup
Salt (optional)	dash	$\frac{1}{8}$ tsp

136. Consider the rational number $0.\overline{9}$.
   a) Use the method from Example 8 on page 233 to convert $0.\overline{9}$ to a quotient of integers.
   b) Find a number halfway between $0.\overline{9}$ and 1 by adding the two numbers and dividing by 2.
   c) Find $\frac{1}{3} + \frac{2}{3}$. Express $\frac{1}{3}$ and $\frac{2}{3}$ as repeating decimals. Now find the same sum using the repeating decimal representation of $\frac{1}{3}$ and $\frac{2}{3}$.
   d) What conclusion can you draw from parts (a), (b), and (c)?

Recreational Mathematics

137. *Paper Folding* Fold a sheet of paper in half. Now unfold the paper. You will see that this one crease divided the paper into two equal regions. Each of these regions represents $\frac{1}{2}$ of the area of the entire sheet of paper. Next, fold a sheet of paper in half and then fold it in half again. Now unfold this piece of paper. You will see that these two creases divided the paper into four equal regions. Each of these regions can be considered $\frac{1}{4}$ of the area of the sheet of paper. Continue this process and answer the following questions.
   a) If you fold the paper in half three times, each region will be what fraction of the area of the sheet of paper?
   b) If you fold the paper in half four times, each region will be what fraction of the area of the sheet of paper?
   c) How many creases will you need to form regions that are $\frac{1}{32}$ of the area of the sheet of paper?
   d) How many creases will you need to form regions that are $\frac{1}{64}$ of the area of the sheet of paper?

Internet/Research Activity

138. The ancient Greeks are often considered the first true mathematicians. Write a report summarizing the ancient Greeks' contributions to rational numbers. Include in your report what they learned and believed about the rational numbers. References include encyclopedias, history of mathematics books, and Internet websites.

5.4 THE IRRATIONAL NUMBERS AND THE REAL NUMBER SYSTEM

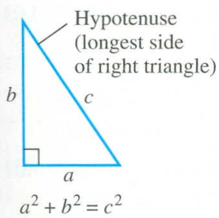


Figure 5.7

Pythagoras (ca. 585–500 B.C.), a Greek mathematician, is credited with providing a written proof that in any *right triangle* (a triangle with a 90° angle; see Fig. 5.7), the square of the length of one side (a^2) added to the square of the length of the other side (b^2) equals the square of the length of the hypotenuse (c^2). The formula $a^2 + b^2 = c^2$ is now known as the ***Pythagorean theorem***.* Pythagoras found that the solution of the formula, where $a = 1$ and $b = 1$, is not a rational number.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + 1^2 &= c^2 \\ 1 + 1 &= c^2 \\ 2 &= c^2 \end{aligned}$$

There is no rational number that when squared will equal 2. This prompted a need for a new set of numbers, the irrational numbers.

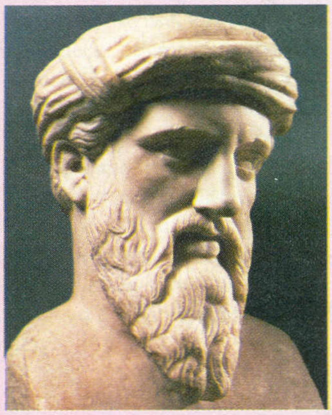
In Section 5.2, we introduced the real number line. The points on the real number line that are not rational numbers are referred to as irrational numbers. Recall that every rational number is either a terminating or a repeating decimal number. Therefore, irrational numbers, when represented as decimal numbers, will be nonterminating, nonrepeating decimal numbers.

An **irrational number** is a real number whose decimal representation is a nonterminating, nonrepeating decimal number.

*The Pythagorean theorem is discussed in more detail in Section 9.3.

PROFILE IN MATHEMATICS

PYTHAGORAS OF SAMOS



Pythagoras of Samos founded a philosophical and religious school in southern Italy in the sixth century B.C. The scholars at the school, known as Pythagoreans, produced important works of mathematics, astronomy, and theory of music. Although the Pythagoreans are credited with proving the Pythagorean theorem, it was known to the ancient Babylonians 1000 years earlier. The Pythagoreans were a secret society that formed a model for many secret societies in existence today. One practice was that students were to spend their first three years of study in silence, while their master, Pythagoras, spoke to them from behind a curtain. Among other philosophical beliefs was "that at its deepest level, reality is mathematical in nature."

A nonrepeating decimal number such as 5.12639537 . . . can be used to indicate an irrational number. Notice that no number or set of numbers repeat on a continuous basis, and the three dots at the end of the number indicate that the number continues indefinitely. Nonrepeating number patterns can be used to indicate irrational numbers. For example, 6.1011011101111 . . . and 0.525225222 . . . are both irrational numbers.

The expression $\sqrt{2}$ is read "the square root of 2" or "radical 2." The symbol $\sqrt{}$ is called the ***radical sign***, and the number or expression inside the radical sign is called the ***radicand***. In $\sqrt{2}$, 2 is the radicand.

The square roots of some numbers are rational, whereas the square roots of other numbers are irrational. The ***principal*** (or ***positive***) ***square root*** of a number n, written $\sqrt{n}$, is the positive number that when multiplied by itself, gives n. Whenever we mention the term "square root" in this text, we mean the principal square root. For example,

$$\sqrt{9} = 3 \quad \text{since} \quad 3 \cdot 3 = 9$$
$$\sqrt{36} = 6 \quad \text{since} \quad 6 \cdot 6 = 36$$

Both $\sqrt{9}$ and $\sqrt{36}$ are examples of numbers that are rational numbers because their square roots, 3 and 6 respectively, are terminating decimal numbers.

Returning to the problem faced by Pythagoras: If $c^2 = 2$, then c has a value of $\sqrt{2}$, but what is $\sqrt{2}$ equal to? The $\sqrt{2}$ is an irrational number, and it cannot be expressed as a terminating or repeating decimal number. It can only be approximated by a decimal number: $\sqrt{2}$ is approximately 1.4142135 (to seven decimal places). Later in this section, we will discuss using a calculator to approximate irrational numbers.

Other irrational numbers include $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{37}$. Another important irrational number used to represent the ratio of a circle's circumference to its diameter is pi, symbolized π. Pi is approximately 3.1415926.

We have discussed procedures for performing the arithmetic operations of addition, subtraction, multiplication, and division with rational numbers. We can perform the same operations with the irrational numbers. Before we can proceed, however, we must understand the numbers called perfect squares. Any number that is the square of a natural number is said to be a ***perfect square***.

Natural numbers	1,	2,	3,	4,	5,	6, . . .
Squares of the natural numbers	1^2,	2^2,	3^2,	4^2,	5^2,	6^2, . . .
or perfect squares	1,	4,	9,	16,	25,	36, . . .

The numbers 1, 4, 9, 16, 25, and 36 are some of the perfect square numbers. Can you determine the next two perfect square numbers? How many perfect square numbers are there? The square root of a perfect square number will be a natural number. For example, $\sqrt{1} = 1$, $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{16} = 4$, $\sqrt{25} = 5$, and so on.

The number that multiplies a radical is called the radical's ***coefficient***. For example, in $3\sqrt{5}$, the 3 is the coefficient of the radical.

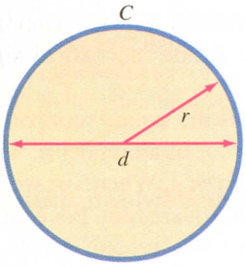
Some irrational numbers can be simplified by determining whether there are any perfect square factors in the radicand. If there are, the following rule can be used to simplify the radical.

Product Rule for Radicals

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}, \qquad a \geq 0, \qquad b \geq 0$$

DID YOU KNOW

A Piece of the Pi



$C = \pi d$
or $\pi = \dfrac{C}{d}$

The American philosopher and psychologist William James wrote in 1909: "The thousandth decimal of π sleeps there though no one may ever try to compute it." It sleeps no more. We now know π, the ratio of a circle's circumference to its diameter, to over 1.2 trillion digits.

In December 2002, Yasumasa Kanada and others at the University of Tokyo announced that they had calculated π to 1,241,100,000,000 decimal places beating their previous record set in 1999. Their computation of π consumed more than 600 hours of time on a Hitachi SR8000 supercomputer.

This record, like the record for the largest prime number, will most likely be broken in the near future (it might already be broken as you read this). Mathematicians and computer scientists continue to improve both their computers and their methods used to find numbers like the most accurate approximation for π or the largest prime number.

To simplify a radical, write the radical as a product of two radicals. One of the radicals should contain the greatest perfect square that is a factor of the radicand in the original expression. Then simplify the radical containing the perfect square factor. For example,

$$\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3 \cdot \sqrt{2} = 3\sqrt{2}$$

and

$$\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5 \cdot \sqrt{3} = 5\sqrt{3}$$

EXAMPLE 1 *Simplifying Radicals*

Simplify
a) $\sqrt{28}$ b) $\sqrt{48}$

SOLUTION:

a) Since 4 is a perfect square factor of 28, we write

$$\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4} \cdot \sqrt{7} = 2 \cdot \sqrt{7} = 2\sqrt{7}$$

Since 7 has no perfect square factors, $\sqrt{7}$ cannot be simplified.

b) Since 16 is a perfect square factor of 48, we write

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4 \cdot \sqrt{3} = 4\sqrt{3}$$

In Example 1(b), you can obtain the correct answer if you start out factoring differently:

$$\sqrt{48} = \sqrt{4 \cdot 12} = \sqrt{4} \cdot \sqrt{12} = 2\sqrt{12}$$

Note that 12 has 4 as a perfect square factor.

$$2\sqrt{12} = 2\sqrt{4 \cdot 3} = 2 \cdot \sqrt{4} \cdot \sqrt{3} = 2 \cdot 2 \cdot \sqrt{3} = 4\sqrt{3}$$

The second method will eventually give the same answer, but it requires more work. It is best to try to factor out the *largest* perfect square factor from the radicand.

Addition and Subtraction of Irrational Numbers

To add or subtract two or more square roots with the same radicand, add or subtract their coefficients. The answer is the sum or difference of the coefficients multiplied by the common radical.

EXAMPLE 2 *Adding and Subtracting Radicals with the Same Radicand*

Simplify
a) $3\sqrt{2} + 6\sqrt{2}$ b) $4\sqrt{7} + \sqrt{7} - 8\sqrt{7}$

SOLUTION:

a) $3\sqrt{2} + 6\sqrt{2} = (3 + 6)\sqrt{2} = 9\sqrt{2}$
b) $4\sqrt{7} + \sqrt{7} - 8\sqrt{7} = (4 + 1 - 8)\sqrt{7} = -3\sqrt{7}$

Note that $\sqrt{7} = 1\sqrt{7}$.

DID YOU KNOW

Is the Computer Always Right?

We often assume so, on the basis of the history and performance of the computer language and programs we use. Obviously, we cannot check calculations that are being made in billionths of a second. We could run a computation twice or on two different systems to be sure we get the same result to a specified number of decimal places. That's what is done with the big supercomputers. Their computation of the value of π often serves as a test of the program's ability to compute accurately.

EXAMPLE 3 *Adding and Subtracting Radicals with Different Radicands*

Simplify $5\sqrt{3} - \sqrt{12}$.

SOLUTION: These radicals cannot be added in their present form because they contain different radicands. When this occurs, determine whether one or more of the radicals can be simplified so that they have the same radicand.

$$\begin{aligned} 5\sqrt{3} - \sqrt{12} &= 5\sqrt{3} - \sqrt{4\cdot 3} \\ &= 5\sqrt{3} - \sqrt{4}\cdot\sqrt{3} \\ &= 5\sqrt{3} - 2\sqrt{3} \\ &= (5-2)\sqrt{3} = 3\sqrt{3} \end{aligned}$$

▲

Multiplication of Irrational Numbers

When multiplying irrational numbers, we again make use of the product rule for radicals. After the radicands are multiplied, simplify the remaining radical when possible.

EXAMPLE 4 *Multiplying Radicals*

Simplify

a) $\sqrt{3}\cdot\sqrt{27}$ b) $\sqrt{3}\cdot\sqrt{7}$ c) $\sqrt{6}\cdot\sqrt{10}$

SOLUTION:

a) $\sqrt{3}\cdot\sqrt{27} = \sqrt{3\cdot 27} = \sqrt{81} = 9$

b) $\sqrt{3}\cdot\sqrt{7} = \sqrt{3\cdot 7} = \sqrt{21}$

c) $\sqrt{6}\cdot\sqrt{10} = \sqrt{6\cdot 10} = \sqrt{60} = \sqrt{4\cdot 15} = \sqrt{4}\cdot\sqrt{15} = 2\sqrt{15}$ ▲

Division of Irrational Numbers

To divide irrational numbers, use the following rule. After performing the division, simplify when possible.

Quotient Rule for Radicals

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}, \qquad a \geq 0, \qquad b > 0$$

EXAMPLE 5 *Dividing Radicals*

Divide

a) $\dfrac{\sqrt{8}}{\sqrt{2}}$ b) $\dfrac{\sqrt{96}}{\sqrt{2}}$

SOLUTION:

a) $\dfrac{\sqrt{8}}{\sqrt{2}} = \sqrt{\dfrac{8}{2}} = \sqrt{4} = 2$

b) $\dfrac{\sqrt{96}}{\sqrt{2}} = \sqrt{\dfrac{96}{2}} = \sqrt{48} = \sqrt{16\cdot 3} = \sqrt{16}\cdot\sqrt{3} = 4\sqrt{3}$ ▲

Rationalizing the Denominator

A denominator is *rationalized* when it contains no radical expressions. To rationalize a denominator that contains only a square root, multiply both the numerator and denominator of the fraction by a number that will result in the radicand in the denominator becoming a perfect square. (This action is the equivalent of multiplying the fraction by 1 because the value of the fraction does not change.) Then simplify the fractions when possible.

EXAMPLE 6 *Rationalizing the Denominator*

Rationalize the denominator of

a) $\dfrac{5}{\sqrt{2}}$ b) $\dfrac{5}{\sqrt{12}}$ c) $\dfrac{\sqrt{5}}{\sqrt{10}}$

SOLUTION:

a) Multiply the numerator and denominator by a number that will make the radicand a perfect square.

$$\frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{\sqrt{4}} = \frac{5\sqrt{2}}{2}$$

Note that the 2's in the answer cannot be divided out because one 2 is a radicand and the other is not.

b) $\dfrac{5}{\sqrt{12}} = \dfrac{5}{\sqrt{12}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{5\sqrt{3}}{\sqrt{36}} = \dfrac{5\sqrt{3}}{6}$

You could have obtained the same answer to this problem by multiplying both the numerator and denominator by $\sqrt{12}$ and then simplifying. Try to do so now.

c) Write $\dfrac{\sqrt{5}}{\sqrt{10}}$ as $\sqrt{\dfrac{5}{10}}$ and reduce the fraction to obtain $\sqrt{\dfrac{1}{2}}$. By the quotient rule for radicals, $\sqrt{\dfrac{1}{2}} = \dfrac{\sqrt{1}}{\sqrt{2}}$ or $\dfrac{1}{\sqrt{2}}$. Now rationalize $\dfrac{1}{\sqrt{2}}$.

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

▲

Approximating Square Roots on a Scientific Calculator

Consider the irrational number the square root of two. We use the symbol $\sqrt{2}$ to represent the *exact value* of this number. Although exact values are important, approximations are also important, especially when working with application problems. We can use a scientific calculator to obtain approximations for square roots. Scientific calculators generally have one of the following square root keys:*

$\boxed{\sqrt{\ }}$ or $\boxed{\sqrt{x}}$

*If your calculator has the $\sqrt{\ }$ symbol printed *above* the key instead of on the face of the key, you can access the square root function by first pressing the "2nd" or the "inverse" key.

For simplicity, we will refer to the square root key with the $\boxed{\sqrt{\ }}$ symbol.

To approximate $\sqrt{2}$, perform the following keystrokes:

$$2 \ \boxed{\sqrt{\ }}$$

or, depending on your model of calculator, you may have to do the following:

$$\boxed{\sqrt{\ }} \ \boxed{2} \ \boxed{\text{ENTER}}$$

The display on your calculator may read 1.414213562. Your calculator may display more or fewer digits. It is important to realize that 1.414213562 is a rational number *approximation* for the irrational number $\sqrt{2}$. The symbol $\approx$ means *is approximately equal to,* and we write

$$\sqrt{2} \approx 1.414213562$$

Exact value (irrational number) Approximation (rational number)

EXAMPLE 7 *Approximating Square Roots*

Use a scientific calculator to approximate the following square roots. Round your answers to two decimal places.

a) $\sqrt{5}$ b) $\sqrt{17}$ c) $\sqrt{91}$ d) $\sqrt{237}$

SOLUTION:

a) $\sqrt{5} \approx 2.24$ b) $\sqrt{17} \approx 4.12$ c) $\sqrt{91} \approx 9.54$ d) $\sqrt{237} \approx 15.39$

SECTION 5.4 EXERCISES

Concept/Writing Exercises

1. Explain the difference between a rational number and an irrational number.
2. What is the principal square root of a number?
3. What is a perfect square?
4. a) State the product rule for radicals.
   b) State the quotient rule for radicals.
5. a) Explain how to add or subtract square roots that have the same radicand.
   b) Using the procedure in part (a), add $3\sqrt{6} + 5\sqrt{6} - 9\sqrt{6}$.
6. What does it mean to rationalize the denominator?
7. a) Explain how to rationalize a denominator that contains a square root.
   b) Using the procedure in part (a), rationalize $\frac{7}{\sqrt{3}}$.
8. a) Explain how to approximate square roots on your calculator.
   b) Using the procedure in part (a), approximate $\sqrt{7}$. Round your answer to the nearest hundredth.

Practice the Skills

In Exercises 9–18, determine whether the number is rational or irrational.

9. $\sqrt{36}$
10. $\sqrt{18}$
11. $\frac{2}{3}$
12. 0.212112111...
13. 3.575775777...
14. π
15. $\frac{22}{7}$
16. 3.14159
17. 3.14159...
18. $\frac{\sqrt{5}}{\sqrt{5}}$

In Exercises 19–28, evaluate the expression.

19. $\sqrt{64}$
20. $\sqrt{144}$
21. $\sqrt{100}$
22. $-\sqrt{144}$
23. $-\sqrt{169}$
24. $\sqrt{25}$
25. $-\sqrt{225}$
26. $-\sqrt{36}$
27. $-\sqrt{100}$
28. $\sqrt{256}$

In Exercises 29–38, classify the number as a member of one or more of the following sets: the rational numbers, the integers, the natural numbers, the irrational numbers.

29. 1
30. -4
31. $\sqrt{49}$
32. $\frac{4}{5}$
33. 0.040040004
34. 2.718
35. $-\frac{7}{8}$
36. 0.123123123
37. $0.\overline{123}$
38. 0.123112311123 . . .

In Exercises 39–48, simplify the radical.

39. $\sqrt{18}$
40. $\sqrt{20}$
41. $\sqrt{48}$
42. $\sqrt{60}$
43. $\sqrt{63}$
44. $\sqrt{75}$
45. $\sqrt{80}$
46. $\sqrt{90}$
47. $\sqrt{162}$
48. $\sqrt{300}$

In Exercises 49–58, perform the indicated operation.

49. $2\sqrt{6} + 5\sqrt{6}$
50. $3\sqrt{17} + \sqrt{17}$
51. $5\sqrt{12} - \sqrt{75}$
52. $2\sqrt{5} + 3\sqrt{20}$
53. $4\sqrt{12} - 7\sqrt{27}$
54. $2\sqrt{7} + 5\sqrt{28}$
55. $5\sqrt{3} + 7\sqrt{12} - 3\sqrt{75}$
56. $13\sqrt{2} + 2\sqrt{18} - 5\sqrt{32}$
57. $\sqrt{8} - 3\sqrt{50} + 9\sqrt{32}$
58. $\sqrt{63} + 13\sqrt{98} - 5\sqrt{112}$

In Exercises 59–68, perform the indicated operation. Simplify the answer when possible.

59. $\sqrt{2}\sqrt{8}$
60. $\sqrt{5}\sqrt{15}$
61. $\sqrt{6}\sqrt{10}$
62. $\sqrt{3}\sqrt{6}$
63. $\sqrt{10}\sqrt{20}$
64. $\sqrt{11}\sqrt{33}$
65. $\frac{\sqrt{8}}{\sqrt{4}}$
66. $\frac{\sqrt{125}}{\sqrt{5}}$
67. $\frac{\sqrt{72}}{\sqrt{8}}$
68. $\frac{\sqrt{136}}{\sqrt{8}}$

In Exercises 69–78, rationalize the denominator.

69. $\frac{1}{\sqrt{2}}$
70. $\frac{3}{\sqrt{3}}$
71. $\frac{\sqrt{3}}{\sqrt{7}}$
72. $\frac{\sqrt{3}}{\sqrt{10}}$
73. $\frac{\sqrt{20}}{\sqrt{3}}$
74. $\frac{\sqrt{50}}{\sqrt{14}}$
75. $\frac{\sqrt{9}}{\sqrt{2}}$
76. $\frac{\sqrt{15}}{\sqrt{3}}$
77. $\frac{\sqrt{10}}{\sqrt{6}}$
78. $\frac{8}{\sqrt{8}}$

Problem Solving

Approximating Radicals *The following diagram shows a 16 in. ruler marked using $\frac{1}{2}$ inches.*

In Exercises 79–84, without using a calculator, indicate between which two adjacent markers each of the following irrational numbers will fall. Explain how you obtained your answer. Support your answer by obtaining an approximation with a calculator.

79. $\sqrt{7}$ in.
80. $\sqrt{37}$ in.
81. $\sqrt{107}$ in.
82. $\sqrt{135}$ in.
83. $\sqrt{170}$ in.
84. $\sqrt{200}$ in.

In Exercises 85–90, determine whether the statement is true or false. Rewrite each false statement to make it a true statement. A false statement can be modified in more than one way to be made a true statement.

85. $\sqrt{p}$ is a rational number for any prime number p.
86. $\sqrt{c}$ is a rational number for any composite number c.
87. The sum of any two rational numbers is always a rational number.
88. The product of any two rational numbers is always a rational number.
89. The product of an irrational and a rational number is always an irrational number.
90. The product of any two irrational numbers is always an irrational number.

In Exercises 91–94, give an example to show that the stated case can occur.

91. The sum of two irrational numbers may be a rational number.
92. The sum of two irrational numbers may be an irrational number.
93. The product of two irrational numbers may be an irrational number.
94. The product of two irrational numbers may be a rational number.
95. Without doing any calculations, determine whether $\sqrt{5} = 2.236$. Explain your answer.
96. Without doing any calculations, determine whether $\sqrt{14} = 3.742$. Explain your answer.
97. The number π is an irrational number. Often the values 3.14 or $\frac{22}{7}$ are used for π. Does π equal either 3.14 or $\frac{22}{7}$? Explain your answer.
98. Give an example to show that $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.

99. Give an example to show that $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.

100. *A Swinging Pendulum* The time T required for a pendulum to swing back and forth may be found by the formula

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the length of the pendulum and g is the acceleration of gravity. Find the time in seconds if $l = 35$ cm and $g = 980$ cm/sec^2. Round answer to the nearest tenth of a second.

101. *Estimating Speed of a Vehicle* The speed a vehicle was traveling, s, in miles per hour, when the brakes were first applied, can be estimated using the formula $s = \sqrt{\frac{d}{0.04}}$ where d is the length of the vehicle's skid marks, in feet.
   a) Determine the speed of a car that made skid marks 4 ft long.
   b) Determine the speed of a car that made skid marks 16 ft long.
   c) Determine the speed of a car that made skid marks 64 ft long.
   d) Determine the speed of a car that made skid marks 256 ft long.

102. *Dropping an Object* The formula $t = \frac{\sqrt{d}}{4}$ can be used to estimate the time, t, in seconds it takes for an object dropped to travel d feet.
   a) Determine the time it takes for an object to drop 100 ft.
   b) Determine the time it takes for an object to drop 400 ft.
   c) Determine the time it takes for an object to drop 900 ft.
   d) Determine the time it takes for an object to drop 1600 ft.

Challenge Problems/Group Activities

103. a) If a radical expression is evaluated on a calculator, explain how you can determine whether the expression is a rational or irrational number.
   b) Is $\sqrt{0.04}$ rational or irrational? Explain.
   c) Is $\sqrt{0.7}$ rational or irrational? Explain.

104. One way to find a rational number between two distinct rational numbers is to add the two distinct rational numbers and divide by 2. Do you think that this method will work for finding an irrational number between two distinct irrational numbers? Explain.

Recreational Mathematics

105. *More Four 4's* In Exercise 82 on page 227, we introduced some of the basic rules of the game Four 4's. We now expand our operations to include square roots. For example, one way to obtain the whole number 8 is $\sqrt{4} + \sqrt{4} + \sqrt{4} + \sqrt{4} = 2 + 2 + 2 + 2 = 8$. Using the rules given on page 227, and using at least one square root of 4, $\sqrt{4}$, play Four 4's to obtain the following whole numbers:
   a) 11
   b) 13
   c) 14
   d) 18

Internet/Research Activities

In Exercises 106 and 107, references include history of mathematics books, encyclopedias, and Internet web sites.

106. Write a report on the history of the development of the irrational numbers.

107. Write a report on the history of pi. In your report, indicate when the symbol π was first used and list the first 10 digits of π.

5.5 REAL NUMBERS AND THEIR PROPERTIES

Now that we have discussed both the rational and irrational numbers, we can discuss the real numbers and the properties of the real number system. The union of the rational numbers and the irrational numbers is the ***set of real numbers***, symbolized by $\mathbb{R}$.

Figure 5.8 illustrates the relationship among various sets of numbers. It shows that the natural numbers are a subset of the whole numbers, the integers, the rational numbers, and the real numbers. For example, since the number 3 is a natural or counting number, it is also a whole number, an integer, a rational number, and a real number. Since the rational number $\frac{1}{4}$ is outside the set of integers, it is not an integer, a whole number, or a natural number. The number $\frac{1}{4}$ is a real number, however, as is the irrational number $\sqrt{2}$. Note that the real numbers are the union of the rational numbers and the irrational numbers.

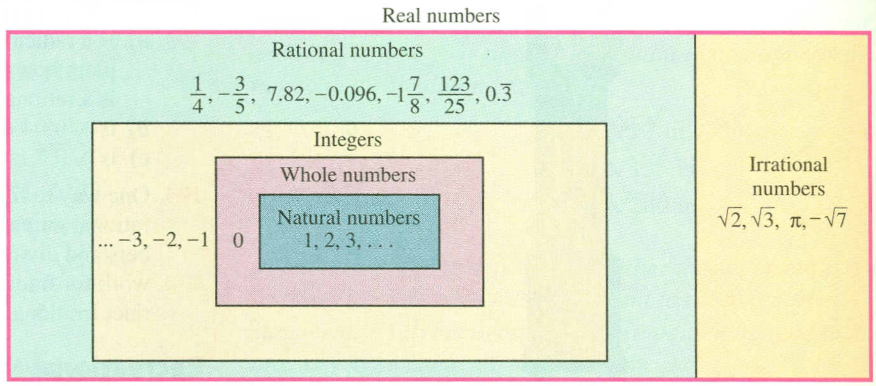


Figure 5.8

The relationship between the various sets of numbers in the real number system can also be illustrated with a tree diagram, as in Fig. 5.9.

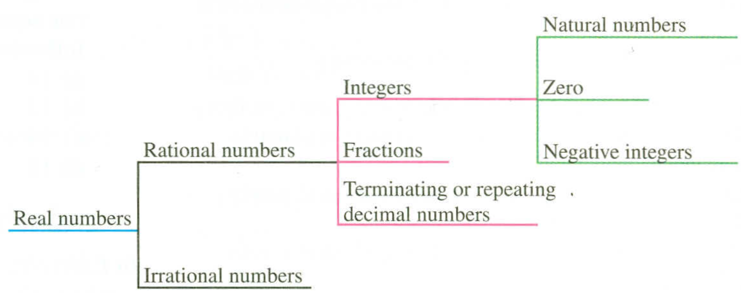


Figure 5.9

Figure 5.9 shows that, for example, the natural numbers are a subset of the integers, the rational numbers, and the real numbers. We can also see, for example, the natural numbers, zero, and the negative integers together form the integers.

Properties of the Real Number System

We are now prepared to consider the properties of the real number system. The first property that we will discuss is *closure.*

> If an operation is performed on any two elements of a set and the result is an element of the set, we say that the set is **closed** under that given operation.

DID YOU KNOW

An Important Number

Some years ago, a group of French mathematicians who worked under the collective pseudonym of "Monsieur Nicholas Bourbaki" embarked on the development of an encyclopedic description of all mathematics. They devoted 200 pages simply to introduce the innocent-looking concept, the number 1.

Is the sum of any two natural numbers a natural number? The answer is yes. Thus, we say that the natural numbers are closed under the operation of addition.

Are the natural numbers closed under the operation of subtraction? If we subtract one natural number from another natural number, must the difference always be a natural number? The answer is no. For example, $3 - 5 = -2$, which is not a natural number. Therefore, the natural numbers are not closed under the operation of subtraction.

EXAMPLE 1 *Closure of Sets*

Determine whether the integers are closed under the operations of (a) multiplication and (b) division.

SOLUTION:

a) If we multiply any two integers, will the product always be an integer? The answer is yes. Thus, the integers are closed under the operation of multiplication.

b) If we divide any two integers, will the quotient always be an integer? The answer is no. For example, $6 \div 5 = \frac{6}{5}$, which is not an integer. Therefore, the integers are not closed under the operation of division. ▲

Next we will discuss three important properties: the commutative property, the associative property, and the distributive property. A knowledge of these properties is essential for the understanding of algebra. We begin with the commutative property.

Commutative Property

ADDITION	MULTIPLICATION
$a + b = b + a$	$a \cdot b = b \cdot a$

for any real numbers a and b.

The commutative property states that the *order* in which two numbers are added or multiplied is not important. For example, $4 + 5 = 5 + 4 = 9$ and $3 \cdot 6 = 6 \cdot 3 = 18$. Note that the commutative property does not hold for the operations of subtraction or division. For example,

$$4 - 7 \neq 7 - 4 \qquad \text{and} \qquad 9 \div 3 \neq 3 \div 9$$

Now we introduce the associative property.

Associative Property

ADDITION	MULTIPLICATION
$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

for any real numbers a, b, and c.

The associative property states that when adding or multiplying three real numbers, we may place parentheses around any two adjacent numbers. For example,

$$\begin{array}{cc} (3 + 4) + 5 = 3 + (4 + 5) & (3 \cdot 4) \cdot 5 = 3 \cdot (4 \cdot 5) \\ 7 + 5 = 3 + 9 & 12 \cdot 5 = 3 \cdot 20 \\ 12 = 12 & 60 = 60 \end{array}$$

The associative property does not hold for the operations of subtraction and division. For example,

$$(10 - 6) - 2 \neq 10 - (6 - 2) \qquad \text{and} \qquad (27 \div 9) \div 3 \neq 27 \div (9 \div 3)$$

Note the difference between the commutative property and the associative property. The commutative property involves a change in *order*, whereas the associative property involves a change in *grouping* (or the *association* of numbers that are grouped together).

Another property of the real numbers is the distributive property of multiplication over addition.

Distributive Property of Multiplication over Addition

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

for any real numbers a, b, and c.

For example, if $a = 3$, $b = 4$, and $c = 5$, then

$$\begin{aligned} 3 \cdot (4 + 5) &= (3 \cdot 4) + (3 \cdot 5) \\ 3 \cdot 9 &= 12 + 15 \\ 27 &= 27 \end{aligned}$$

This result indicates that, when using the distributive property, you may either add first and then multiply or multiply first and then add. Note that the distributive property involves two operations, addition and multiplication. Although positive integers were used in the example, any real numbers could have been used.

We frequently use the commutative, associative, and distributive properties without realizing that we are doing so. To add $13 + 4 + 6$, we may add the $4 + 6$ first to get 10. To this sum we then add 13 to get 23. Here we have done the equivalent of placing parentheses around the $4 + 6$. We can do so because of the associative property of addition.

To multiply 102×11 in our heads, we might multiply $100 \times 11 = 1100$ and $2 \times 11 = 22$ and add these two products to get 1122. We are permitted to do so because of the distributive property.

$$\begin{aligned} 102 \times 11 = (100 + 2) \times 11 &= (100 \times 11) + (2 \times 11) \\ &= 1100 + 22 = 1122 \end{aligned}$$

EXAMPLE 2 *Identifying Properties of Real Numbers*

Name the property illustrated.

a) $2 + 5 = 5 + 2$

b) $(x + 3) + 5 = x + (3 + 5)$

c) $4 \cdot (3 \cdot y) = (4 \cdot 3) \cdot y$

d) $9(w + 3) = 9 \cdot w + 9 \cdot 3$
e) $5 + (z + 3) = 5 + (3 + z)$
f) $(2p) \cdot 5 = 5 \cdot (2p)$

SOLUTION:

a) Commutative property of addition
b) Associative property of addition
c) Associative property of multiplication
d) Distributive property of multiplication over addition
e) The only change between the left and right sides of the equal sign is the order of the z and 3 within the parentheses. The order is changed from $z + 3$ to $3 + z$ using the commutative property of addition.
f) The order of 5 and $(2p)$ is changed by using the commutative property of multiplication.

EXAMPLE 3 *Simplifying by Using the Distributive Property*

Use the distributive property to simplify

a) $2(3 + \sqrt{5})$ b) $\sqrt{2}(7 + \sqrt{3})$

SOLUTION:

a) $$2(3 + \sqrt{5}) = (2 \cdot 3) + (2 \cdot \sqrt{5}) \\ = 6 + 2\sqrt{5}$$

b) $$\sqrt{2}(7 + \sqrt{3}) = (\sqrt{2} \cdot 7) + (\sqrt{2} \cdot \sqrt{3}) \\ = 7\sqrt{2} + \sqrt{6}$$

Note that $\sqrt{2} \cdot 7$ is written $7\sqrt{2}$.

EXAMPLE 4 *Distributive Property*

Use the distributive property to multiply $5(q + 7)$. Then simplify the result.

SOLUTION:

$$5(q + 7) = 5 \cdot q + 5 \cdot 7 \\ = 5q + 35$$

We summarize the properties mentioned in this section as follows, where a, b, and c are any real numbers.

Commutative property of addition	$a + b = b + a$
Commutative property of multiplication	$a \cdot b = b \cdot a$
Associative property of addition	$(a + b) + c = a + (b + c)$
Associative property of multiplication	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Distributive property of multiplication over addition	$a \cdot (b + c) = a \cdot b + a \cdot c$

SECTION 5.5 EXERCISES

Concept/Writing Exercises

1. What are the real numbers?
2. What symbol is used to represent the set of real numbers?
3. What does it mean if a set is closed under a given operation?
4. Give the commutative property of multiplication, explain what it means, and give an example illustrating it.
5. Give the commutative property of addition, explain what it means, and give an example illustrating it.
6. Give the associative property of addition, explain what it means, and give an example illustrating it.
7. Give the associative property of multiplication, explain what it means, and give an example illustrating it.
8. Give the distributive property of multiplication over addition, explain what it means, and give an example illustrating it.

Practice the Skills

In Exercises 9–12, determine whether the natural numbers are closed under the given operation.

9. Addition
10. Subtraction
11. Division
12. Multiplication

In Exercises 13–16, determine whether the integers are closed under the given operation.

13. Subtraction
14. Addition
15. Division
16. Multiplication

In Exercises 17–20, determine whether the rational numbers are closed under the given operation.

17. Addition
18. Subtraction
19. Multiplication
20. Division

In Exercises 21–24, determine whether the irrational numbers are closed under the given operation.

21. Addition
22. Subtraction
23. Multiplication
24. Division

In Exercises 25–28, determine whether the real numbers are closed under the given operation.

25. Addition
26. Subtraction
27. Division
28. Multiplication

29. Does $x + (3 + 4) = (3 + 4) + x$ illustrate the commutative property or the associative property? Explain your answer.
30. Does $4 + (5 + 6) = 4 + (6 + 5)$ illustrate the commutative property or the associative property? Explain your answer.
31. Give an example to show that the commutative property of multiplication may be true for the negative integers.
32. Give an example to show that the commutative property of addition may be true for the negative integers.
33. Does the commutative property hold for the rational numbers under the operation of division? Give an example to support your answer.
34. Does the commutative property hold for the integers under the operation of subtraction? Give an example to support your answer.
35. Give an example to show that the associative property of multiplication may be true for the negative integers.
36. Give an example to show that the associative property of addition may be true for the negative integers.
37. Does the associative property hold for the integers under the operation of division? Give an example to support your answer.
38. Does the associative property hold for the integers under the operation of subtraction? Give an example to support your answer.
39. Does the associative property hold for the real numbers under the operation of division? Give an example to support your answer.
40. Does $a + (b \cdot c) = (a + b) \cdot (a + c)$? Give an example to support your answer.

In Exercises 41–56, state the name of the property illustrated.

41. $24 + 7 = 7 + 24$
42. $5(x + 3) = 5 \cdot x + 5 \cdot 3$
43. $(7 \cdot 4) \cdot 5 = 7 \cdot (4 \cdot 5)$
44. $v + w = w + v$
45. $(24 + 7) + 3 = 24 + (7 + 3)$
46. $4 \cdot (11 \cdot x) = (4 \cdot 11) \cdot x$
47. $\sqrt{3} \cdot 7 = 7 \cdot \sqrt{3}$
48. $\frac{3}{8} + \left(\frac{1}{8} + \frac{3}{2}\right) = \left(\frac{3}{8} + \frac{1}{8}\right) + \frac{3}{2}$
49. $8 \cdot (7 + \sqrt{2}) = 8 \cdot 7 + 8 \cdot \sqrt{2}$
50. $\sqrt{5} \cdot \frac{2}{3} = \frac{2}{3} \cdot \sqrt{5}$
51. $(1 + 10) + 100 = (10 + 1) + 100$

52. $(r + s) + t = t + (r + s)$
53. $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$
54. $g \cdot (h + i) = (h + i) \cdot g$
55. $(p + q) + (r + s) = (r + s) + (p + q)$
56. $(a \cdot b) + (c \cdot d) = (b \cdot a) + (c \cdot d)$

In Exercises 57–68, use the distributive property to multiply. Then, if possible, simplify the resulting expression.

57. $2(c + 7)$
58. $-3(d - 1)$
59. $\frac{2}{3}(x - 6)$
60. $-\frac{5}{8}(k + 8)$
61. $6\left(\frac{x}{2} + \frac{2}{3}\right)$
62. $24\left(\frac{x}{3} - \frac{1}{8}\right)$
63. $32\left(\frac{1}{16}x - \frac{1}{32}\right)$
64. $15\left(\frac{2}{3}x - \frac{4}{5}\right)$
65. $3(5 - \sqrt{5})$
66. $-7(2 + \sqrt{11})$
67. $\sqrt{2}(\sqrt{2} + \sqrt{3})$
68. $\sqrt{3}(\sqrt{15} + \sqrt{21})$

In Exercises 69–74, name the property used to go from step to step. You only need to name the properties indicated by an a), b), c), or d). For example, in Exercise 69, you need to supply an answer for part a), and an answer to go from the second step of part a) to part b).

69. a) $7(x + 2) + 3 = (7 \cdot x + 7 \cdot 2) + 3$
$= (7x + 14) + 3$
b) $= 7x + (14 + 3)$
$= 7x + 17$

70. a) $3(n + 5) + 6 = (3 \cdot n + 3 \cdot 5) + 6$
$= (3n + 15) + 6$
b) $= 3n + (15 + 6)$
$= 3n + 21$

71. a) $7(k + 1) + 2k = (7 \cdot k + 7 \cdot 1) + 2k$
$= (7k + 7) + 2k$
b) $= 7k + (7 + 2k)$
c) $= 7k + (2k + 7)$
d) $= (7k + 2k) + 7$
$= 9k + 7$

72. a) $11(h + 6) + 5h = (11 \cdot h + 11 \cdot 6) + 5h$
$= (11h + 66) + 5h$
b) $= 11h + (66 + 5h)$
c) $= 11h + (5h + 66)$
d) $= (11h + 5h) + 66$
$= 16h + 66$

73. a) $9 + 2(t + 3) + 7t = 9 + (2 \cdot t + 2 \cdot 3) + 7t$
$= 9 + (2t + 6) + 7t$
b) $= 9 + (6 + 2t) + 7t$
c) $= (9 + 6) + 2t + 7t$
$= 15 + 9t$
d) $= 9t + 15$

74. a) $7 + 5(s + 4) + 3s = 7 + (5 \cdot s + 5 \cdot 4) + 3s$
$= 7 + (5s + 20) + 3s$
b) $= 7 + (20 + 5s) + 3s$
c) $= (7 + 20) + 5s + 3s$
$= 27 + 8s$
d) $= 8s + 27$

In Exercises 75–80, determine whether the activity can be used to illustrate the commutative property. For the property to hold, the end result must be identical, regardless of the order in which the actions are performed.

75. Putting on your seat belt and locking your car door

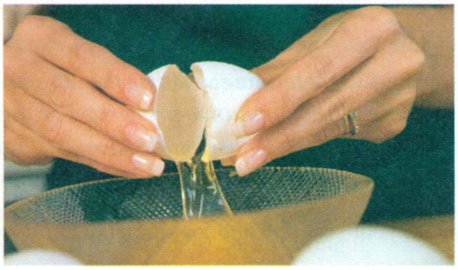
76. Putting on your left shoe and putting on your right shoe
77. Washing clothes and drying clothes
78. Turning on a computer and typing a term paper on the computer
79. Filling your car with gasoline and washing the windshield
80. Turning on a lamp and reading a book

In Exercises 81–88, determine whether the activity can be used to illustrate the associative property. For the property to hold, doing the first two actions followed by the third would produce the same end result as doing the second and third actions followed by the first.

81. Washing the exterior of your car, vacuuming out the interior, and checking the oil
82. Reading a novel, writing a book report on the novel, and making a presentation to your class about your book report
83. Sending a holiday card to your grandmother, sending one to your parents, and sending one to your teacher
84. Mowing the lawn, trimming the bushes, and removing dead limbs from trees

85. Brushing your teeth, washing your face, and combing your hair

86. Cracking an egg, pouring out the egg, and cooking the egg



87. Taking a bath, brushing your teeth, and taking your vitamins

88. While making meatloaf, mixing in the milk, mixing in the spices, and mixing in the bread crumbs

Challenge Problems/Group Activities

89. Describe two other activities that can be used to illustrate the commutative property (see Exercises 75–80).

90. Describe three other activities that can be used to illustrate the associative property (see Exercises 81–88).

91. Does $0 \div a = a \div 0$ (assume $a \neq 0$)? Explain.

Recreational Mathematics

92. **a)** Consider the three words *man eating tiger.* Does (*man eating*) *tiger* mean the same as *man* (*eating tiger*)?
  b) Does (*horse riding*) *monkey* mean the same as *horse* (*riding monkey*)?
  c) Can you find three other nonassociative word triples?

Internet/Research Activity

93. A set of numbers that was not discussed in this chapter is the set of *complex numbers.* Write a report on complex numbers. Include their relationship to the real numbers.

5.6 RULES OF EXPONENTS AND SCIENTIFIC NOTATION

An understanding of exponents is important in solving problems in algebra. In the expression 5^2, the 2 is referred to as the ***exponent*** and the 5 is referred to as the ***base***. We read 5^2 as 5 to the second power, or 5 squared, which means

$$5^2 = \underbrace{5 \cdot 5}_{\text{2 factors of 5}}$$

The number 5 to the third power, or 5 cubed, written 5^3, means

$$5^3 = \underbrace{5 \cdot 5 \cdot 5}_{\text{3 factors of 5}}$$

In general, the number b to the nth power, written b^n, means

$$b^n = \underbrace{b \cdot b \cdot b \cdot \cdots \cdot b}_{n \text{ factors of } b}$$

EXAMPLE 1 *Evaluating the Power of a Number*

Evaluate the following.

a) 4^2 b) $(-5)^2$ c) 5^3 d) 1^{1000} e) 7^1

SOLUTION:

a) $4^2 = 4 \cdot 4 = 16$

b) $(-5)^2 = (-5)(-5) = 25$

c) $5^3 = 5 \cdot 5 \cdot 5 = 125$

d) $1^{1000} = 1$. (The number 1 times itself any number of times equals 1.)

e) $7^1 = 7$. (Any number with an exponent of 1 equals the number itself.)

EXAMPLE 2 *The Importance of Parentheses*

Evaluate the following.

a) $(-2)^4$ b) -2^4 c) $(-2)^5$ d) -2^5

SOLUTION:

a) $(-2)^4 = (-2)(-2)(-2)(-2) = 4(-2)(-2) = -8(-2) = 16$

b) -2^4 means take the opposite of 2^4 or $-1 \cdot 2^4$.

$-1 \cdot 2^4 = -1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = -1 \cdot 16 = -16$

c) $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = 4(-2)(-2)(-2) = -8(-2)(-2)$
$= 16(-2) = -32$

d) $-2^5 = -1 \cdot 2^5 = -1 \cdot 32 = -32$

From Example 2, we can see that $(-x)^n \neq -x^n$, where n is an even natural number.

Rules of Exponents

Now that we know how to evaluate powers of numbers we can discuss the rules of exponents. Consider

$$2^2 \cdot 2^3 = \underbrace{2 \cdot 2}_{\text{2 factors}} \cdot \underbrace{2 \cdot 2 \cdot 2}_{\text{3 factors}} = 2^5$$

This example illustrates the product rule for exponents.

Product Rule for Exponents

$$a^m \cdot a^n = a^{m+n}$$

Therefore, by using the product rule, $2^2 \cdot 2^3 = 2^{2+3} = 2^5$.

EXAMPLE 3 *Using the Product Rule for Exponents*

Use the product rule to simplify.

a) $3^4 \cdot 3^5$ b) $7^2 \cdot 7^6$

SOLUTION:

a) $3^4 \cdot 3^5 = 3^{4+5} = 3^9$ b) $7^2 \cdot 7^6 = 7^{2+6} = 7^8$

DID YOU KNOW

A Very Large Number

170,141,183,460,469,231,731,687, 303,715,884,105,727 is a very large number. How would you read this number? Take a breath.

170 undecillion, 141 decillion, 183 nonillion, 460 octillion, 469 septillion, 231 sextillion, 731 quintillion, 687 quadrillion, 303 trillion, 715 billion, 884 million, 105 thousand, 727.

Often numbers this large can be represented with an approximation involving scientific notation. This number, however, is a prime number, and its *exact* representation is very important. Mathematicians frequently represent such large numbers by using exponents. Here, another exact, and more efficient, representation of this number is $2^{127} - 1$.

Consider

$$\frac{2^5}{2^2} = \frac{2 \cdot 2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2}} = 2 \cdot 2 \cdot 2 = 2^3$$

This example illustrates the quotient rule for exponents.

Quotient Rule for Exponents

$$\frac{a^m}{a^n} = a^{m-n}, \qquad a \neq 0$$

Therefore, $\frac{2^5}{2^2} = 2^{5-2} = 2^3$.

EXAMPLE 4 *Using the Quotient Rule for Exponents*

Use the quotient rule to simplify.

a) $\frac{5^8}{5^5}$ b) $\frac{8^{12}}{8^5}$

SOLUTION:

a) $\frac{5^8}{5^5} = 5^{8-5} = 5^3$ b) $\frac{8^{12}}{8^5} = 8^{12-5} = 8^7$

Consider $2^3 \div 2^3$. The quotient rule gives

$$\frac{2^3}{2^3} = 2^{3-3} = 2^0$$

But $\frac{2^3}{2^3} = \frac{8}{8} = 1$. Therefore, 2^0 must equal 1. This example illustrates the zero exponent rule.

Zero Exponent Rule

$$a^0 = 1, \qquad a \neq 0$$

Note that 0^0 is not defined by the zero exponent rule.

EXAMPLE 5 *The Zero Power*

Use the zero exponent rule to simplify.

a) 2^0 b) $(-2)^0$ c) -2^0 d) $(5x)^0$ e) $5x^0$

SOLUTION:

a) $2^0 = 1$ b) $(-2)^0 = 1$ c) $-2^0 = -1 \cdot 2^0 = -1 \cdot 1 = -1$

d) $(5x)^0 = 1$ e) $5x^0 = 5 \cdot x^0 = 5 \cdot 1 = 5$

Consider $2^3 \div 2^5$. The quotient rule yields

$$\frac{2^3}{2^5} = 2^{3-5} = 2^{-2}$$

But $\frac{2^3}{2^5} = \frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2} = \frac{1}{2^2}$. Since $\frac{2^3}{2^5}$ equals both 2^{-2} and $\frac{1}{2^2}$, then 2^{-2} must equal $\frac{1}{2^2}$. This example illustrates the negative exponent rule.

Negative Exponent Rule

$$a^{-m} = \frac{1}{a^m}, \qquad a \neq 0$$

EXAMPLE 6 *Using the Negative Exponent Rule*

Use the negative exponent rule to simplify.

a) 5^{-2} b) 8^{-1}

SOLUTION:

a) $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$ b) $8^{-1} = \frac{1}{8^1} = \frac{1}{8}$

Consider $(2^3)^2$.

$$(2^3)^2 = (2^3)(2^3) = 2^{3+3} = 2^6$$

This example illustrates the power rule for exponents.

Power Rule for Exponents

$$(a^m)^n = a^{m \cdot n}$$

Thus, $(2^3)^2 = 2^{3\cdot 2} = 2^6$.

EXAMPLE 7 *Evaluating a Power Raised to Another Power*

Use the power rule to simplify.

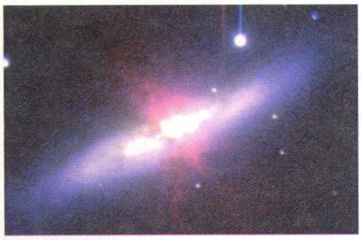
a) $(5^4)^3$ b) $(7^2)^5$

SOLUTION:

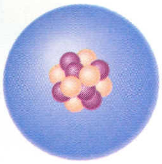
a) $(5^4)^3 = 5^{4\cdot 3} = 5^{12}$ b) $(7^2)^5 = 7^{2\cdot 5} = 7^{10}$

DID YOU KNOW

Large and Small Numbers



Diameter of a galaxy may be 1×10^5 light-years



Diameter of an atom may be 1×10^{-10} meter

Our everyday activities don't require us to deal with quantities much above those in the thousands: \$6.95 for lunch, 100 meters to a lap, a \$15,000 car loan, and so on. Yet as modern technology has developed, so has our ability to study all aspects of the universe we live in, from the very large to the very small. Modern technology can be used with the rules of exponents and scientific notation to study everything from the diameter of a galaxy to the diameter of an atom.

Summary of the Rules of Exponents

$a^m \cdot a^n = a^{m+n}$	Product rule for exponents
$\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	Quotient rule for exponents
$a^0 = 1, \quad a \neq 0$	Zero exponent rule
$a^{-m} = \dfrac{1}{a^m}, \quad a \neq 0$	Negative exponent rule
$(a^m)^n = a^{m \cdot n}$	Power rule for exponents

Scientific Notation

Often scientific problems deal with very large and very small numbers. For example, the distance from Earth to the sun is about 93,000,000 miles. The wavelength of a yellow color of light is about 0.0000006 meter. Because working with many zeros is difficult, scientists developed a notation that expresses such numbers with exponents. For example, consider the distance from Earth to the sun, 93,000,000 miles.

$$\begin{aligned} 93{,}000{,}000 &= 9.3 \times 10{,}000{,}000 \\ &= 9.3 \times 10^7 \end{aligned}$$

The wavelength of a yellow color of light is about 0.0000006 meter.

$$\begin{aligned} 0.0000006 &= 6.0 \times 0.0000001 \\ &= 6.0 \times 10^{-7} \end{aligned}$$

The numbers 9.3×10^7 and 6.0×10^{-7} are written in a form called ***scientific notation***. Each number written in scientific notation is written as a number greater than or equal to 1 and less than 10 multiplied by some power of 10.

Some examples of numbers in scientific notation are

$$3.7 \times 10^3, \qquad 2.05 \times 10^{-3}, \qquad 5.6 \times 10^8, \qquad \text{and} \qquad 1.00 \times 10^{-5}$$

The following is a procedure for writing a number in scientific notation.

To Write a Number in Scientific Notation

1. Move the decimal point in the original number to the right or left until you obtain a number greater than or equal to 1 and less than 10.
2. Count the number of places you have moved the decimal point to obtain the number in step 1. If the decimal point was moved to the left, the count is to be considered positive. If the decimal point was moved to the right, the count is to be considered negative.
3. Multiply the number obtained in step 1 by 10 raised to the count found in step 2. (Note that the count determined in step 2 is the exponent on the base 10.)

DID YOU KNOW

Catalan's Conjecture

Eugène Catalan

Preda Mihailescu

Consider all the squares of positive integers:

$1^2 = 1,\quad 2^2 = 4,\quad 3^2 = 9,$
$4^2 = 16,\quad$ and so on

Now consider all the cubes of positive integers:

$1^3 = 1,\quad 2^3 = 8,\quad 3^3 = 27,$
$4^3 = 64,\quad$ and so on

Next, consider all the fourth powers of positive integers:

$1^4 = 1,\quad 2^4 = 16,\quad 3^4 = 81,$
$4^4 = 256,\quad$ and so on

And so on. If we were to put *all* powers of the positive integers into one set, the set would begin as follows: $\{1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, \ldots\}$. In 1844, Belgian mathematician Eugène Catalan proposed that of all the numbers in this infinite set, only 8 and 9 are consecutive integers. Although this conjecture was easily stated, the formal proof of it eluded mathematicians for over 150 years. In April 2002, Preda Mihailescu announced that he had proved the Catalan conjecture.

EXAMPLE 8 *Converting from Decimal Notation to Scientific Notation*

Write each number in scientific notation.

a) In 2002, the population of the United States was about 288,000,000.
b) In 2002, the population of China was about 1,283,000,000.
c) In 2002, the population of the world was about 6,251,000,000.
d) The diameter of a hydrogen atom nucleus is about 0.0000000000011 millimeter.
e) The wavelength of an x-ray is about 0.000000000492 meter.

SOLUTION

a) $288{,}000{,}000 = 2.88 \times 10^8$
b) $1{,}283{,}000{,}000 = 1.283 \times 10^9$
c) $6{,}251{,}000{,}000 = 6.251 \times 10^9$
d) $0.0000000000011 = 1.1 \times 10^{-12}$
e) $0.000000000492 = 4.92 \times 10^{-10}$

To convert from a number given in scientific notation to decimal notation we reverse the procedure.

To Change a Number in Scientific Notation to Decimal Notation

1. Observe the exponent on the 10.
2. a) If the exponent is positive, move the decimal point in the number to the right the same number of places as the exponent. Adding zeros to the number might be necessary.
   b) If the exponent is negative, move the decimal point in the number to the left the same number of places as the exponent. Adding zeros might be necessary.

EXAMPLE 9 *Converting from Scientific Notation to Decimal Notation*

Write each number in decimal notation.

a) The average distance from Earth to the sun is about 9.3×10^7 miles.
b) The half-life of uranium 235 is about 4.5×10^9 years.
c) The average grain size in siltstone is 1.35×10^{-3} inch.
d) A *millimicron* is a unit of measure used for very small distances. One millimicron is about 3.94×10^{-8} inch.

SOLUTION:

a) $9.3 \times 10^7 = 93{,}000{,}000$
b) $4.5 \times 10^9 = 4{,}500{,}000{,}000$
c) $1.35 \times 10^{-3} = 0.00135$
d) $3.94 \times 10^{-8} = 0.0000000394$

In scientific journals and books, we occasionally see numbers like 10^{15} and 10^{-6}. We interpret these numbers as 1×10^{15} and 1×10^{-6}, respectively, when converting the numbers to decimal form.

EXAMPLE 10 *Multiplying Numbers in Scientific Notation*

Multiply $(2.1 \times 10^5)(9 \times 10^{-3})$. Write the answer in scientific notation and in decimal notation.

SOLUTION:

$$\begin{aligned}(2.1 \times 10^5)(9 \times 10^{-3}) &= (2.1 \times 9)(10^5 \times 10^{-3})\\ &= 18.9 \times 10^2\\ &= 1.89 \times 10^3 &&\text{Scientific notation}\\ &= 1{,}890 &&\text{Decimal notation}\end{aligned}$$

EXAMPLE 11 *Dividing Numbers Using Scientific Notation*

Divide $\dfrac{0.000000000048}{24{,}000{,}000{,}000}$. Write the answer in scientific notation.

SOLUTION: First write each number in scientific notation.

$$\begin{aligned}\frac{0.000000000048}{24{,}000{,}000{,}000} = \frac{4.8 \times 10^{-11}}{2.4 \times 10^{10}} &= \left(\frac{4.8}{2.4}\right)\left(\frac{10^{-11}}{10^{10}}\right)\\ &= 2.0 \times 10^{-11-10}\\ &= 2.0 \times 10^{-21}\end{aligned}$$

Scientific Notation on the Scientific Calculator

One of the advantages of using scientific notation when working with very large and very small numbers is the ease with which you can perform operations. Performing these operations is even easier with the use of a scientific calculator. Most scientific calculators have a scientific notation key labeled "Exp," "EXP," or "EE." We will refer to the scientific notation key as $\boxed{\text{EXP}}$. The following keystrokes can be used to enter the number 4.3×10^6

Keystroke(s)	Calculator display
4.3	4.3
$\boxed{\text{EXP}}$	4.3^{00}
6	4.3^{06}

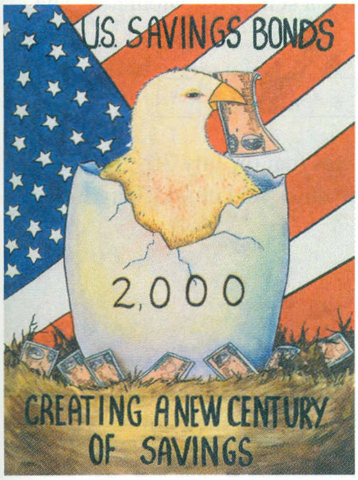
Your calculator may have some slight variations to the display shown here. The display 4.3^{06} means 4.3×10^6. We now will use our calculators to perform some computations using scientific notation.

EXAMPLE 12 *Use Scientific Notation on a Calculator to Find a Product*

Multiply $(4.3 \times 10^6)(2 \times 10^{-4})$ using a scientific calculator. Write the answer in decimal notation.

DID YOU KNOW

What's the Difference Between Debt and Deficit?



We often hear economists and politicians talk about such things as revenue, expenditures, deficit, surplus, and national debt. Revenue is the money the government collects annually, mostly through taxes. Expenditures are the money the government spends annually. If revenue exceeds expenditures, a surplus occurs; if expenditures exceed revenue, a deficit occurs. The national debt is the total of all the budget deficits and (the few) surpluses encountered by the federal government for over 200 years. How big is the national debt? Search the Internet for the latest figures. As of July 30, 2003, the national debt was $6,737,879,490,363.90. This amount is owed to investors worldwide—perhaps including you—who own U.S. government bonds.

SOLUTION: Our sequence of keystrokes is as follows:

Keystroke(s)	Display	
4.3	4.3	
$\boxed{\text{EXP}}$	4.3^{00}	
6	4.3^{06}	
$\boxed{\times}$	4300000	4.3×10^6 is now entered in the calculator. Most calculators will convert to decimal notation (if the display can show the number)
2	2.	
$\boxed{\text{EXP}}$	$2.^{00}$	
4	$2.^{04}$	Enter the positive form of the exponent
$\boxed{+/-}$	$2.^{-04}$	Make the exponent negative*
$\boxed{=}$	860.	Press $\boxed{=}$ to obtain the answer of 860†

EXAMPLE 13 *Use Scientific Notation on a Calculator to Find a Quotient*

Divide $\dfrac{0.000000000048}{24{,}000{,}000{,}000}$ using a scientific calculator. Write the answer in scientific notation.

SOLUTION: We first rewrite the numerator and denominator using scientific notation. See Example 11.

$$\frac{0.000000000048}{24{,}000{,}000{,}000} = \frac{4.8 \times 10^{-11}}{2.4 \times 10^{10}}$$

Next we use a scientific calculator to perform the computation. The keystrokes are as follows

$$4.8 \ \boxed{\text{EXP}} \ 11 \ \boxed{+/-} \ \boxed{\div} \ 2.4 \ \boxed{\text{EXP}} \ 10 \ \boxed{=}$$

The display on the calculator is $2.^{-21}$, which means 2.0×10^{-21}.

EXAMPLE 14 *U.S. Debt per Person*

On July 30, 2003, the U.S. Department of the Treasury estimated the U.S. federal debt to be about $6.74 trillion. On this same day, the U.S. Bureau of the Census estimated the U.S. population to be about 292 million people. Determine the average debt, per person, by dividing the U.S. federal debt by the U.S. population.

*Some calculators will require you to enter the negative sign before entering the exponent.

†Some calculators will display the answer in scientific notation. In this case, the display will show $8.6^{\ 02}$, which means 8.6×10^2 and equals 860.

SOLUTION: First, we will write the numbers involved using decimal notation and then convert them to scientific notation.

$$6.74 \text{ trillion} = 6{,}740{,}000{,}000{,}000 = 6.74 \times 10^{12}$$
$$292 \text{ million} = 292{,}000{,}000 = 2.92 \times 10^{8}$$

Now we will divide 6.74×10^{12} by 2.92×10^{8} using a scientific calculator. The keystrokes are

6.74 $\boxed{\text{EXP}}$ 12 $\boxed{\div}$ 2.92 $\boxed{\text{EXP}}$ 8 $\boxed{=}$

The display shows 23,082.1918. This number indicates that on July 30, 2003, the U.S. government owed about \$23,082.19 per man, woman, and child living in the United States.

SECTION 5.6 EXERCISES

Concept/Writing Exercises

1. In the expression 2^3, what is the name given to the 2, and what is the name given to the 3?
2. Explain the meaning of b^n.
3. a) Explain the product rule for exponents.
   b) Use the product rule to simplify $2^3 \cdot 2^4$.
4. a) Explain the quotient rule for exponents.
   b) Use the quotient rule to simplify $\frac{5^6}{5^4}$.
5. a) Explain the zero exponent rule.
   b) Use the zero exponent rule to simplify 7^0.
6. a) Explain the negative exponent rule.
   b) Use the negative exponent rule to simplify 2^{-3}.
7. a) Explain the power rule for exponents.
   b) Use the power rule to simplify $(3^2)^4$.
8. Explain how you can simplify the expression 1^{500}.
9. Explain how you can simplify the following expressions and then simplify the expression.
   a) -1^{500}
   b) $(-1)^{500}$
   c) -1^{501}
   d) $(-1)^{501}$
10. a) In your own words, explain how to change a number in decimal notation to scientific notation.
    b) Using the procedure in part (a), change 0.000426 to scientific notation.
11. a) In your own words, explain how to change a number in scientific notation to decimal notation.
    b) Using the procedure in part (a), change 5.76×10^{-4} to decimal notation.
12. A number is given in scientific notation. What does it indicate about the number when the exponent on the 10 is (a) positive, (b) zero, and (c) negative?

Practice the Skills

In Exercises 13–44, evaluate the expression.

13. 5^2
14. 3^4
15. $(-2)^4$
16. -2^4
17. -3^2
18. $(-3)^2$
19. $\left(\frac{2}{3}\right)^2$
20. $\left(-\frac{7}{8}\right)^2$
21. $(-5)^2$
22. -5^2
23. $2^3 \cdot 3^2$
24. $\frac{15^2}{3^2}$
25. $\frac{5^7}{5^5}$
26. $3^3 \cdot 3^4$
27. $\frac{7}{7^3}$
28. $3^4 \cdot 7^0$
29. $(-13)^0$
30. $(-3)^4$
31. 3^4
32. -3^4
33. 3^{-2}
34. 3^{-3}
35. $(2^3)^4$
36. $(1^{12})^{13}$
37. $\frac{11^{25}}{11^{23}}$
38. $5^2 \cdot 5$
39. $(-4)^2$
40. 4^{-2}
41. -4^2
42. $(4^3)^2$
43. $(2^2)^{-3}$
44. $3^{-3} \cdot 3$

In Exercises 45–60, express the number in scientific notation.

45. 231,000
46. 297,000,000
47. 15
48. 0.000034
49. 0.56
50. 0.00467

51. 19,000 **52.** 1,260,000,000 **53.** 0.000186
54. 0.0003 **55.** 0.00000423 **56.** 54,000
57. 711 **58.** 0.02 **59.** 0.153
60. 416,000

In Exercises 61–76, express the number in decimal notation.

61. 2.3×10^3 **62.** 4.78×10^5 **63.** 3.901×10^{-3}
64. 1.764×10^7 **65.** 8.62×10^{-5} **66.** 2.19×10^{-4}
67. 3.12×10^{-1} **68.** 4.6×10^1 **69.** 9×10^6
70. 7.3×10^4 **71.** 2.31×10^2 **72.** 1.04×10^{-2}
73. 3.5×10^4 **74.** 2.17×10^{-6} **75.** 1×10^4
76. 1×10^{-3}

In Exercises 77–86, (a) perform the indicated operation without the use of a calculator and express each answer in decimal notation. (b) Confirm your answer from part (a) by using a scientific calculator to perform the operations. If the calculator displays the answer in scientific notation, convert the answer to decimal notation.

77. $(2 \times 10^3)(4 \times 10^2)$ **78.** $(4.1 \times 10^{-3})(2 \times 10^3)$
79. $(5.1 \times 10^1)(3 \times 10^{-4})$
80. $(1.6 \times 10^{-2})(4 \times 10^{-3})$
81. $\dfrac{6.4 \times 10^5}{2 \times 10^3}$ **82.** $\dfrac{8 \times 10^{-3}}{2 \times 10^1}$
83. $\dfrac{8.4 \times 10^{-6}}{4 \times 10^{-3}}$ **84.** $\dfrac{25 \times 10^3}{5 \times 10^{-2}}$
85. $\dfrac{4 \times 10^5}{2 \times 10^4}$ **86.** $\dfrac{16 \times 10^3}{8 \times 10^{-3}}$

In Exercises 87–96, (a) perform the indicated operation without the use of a calculator and express each answer in scientific notation. (b) Confirm your answer from part (a) by using a scientific calculator to perform the operations. If the calculator displays the answer in decimal notation, convert the answer to scientific notation.

87. (300,000)(2,000,000) **88.** (0.000041)(3000)
89. (0.003)(0.00015) **90.** (230,000)(3000)
91. $\dfrac{1{,}400{,}000}{700}$ **92.** $\dfrac{20{,}000}{0.0005}$
93. $\dfrac{0.00004}{200}$ **94.** $\dfrac{0.0012}{0.000006}$
95. $\dfrac{150{,}000}{0.0005}$ **96.** $\dfrac{24{,}000}{8{,}000{,}000}$

Problem Solving

In Exercises 97–100, list the numbers from smallest to largest.

97. 5.8×10^5; 3.2×10^{-1}; 4.6; 8.3×10^{-4}
98. 8.5×10^{-5}; 8.2×10^3; 1.3×10^{-1}; 6.2×10^4
99. 40,000; 4.1×10^3; 0.00079; 8.3×10^{-5}
100. 267,000,000; 3.14×10^7; 1,962,000; 4.79×10^6

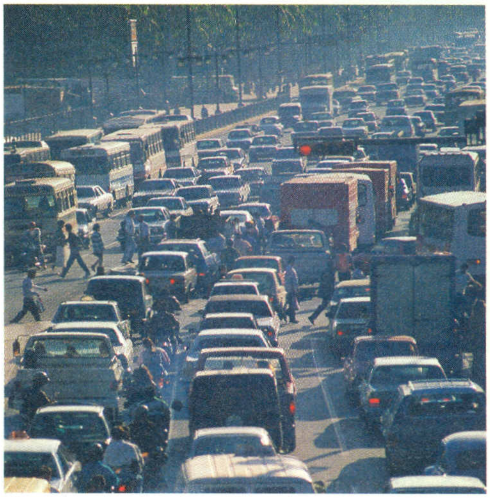
In Exercises 101–107, express your answer (a) using decimal notation and (b) using scientific notation. You may use a scientific calculator to perform the necessary operations.

101. *Gross Domestic Product* The gross domestic product (GDP) of a country is the total national output of goods and services produced within that country. In 2001, the GDP of the United States was about \$10.1432 trillion and the U.S. population was about 285 million people. Determine the U.S. GDP per person by dividing the GDP by the population.

102. *Japan's GDP* In 2001, the GDP (see Exercise 101) of Japan was about \$4.1468 trillion and the population of Japan was about 127 million people. Determine Japan's GDP per person by dividing the GDP by the population.

103. *Computer Speed* On April 20, 2002, the NEC Earth Simulator computer—located in Yokohama, Japan—broke the world's record for being the fastest supercomputer. This computer is capable of performing 36.6 trillion calculations per second. At this rate, how long would it take to perform a task requiring 7.69×10^{33} calculations?

104. *World Population* According to the U.S. Bureau of the Census, the population of the world in 2002 was approximately 6.251×10^9 and the population of China was about 1.283×10^9. How many people lived outside China?



105. *Traveling to Jupiter* The distance from Earth to the planet Jupiter is approximately 4.5×10^8 mi. If a spacecraft traveled at a speed of 25,000 mph, how many hours would the spacecraft need to travel from Earth to Jupiter? Use distance = rate × time.

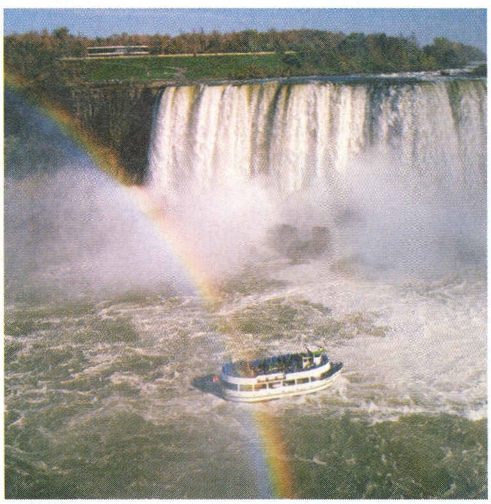
106. *Traveling to the Moon* The distance from Earth to the moon is approximately 239,000 mi. If a spacecraft travels at a speed of 20,000 mph, how many hours would the spacecraft need to travel from Earth to the moon? Use distance = rate × time.

107. *Bucket Full of Molecules* A drop of water contains about 40 billion molecules. If a bucket has half a million drops of water in it, how many molecules of water are in the bucket?

108. *Blood Cells in a Cubic Millimeter* If a cubic millimeter of blood contains 5,800,000 red blood cells, how many red blood cells are contained in 50 cubic millimeters of blood?

109. *Radioactive Isotopes* The half-life of a radioactive isotope is the time required for half the quantity of the isotope to decompose. The half-life of uranium 238 is 4.5×10^9 years, and the half-life of uranium 234 is 2.5×10^5 years. How many times greater is the half-life of uranium 238 than uranium 234?

110. *1950 Niagara Treaty* The 1950 Niagara Treaty between the United States and Canada requires that during the tourist season a minimum of 100,000 cubic feet of water per second (ft^3/sec) flow over Niagara Falls (another 130,000–160,000 ft^3/sec are diverted for power generation). Find the minimum amount of water that will flow over the falls in a 24-hour period during the tourist season.



111. *U.S. Debt per Person: 1993 versus 2003* In Example 14, the U.S. government debt was discussed. It was found that the debt in 2003 was $23,082.19 per person. In 1993, the U.S. government debt was about $4.41 trillion, and the population of the United States was about 258 million people.

a) Determine the amount of debt per person in the United States in 1993.

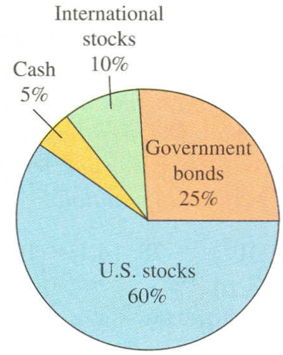
b) How much more per person did the U.S. government owe in 2003 than in 1993?

112. *Disposable Diaper Quantity* Laid end to end, the 18 billion disposable diapers thrown away in the United States each year would reach the moon and back seven times.

a) Write 18 billion in scientific notation.

b) If the distance from Earth to the moon is 2.38×10^5 miles, what is the length of all these diapers placed end to end? Write the answer in decimal notation.

113. *Mutual Fund Manager* Lauri Mackey is the fund manager for the Mackey Mutual Fund. This mutual fund has total assets of $1.2 billion. Lauri wants to maintain the investments in this fund according to the following pie chart.



Source: Ibbotson Associates

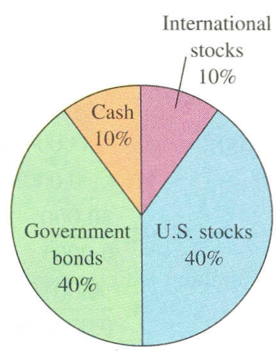
a) How much of the $1.2 billion should be invested in U.S. stocks?

b) How much should be invested in government bonds?

c) How much should be invested in international stocks?

d) How much should remain in cash?

114. *Another Mutual Fund Manager* Susan Dratch is the fund manager for the Dratch Mutual Fund. This mutual fund has total assets of $3.4 billion. Susan wants to maintain the investments in this fund according to the following pie chart.



Source: Ibbotson Associates

a) How much of the $3.4 billion should be invested in U.S. stocks?

b) How much should be invested in government bonds?
c) How much should be invested in international stocks?
d) How much should remain in cash?

115. *Metric System Comparison* In the metric system, 1 meter $= 10^3$ millimeters. How many times greater is a meter than a millimeter? Explain how you determined your answer.

116. In the metric system, 1 gram $= 10^3$ millimeters and 1 gram $= 10^{-3}$ kilogram. What is the relationship between milligrams and kilograms? Explain how you determined your answer.

117. *Earth to Sun Comparison* The mass of the sun is approximately 2×10^{30} kilograms, and the mass of Earth is approximately 6×10^{24} kilograms. How many times greater is the mass of the sun than the mass of Earth? Write your answer in decimal notation.

118. *The Day of Six Billion* The United Nations declared October 12, 1999, the *Day of Six Billion*. On this day, Earth's population was estimated to reach 6 billion. Currently, Earth's population is doubling about every 35 years.
a) Using this figure, estimate the world's population in the year 2034.
b) Assuming 365 days in a year, estimate the average number of additional people added to Earth's population each day between 1999 and 2034.

119. *Computer Calculation Speed* The IBM Blue Pacific computer is capable of operating at a peak speed of about 3.9 trillion (3,900,000,000,000) calculations per second. At this rate, how long would it take to perform a task requiring 897 quadrillion (897,000,000,000,000,000) calculations?

Challenge Problems/Group Activities

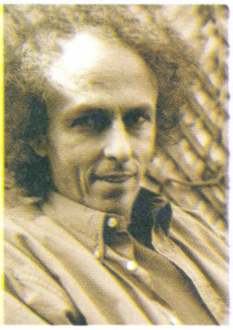
120. *Comparing a Million to a Billion* Many people have no idea of the difference in size between a million (1,000,000), a billion (1,000,000,000), and a trillion (1,000,000,000,000).
a) Write a million, a billion, and a trillion in scientific notation.
b) Determine how long it would take to spend a million dollars if you spent \$1000 a day.
c) Repeat part (b) for a billion dollars.
d) Repeat part (b) for a trillion dollars.
e) How many times greater is a billion dollars than a million dollars?

121. *Speed of Light*
a) Light travels at a speed of 1.86×10^5 mi/sec. A *light-year* is the distance that light travels in 1 year. Determine the number of miles in a light year.
b) Earth is approximately 93,000,000 mi from the sun. How long does it take light from the sun to reach Earth?

122. *Bacteria in a Culture* The exponential function $E(t) = 2^{10} \cdot 2^t$ approximates the number of bacteria in a certain culture after t hours.
a) The initial number of bacteria is determined when $t = 0$. What is the initial number of bacteria?
b) How many bacteria are there after $\frac{1}{2}$ hour?

Internet/Research Activities

123. John Allen Paulos of Temple University has written many entertaining books about mathematics for nonmathematicians. Included among these are *Mathematics and Humor* (1980); *I Think, Therefore I Laugh* (1985); *Innumeracy—Mathematical Illiteracy and Its Consequences* (1989); *Beyond Numeracy—Ruminations of a Numbers Man* (1991); *A Mathematician Reads the Newspaper* (1995); *Once Upon a Number* (1998); *A Mathematician Plays the Stock Market* (2003); and *Music and Gender* (2003). Read one of Paulos's books and write a 500-word report on it.



John Allen Paulos

124. Obtain data from the U.S. Department of the Treasury and from the U.S. Bureau of the Census Internet web sites to calculate the current U.S. government debt per person. Write a report in which you compare your figure with those obtained in Exercise 111 and Example 14. Include in your report definitions of the following terms: revenues, expenditures, deficit, and surplus.

125. Find an article in a newspaper or magazine that contains scientific notation. Write a paragraph explaining how scientific notation was used. Attach a copy of the article to your report.

5.7 ARITHMETIC AND GEOMETRIC SEQUENCES

Now that you can recognize the various sets of real numbers and know how to add, subtract, multiply, and divide real numbers, we can discuss sequences. A ***sequence*** is a list of numbers that are related to each other by a rule. The numbers that form the sequence are called its ***terms***. If your salary increases or decreases by a fixed amount over a period of time, the listing of the amounts, over time, would form an arithmetic sequence. When interest in a savings account is compounded at regular intervals, the listing of the amounts in the account over time will be a geometric sequence.

Arithmetic Sequences

A sequence in which each term after the first term differs from the preceding term by a constant amount is called an ***arithmetic sequence***. The amount by which each pair of successive terms differs is called the ***common difference***, d. The common difference can be found by subtracting any term from the term that directly follows it.

Examples of arithmetic sequences	Common differences
$1, 5, 9, 13, 17, \ldots$	$d = 5 - 1 = 4$
$-7, -5, -3, -1, 1, \ldots$	$d = -5 - (-7) = -5 + 7 = 2$
$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots$	$d = \frac{3}{2} - \frac{5}{2} = -\frac{2}{2} = -1$

EXAMPLE 1 *The First Five Terms of an Arithmetic Sequence*

Write the first five terms of the arithmetic sequence with first term 5 and a common difference of 4.

SOLUTION: The first term is 5. The second term is $5 + 4$ or 9. The third term is $9 + 4$ or 13. The fourth term is $13 + 4$ or 17. The fifth term is $17 + 4$ or 21. Thus, the first five terms of the sequence are 5, 9, 13, 17, 21.

EXAMPLE 2 *An Arithmetic Sequence with a Negative Difference*

Write the first five terms of the arithmetic sequence with first term 10 and common difference of -3.

SOLUTION: The sequence is

$$10, 7, 4, 1, -2$$

When discussing a sequence, we often represent the first term as a_1 (read "a sub 1"), the second term as a_2, the fifteenth term as a_{15}, and so on. We use the notation a_n to represent the general or nth term of a sequence. Thus a sequence may be symbolized as

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

For example, in the sequence $2, 5, 8, 11, 14, \ldots$, we have

$$a_1 = 2, a_2 = 5, a_3 = 8, a_4 = 11, a_5 = 14, \ldots.$$

When we know the first term of an arithmetic sequence and the common difference, we can use the following formula to find the value of any specific term.

General or nth Term of an Arithmetic Sequence

$$a_n = a_1 + (n - 1)d$$

EXAMPLE 3 *Finding the Seventh Term of an Arithmetic Sequence*

Find the seventh term of the arithmetic sequence whose first term is 3 and whose common difference is -6.

SOLUTION: To find the seventh term, or a_7, replace n in the formula with 7, a_1 with 3, and d with -6.

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ a_7 &= 3 + (7 - 1)(-6) \\ &= 3 + (6)(-6) \\ &= 3 - 36 \\ &= -33 \end{aligned}$$

The seventh term is -33. As a check, we have listed the first seven terms of the sequence: $3, -3, -9, -15, -21, -27, -33$.

EXAMPLE 4 *Finding the nth Term of an Arithmetic Sequence*

Write an expression for the general or nth term, a_n, for the sequence $1, 6, 11, 16, \ldots$.

SOLUTION: In this sequence, the first term a_1, is 1, and the common difference, d, is 5. We substitute these values into $a_n = a_1 + (n - 1)d$ to obtain an expression for the nth term, a_n.

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ &= 1 + (n - 1)5 \\ &= 1 + 5n - 5 \\ &= 5n - 4 \end{aligned}$$

Note that when $n = 1$, the first term is $5(1) - 4 = 1$. When $n = 2$, the second term is $5(2) - 4 = 6$, and so on.

We can use the following formula to find the sum of the first n terms in an arithmetic sequence.

Sum of the First n Terms in an Arithmetic Sequence

$$s_n = \frac{n(a_1 + a_n)}{2}$$

PROFILE IN MATHEMATICS

CARL FRIEDRICH GAUSS

Carl Friedrich Gauss (1777–1855), often called the "Prince of Mathematicians," made significant contributions to the fields of algebra, geometry, and number theory. Gauss was only 22 years old when he proved the fundamental theorem of algebra for his doctoral dissertation.

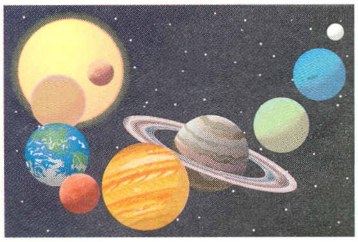
When Gauss was only 10, his mathematics teacher gave him the problem of finding the sum of the first 100 natural numbers, thinking that this would keep him busy for a while. Gauss recognized a pattern in the sequence of numbers when he considered the sum of the following numbers.

$$\begin{array}{rcrcrcccrcr} 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 2 & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101 \end{array}$$

He had the required answer in no time at all. When he added, he had one hundred 101's. Therefore, the sum is $\frac{1}{2}(100)(101) = 5050$.

DID YOU KNOW

A Heavenly Sequence



The sequence 0, 3, 6, 12, 24, 48, 96, 192, . . . , which resembles a geometric sequence, is known collectively as the **Titius–Bode law**. The sequence, discovered in 1766 by the two German astronomers, was of great importance to astronomy in the eighteenth and nineteenth centuries. When 4 was added to each term and the result was divided by 10, the sequence closely corresponded with the observed mean distance from the sun to the known principal planets of the solar system (in astronomical units, or au): Mercury (0.4), Venus (0.7), Earth (1.0), Mars (1.6), missing planet (2.8), Jupiter (5.2), Saturn (10.0), missing planet (19.6). The exciting discovery of Uranus in 1781 with a mean distance of 19.2 astronomical units was in near agreement with the Titius–Bode law and stimulated the search for an undiscovered planet at a predicted 2.8 astronomical units. This effort led to the discovery of Ceres and other members of the asteroid belt. Although the Titius–Bode law broke down after discoveries of Neptune (30.1 au) and Pluto (39.5 au), many scientists still believe that other applications of the Titius–Bode law will emerge in the future.

In this formula, s_n represents the sum of the first n terms, a_1 is the first term, a_n is the nth term, and n is the number of terms in the sequence from a_1 to a_n.

EXAMPLE 5 *Finding the Sum of a Sequence*

Find the sum of the first 25 natural numbers.

SOLUTION: The sequence we are discussing is

$$1, 2, 3, 4, 5, \ldots, 25$$

In this sequence, $a_1 = 1$, $a_{25} = 25$, and $n = 25$. Thus, the sum of the first 25 terms is

$$\begin{aligned} s_n &= \frac{n(a_1 + a_n)}{2} \\ s_{25} &= \frac{25(1 + 25)}{2} \\ &= \frac{25(26)}{2} = 325 \end{aligned}$$

Thus, the sum of the terms $1 + 2 + 3 + 4 + \cdots + 25$ is 325.

Geometric Sequences

The next type of sequence we will discuss is the geometric sequence. A ***geometric sequence*** is one in which the ratio of any term to the term that directly precedes it is a constant. This constant is called the ***common ratio***. The common ratio, r, can be found by taking any term except the first and dividing that term by the preceding term.

Examples of geometric sequences	Common ratios
$2, 4, 8, 16, 32, \ldots$	$r = 4 \div 2 = 2$
$-3, 6, -12, 24, -48, \ldots$	$r = 6 \div (-3) = -2$
$\frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \ldots$	$r = \frac{2}{9} \div \frac{2}{3} = \left(\frac{2}{9}\right)\left(\frac{3}{2}\right) = \frac{1}{3}$

To construct a geometric sequence when the first term, a_1, and common ratio are known, multiply the first term by the common ratio to get the second term. Then multiply the second term by the common ratio to get the third term, and so on.

EXAMPLE 6 *The First Five Terms of a Geometric Sequence*

Write the first five terms of the geometric sequence whose first term, a_1, is 3 and whose common ratio, r, is 4.

SOLUTION: The first term is 3. The second term, found by multiplying the first term by 4, is $3 \cdot 4$ or 12. The third term is $12 \cdot 4$ or 48. The fourth term is $48 \cdot 4$ or 192. The fifth term is $192 \cdot 4$ or 768. Thus, the first five terms of the sequence are 3, 12, 48, 192, 768.

When we know the first term of a geometric sequence and the common ratio, we can use the following formula to find the value of the general or nth term, a_n.

General or nth Term of a Geometric Sequence

$$a_n = a_1 r^{n-1}$$

EXAMPLE 7 *Finding the Seventh Term of a Geometric Sequence*

Find the seventh term of the geometric sequence whose first term is -3 and whose common ratio is -2.

SOLUTION: In this sequence, $a_1 = -3$, $r = -2$, and $n = 7$. Substituting the values, we obtain

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_7 &= -3(-2)^{7-1} \\ &= -3(-2)^6 \\ &= -3(64) \\ &= -192 \end{aligned}$$

As a check, we have listed the first seven terms of the sequence: $-3, 6, -12, 24, -48, 96, -192$.

EXAMPLE 8 *Finding the nth Term of a Geometric Sequence*

Write an expression for the general or nth term, a_n, of the sequence $2, 6, 18, 54, \ldots$.

SOLUTION: In this sequence, $a_1 = 2$ and $r = 3$. We substitute these values into $a_n = a_1 r^{n-1}$ to obtain an expression for the nth term, a_n.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ &= 2(3)^{n-1} \end{aligned}$$

Note than when $n = 1$, $a_1 = 2(3)^0 = 2(1) = 2$. When $n = 2$, $a_2 = 2(3)^1 = 6$, and so on.

We can use the following formula to find the sum of the first n terms of a geometric sequence.

Sum of the First n Terms of a Geometric Sequence

$$s_n = \frac{a_1(1 - r^n)}{1 - r}, \qquad r \neq 1$$

EXAMPLE 9 *Adding the First n Terms of a Geometric Sequence*

Find the sum of the first five terms in the geometric sequence whose first term is 4 and whose common ratio is 2.

SOLUTION: In this sequence, $a_1 = 4$, $r = 2$, and $n = 5$. Substituting these values into the formula, we get

$$s_n = \frac{a_1(1 - r^n)}{1 - r}$$

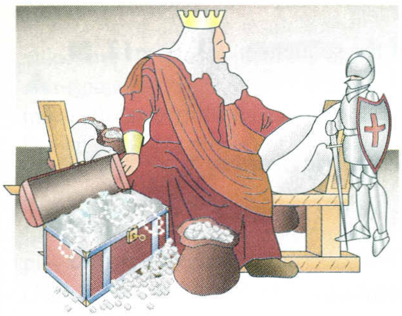
$$s_5 = \frac{4[1 - (2)^5]}{1 - 2}$$

$$= \frac{4(1 - 32)}{-1}$$

$$= \frac{4(-31)}{-1} = \frac{-124}{-1} = 124$$

The sum of the first five terms of the sequence is 124. The first five terms of the sequence are 4, 8, 16, 32, 64. If you add these five numbers, you will obtain the sum 124.

EXAMPLE 10 *Pounds and Pounds of Silver*



As a reward for saving his kingdom from a band of thieves, a king offered a knight one of two options. The knight's first option was to be paid 100,000 pounds of silver all at once. The second option was to be paid over the course of a month. On the first day, he would receive one pound of silver. On the second day, he would receive two pounds of silver. On the third day, he would receive four pounds of silver, and so on, each day receiving double the amount given on the previous day. Assuming the month is 30 days, which option would pay the knight more silver?

SOLUTION: The first option pays the knight 100,000 pounds of silver. The second option pays according to the geometric sequence 1, 2, 4, 8, 16, ... In this sequence, $a_1 = 1$, $r = 2$, and $n = 30$. The sum of this sequence can be found by substituting these values into the formula to obtain

$$s_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$s_{30} = \frac{1(1 - 2^{30})}{1 - 2}$$

$$= \frac{1 - 1{,}073{,}741{,}824}{-1}$$

$$= \frac{-1{,}073{,}741{,}823}{-1}$$

$$= 1{,}073{,}741{,}823$$

Thus, the knight would get paid 1,073,741,823 pounds of silver with the second option. The second option pays 1,073,641,823 more pounds of silver than the first option.

SECTION 5.7 EXERCISES

Concept/Writing Exercises

1. State the definition of *sequence* and give an example.
2. What are the numbers that make up a sequence called?
3. a) State the definition of *arithmetic sequence* and give an example.
   b) State the definition of *geometric sequence* and give an example.
4. a) In the arithmetic sequence 2, 5, 8, 11, 14, . . . , state the common difference, d.
   b) In the geometric sequence 3, 6, 12, 24, 48, . . . , state the common ratio, r.
5. For an arithmetic sequence, state the meaning of each of the following symbols.
   a) a_n b) a_1 c) d d) s_n
6. For a geometric sequence, state the meaning of each of the following symbols.
   a) a_n b) a_1 c) r d) s_n

Practice the Skills

In Exercises 7–14, write the first five terms of the arithmetic sequence with the first term, a_1, and common difference, d.

7. $a_1 = 3, d = 2$
8. $a_1 = 1, d = 3$
9. $a_1 = -5, d = 3$
10. $a_1 = -11, d = 5$
11. $a_1 = 5, d = -2$
12. $a_1 = -3, d = -4$
13. $a_1 = \frac{1}{2}, d = \frac{1}{2}$
14. $a_1 = \frac{5}{2}, d = -\frac{3}{2}$

In Exercises 15–22, find the indicated term for the arithmetic sequence with the first term, a_1, and common difference, d.

15. Find a_6 when $a_1 = 2, d = 3$.
16. Find a_9 when $a_1 = 3$ and $d = -2$.
17. Find a_{10} when $a_1 = -5, d = 2$.
18. Find a_{12} when $a_1 = 7, d = -3$.
19. Find a_{20} when $a_1 = \frac{4}{5}, d = -1$.
20. Find a_{15} when $a_1 = -\frac{1}{2}, d = -2$.
21. Find a_{11} when $a_1 = 4, d = \frac{1}{2}$.
22. Find a_{15} when $a_1 = \frac{4}{3}, d = \frac{1}{3}$.

In Exercises 23–30, write an expression for the general or nth term, a_n, for the arithmetic sequence.

23. $1, 2, 3, 4, \ldots$
24. $1, 3, 5, 7, \ldots$
25. $2, 4, 6, 8, \ldots$
26. $3, 1, -1, -3, \ldots$
27. $-\frac{5}{3}, -\frac{4}{3}, -1, -\frac{2}{3}, \ldots$
28. $-15, -10, -5, 0, \ldots$
29. $-3, -\frac{3}{2}, 0, \frac{3}{2}, \ldots$
30. $-5, -2, 1, 4, \ldots$

In Exercises 31–38, find the sum of the terms of the arithmetic sequence. The number of terms, n, is given.

31. $1, 2, 3, 4, \ldots, 50; n = 50$
32. $2, 4, 6, 8, \ldots, 100; n = 50$
33. $1, 3, 5, 7, \ldots, 99; n = 50$
34. $-4, -7, -10, -13, \ldots, -28; n = 9$
35. $11, 6, 1, -4, \ldots, -24; n = 8$
36. $-9, -\frac{17}{2}, -8, -\frac{15}{2}, \ldots, -\frac{1}{2}; n = 18$
37. $\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \frac{13}{2}, \ldots, \frac{29}{2}; n = 8$
38. $\frac{3}{5}, \frac{4}{5}, 1, \frac{6}{5}, \ldots, 4; n = 18$

In Exercises 39–46, write the first five terms of the geometric sequence with the first term, a_1, and common ratio, r.

39. $a_1 = 3, r = 2$
40. $a_1 = 6, r = 3$
41. $a_1 = 2, r = -2$
42. $a_1 = 8, r = \frac{1}{2}$
43. $a_1 = -3, r = -1$
44. $a_1 = -6, r = -2$
45. $a_1 = -16, r = -\frac{1}{2}$
46. $a_1 = 5, r = \frac{3}{5}$

In Exercises 47–54, find the indicated term for the geometric sequence with the first term, a_1, and common ratio, r.

47. Find a_6 when $a_1 = 3, r = 4$.
48. Find a_5 when $a_1 = 2, r = 2$.
49. Find a_3 when $a_1 = 3, r = \frac{1}{2}$.
50. Find a_7 when $a_1 = -3, r = -3$.
51. Find a_5 when $a_1 = \frac{1}{2}, r = 2$.
52. Find a_{25} when $a_1 = 1, r = 2$.
53. Find a_{10} when $a_1 = -2, r = 3$.
54. Find a_{18} when $a_1 = -5, r = -2$.

In Exercises 55–62, write an expression for the general or nth term, a_n, for the geometric sequence.

55. $1, 2, 4, 8, \ldots$
56. $3, 6, 12, 24, \ldots$
57. $3, -3, 3, -3, \ldots$
58. $-16, -8, -4, -2, \ldots$
59. $\frac{1}{4}, \frac{1}{2}, 1, 2, \ldots$
60. $-3, 6, -12, 24, \ldots$
61. $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \ldots$
62. $-4, -\frac{8}{3}, -\frac{16}{9}, -\frac{32}{27}, \ldots$

In Exercises 63–70, find the sum of the first n terms of the geometric sequence for the values of a_1 and r.

63. $n = 4, a_1 = 3, r = 2$
64. $n = 5, a_1 = 2, r = 3$
65. $n = 7, a_1 = 5, r = 4$

66. $n = 9, a_1 = -3, r = 5$

67. $n = 11, a_1 = -7, r = 3$

68. $n = 15, a_1 = -1, r = 2$

69. $n = 15, a_1 = -1, r = -2$

70. $n = 10, a_1 = 512, r = \frac{1}{2}$

Problem Solving

71. Find the sum of the first 100 natural numbers.

72. Find the sum of the first 100 even natural numbers.

73. Find the sum of the first 100 odd natural numbers.

74. Find the sum of the first 50 multiples of 3.

75. *Annual Pay Raises* Rita Fernandez is given a starting salary of \$20,200 and promised a \$1200 raise per year after each of the next eight years.
    a) Determine her salary during her eighth year of work.
    b) Determine the total salary she received over the 8 years.

76. *Pendulum Movement* Each swing of a pendulum (from far left to far right) is 3 in. shorter than the preceding swing. The first swing is 8 ft.
    a) Find the length of the twelfth swing.
    b) Determine the total distance traveled by the pendulum during the first 12 swings.

77. *A Bouncing Ball* Each time a ball bounces, the height attained by the ball is 6 in. less than the previous height attained. If on the first bounce the ball reaches a height of 6 ft, find the height attained on the eleventh bounce.

78. *Clock Strikes* A clock strikes once at 1 o'clock, twice at 2 o'clock, and so on. How many times does it strike over a 12 hr period?

79. *Squirrels and Pinecones* A tree squirrel cut down 1 pinecone on the first day of October, 2 pinecones on the second day, 3 pinecones on the third day, and so on. How many pinecones did this squirrel cut down during the month of October, which contains 31 days?

80. *Enrollment Increase* The enrollment at Loras College in 2001 was 8000 students. If the enrollment increases by 8% per year, determine the enrollment 10 years later.

81. *Decomposing Substance* A certain substance decomposes and loses 20% of its weight each hour. If there are originally 200 g of the substance, how much remains after 6 hr?

82. *Samurai Sword Construction* While making a traditional Japanese samurai sword, the master sword maker prepares the blade by heating a bar of iron until it is white hot. He then folds it over and pounds it smooth. Therefore, after each folding, the number of layers of steel is doubled. Assuming the sword maker starts with a bar of one layer and folds it 15 times, how many layers of steel will the finished sword contain?

83. *Salary Increase* If your salary were to increase at a rate of 6% per year, find your salary during your 15th year if your original salary is \$20,000.

84. *A Bouncing Ball* When dropped, a ball rebounds to four-fifths of its original height. How high will the ball rebound after the fourth bounce if it is dropped from a height of 30 ft?

85. *Value of a Stock* Ten years ago, Nancy Hart purchased \$2,000 worth of shares in RCF, Inc. Since then, the price of the stock has roughly tripled every two years. Approximately how much are Nancy's shares worth today?

86. *A Baseball Game* During a baseball game, the visiting team scored 1 run in the first inning, 2 runs in the second inning, 3 runs in the third inning, 4 runs in the fourth inning, and so on. The home team scored 1 run in the first inning, 2 runs in the second inning, 4 runs in the third inning, 8 runs in the fourth inning, and so on. What is the score of the game after eight innings?

Inning	1	2	3	4	5	6	7	8	9
Visitors	1	2	3	4					
Home	1	2	4	8					

Challenge Problems/Group Activities

87. A geometric sequence has $a_1 = 82$ and $r = \frac{1}{2}$; find s_6.

88. *Sums of Interior Angles* The sums of the interior angles of a triangle, a quadrilateral, a pentagon, and a sextagon are 180°, 360°, 540°, and 720°, respectively. Use this pattern to

find a formula for the general term, a_n, where a_n represents the sum of the interior angles of an n-sided quadrilateral.

89. *Divisibility by 6* Determine how many numbers between 7 and 1610 are divisible by 6.

90. Find r and a_1 for the geometric sequence with $a_2 = 24$ and $a_5 = 648$.

91. *Total Distance Traveled by a Bouncing Ball* A ball is dropped from a height of 30 ft. On each bounce it attains a height four-fifths of its original height (or of the previous bounce). Find the total vertical distance traveled by the ball after it has completed its fifth bounce (therefore has hit the ground six times).

Recreational Mathematics

92. *A Wagering Strategy* The following is a strategy used by some people involved in games of chance. A player begins by betting a standard bet, say \$1. If the player wins, the player again bets \$1 in the next round. If the player loses, the player bets \$2 in the next round. Next, if the player wins, the player again bets \$1; if the player loses, the player now bets \$4 in the next round. The process continues as long as the player keeps playing, betting \$1 after a win or doubling the previous bet after a loss.

a) Assume a player is using a \$1 standard bet and loses five times in a row. How much money should the player bet in the sixth round? How much money has the player lost at the end of the fifth round?

b) Assume a player is using a \$10 standard bet and loses five times in a row. How much money should the player bet in the sixth round? How much money has the player lost at the end of the fifth round?

c) Assume a player is using a \$1 standard bet and loses 10 times in a row. How much money should the player bet in the 11th round? How much money has the player lost at the end of the 10th round?

d) Assume a player is using a \$10 standard bet and loses 10 times in a row. How much money should the player bet in the 11th round? How much money has the player lost at the end of the 10th round?

e) Why is this a dangerous strategy?

Internet/Research Activity

93. A topic generally associated with sequences is *series.*

a) Research *series* and explain what a series is and how it differs from a sequence. Also write a formal definition of series. Give examples of different kinds of series.

b) Write the arithmetic series associated with the arithmetic sequence 1, 4, 7, 10, 13, ...

c) Write the geometric series associated with the geometric sequence 3, 6, 12, 24, 48, ...

d) What is an infinite geometric series?

e) Find the sum of the terms of the infinite geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$.

5.8 FIBONACCI SEQUENCE

Our discussion of sequences would not be complete without mentioning a sequence known as the *Fibonacci sequence*. The sequence is named after Leonardo of Pisa, also known as Fibonacci. He was one of the most distinguished mathematicians of the Middle Ages. This sequence is first mentioned in his book *Liber Abacci* (Book of the Abacus), which contained many interesting problems, such as: "A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is assumed that every month each pair begets a new pair which from the second month becomes productive?"

The solution to this problem (Fig. 5.10 on page 276) led to the development of the sequence that bears its author's name: the Fibonacci sequence. The sequence is shown in Table 5.1 on page 276. The numbers in the columns titled *Pairs of Adults* form the Fibonacci sequence.

PROFILE IN MATHEMATICS

FIBONACCI

Leonardo of Pisa (1170–1250) is considered one of the most distinguished mathematicians of the Middle Ages. He was born in Italy and was sent by his father to study mathematics with an Arab master. When he began writing, he referred to himself as Fibonacci, or "son of Bonacci," the name by which he is known today. In addition to the famous sequence bearing his name, Fibonacci is also credited with introducing the Hindu–Arabic number system into Europe. His 1202 book, *Liber Abacci* (Book of the Abacus), explained the use of this number system and emphasized the importance of the number zero.

TABLE 5.1

Month	Pairs of Adults	Pairs of Babies	Total Pairs
1	1	0	1
2	1	1	2
3	2	1	3
4	3	2	5
5	5	3	8
6	8	5	13
7	13	8	21
8	21	13	34
9	34	21	55
10	55	34	89
11	89	55	144
12	144	89	233

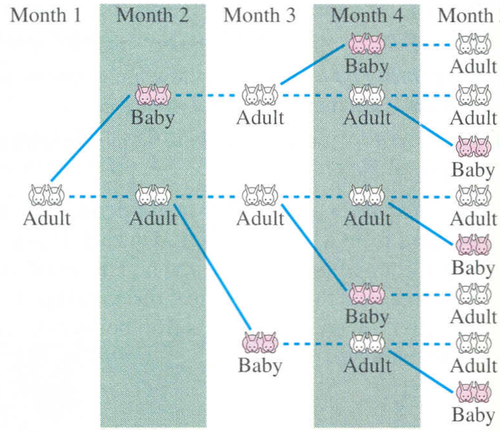


Figure 5.10

Fibonacci Sequence

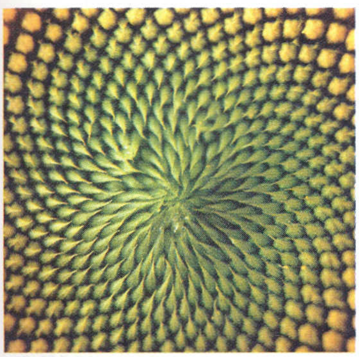
1, 1, 2, 3, 5, 8, 13, 21, . . .

In the Fibonacci sequence, the first and second terms are 1. The sum of these two terms is the third term. The sum of the second and third terms is the fourth term, and so on.

In the middle of the nineteenth century, mathematicians made a serious study of this sequence and found strong similarities between it and many natural phenomena. Fibonacci numbers appear in the seed arrangement of many species of plants and in the petal counts of various flowers. For example, when the flowering head of the sunflower matures to seed, the seeds' spiral arrangement becomes clearly visible. A typical count of these spirals may give 89 steeply curving to the right, 55 curving more shallowly to the left, and 34 again shallowly to the right. The largest known specimen to be examined had spiral counts of 144 right, 89 left, and 55 right. These numbers, like the other three mentioned, are consecutive terms of the Fibonacci sequence.

On the heads of many flowers, petals surrounding the central disk generally yield a Fibonacci number. For example, some daisies contain 21 petals, and others contain 34, 55, or 89 petals. (People who use a daisy to play the "love me, love me not" game will likely pluck 21, 34, 55, or 89 petals before arriving at an answer.)

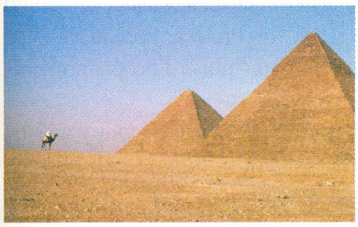
Fibonacci numbers are also observed in the structure of pinecones and pineapples. The tablike or scalelike structures called bracts that make up the main body of the pinecone form a set of spirals that start from the cone's attachment to the branch. Two sets of oppositely directed spirals can be observed, one steep and the other more gradual. A count on the steep spiral will reveal a Fibonacci number, and a count on the gradual one will be the adjacent smaller Fibonacci number, or if not, the next smaller



The head of a sunflower

TABLE 5.2

Numbers	Ratio
1, 1	$\frac{1}{1} = 1$
1, 2	$\frac{2}{1} = 2$
2, 3	$\frac{3}{2} = 1.5$
3, 5	$\frac{5}{3} = 1.666\ldots$
5, 8	$\frac{8}{5} = 1.6$
8, 13	$\frac{13}{8} = 1.625$
13, 21	$\frac{21}{13} \approx 1.615$
21, 34	$\frac{34}{21} \approx 1.619$
34, 55	$\frac{55}{34} \approx 1.618$
55, 89	$\frac{89}{55} \approx 1.618$



The Great Pyramid of Gizeh

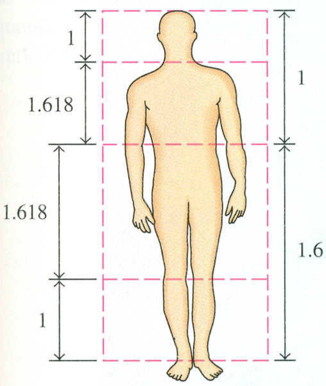


Figure 5.12

Fibonacci number. One investigation of 4290 pinecones from 10 species of pine trees found in California revealed that only 74 cones, or 1.7%, deviated from this Fibonacci pattern.

Like pinecone bracts, pineapple scales are patterned into spirals, and because they are roughly hexagonal in shape, three distinct sets of spirals can be counted.

Fibonacci Numbers and Divine Proportions

In 1753, while studying the Fibonacci sequence, Robert Simson, a mathematician at the University of Glasgow, noticed that when he took the ratio of any term to the term that immediately preceded it, the value he obtained remained in the vicinity of one specific number. To illustrate this, we indicate in Table 5.2 the ratio of various pairs of sequential Fibonacci numbers.

The ratio of the 50th term to the 49th term is 1.6180. Simson proved that the ratio of the $(n + 1)$ term to the nth term as n gets larger and larger is the irrational number $(\sqrt{5} + 1)/2$, which begins 1.61803. . . . This number was already well known to mathematicians at that time as the ***golden number***.

Many years earlier, the Bavarian astronomer and mathematician Johannes Kepler wrote that for him the golden number symbolized the Creator's intention "to create like from like." The golden number $(\sqrt{5} + 1)/2$ is frequently referred to as "phi," symbolized by the Greek letter Φ.

Figure 5.11

The ancient Greeks, in about the sixth century B.C., sought unifying principles of beauty and perfection, which they believed could be described by using mathematics. In their study of beauty, the Greeks used the term *golden ratio.* To understand the golden ratio, let's consider the line segment AB in Fig. 5.11. When this line segment is divided at a point C, such that the ratio of the whole, AB, to the larger part, AC, is equal to the ratio of the larger part, AC, to the smaller part, CB, then each ratio AB/AC and AC/CB is referred to as a ***golden ratio***. The proportion they form, $AB/AC = AC/CB$, is called the ***golden proportion***. Furthermore, each ratio in the proportion will have a value equal to the golden number, $(\sqrt{5} + 1)/2$.

$$\frac{AB}{AC} = \frac{AC}{CB} = \frac{\sqrt{5} + 1}{2} \approx 1.618$$

The Great Pyramid of Gizeh in Egypt, built about 2600 B.C., is the earliest known example of use of the golden ratio in architecture. The ratio of any of its sides of the square base (775.75 ft) to its altitude (481.4 ft) is about 1.611. Other evidence of the use of the golden ratio appears in other Egyptian buildings and tombs.

In medieval times, people referred to the golden proportion as the ***divine proportion***, reflecting their belief in its relationship to the will of God.

The twentieth-century architect Le Corbusier developed a scale of proportions for the human body that he called the Modulor (Fig. 5.12). Note that the navel separates the entire body into golden proportions, as does the neck and knee.

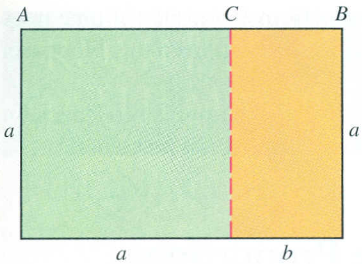


Figure 5.13

From the golden proportion, the ***golden rectangle*** can be formed, as shown in Fig. 5.13.

$$\frac{\text{Length}}{\text{Width}} = \frac{a+b}{a} = \frac{a}{b} = \frac{\sqrt{5}+1}{2}$$

Note that when a square is cut off one end of a golden rectangle, as in Fig. 5.13, the remaining rectangle has the same properties as the original golden rectangle (creating "like from like" as Johannes Kepler had written) and is therefore itself a golden rectangle. Interestingly, the curve derived from a succession of diminishing golden rectangles, as shown in Fig. 5.14, is the same as the spiral curve of the chambered nautilus. The same curve appears on the horns of rams and some other animals. It is the same curve that is observed in the plant structures mentioned earlier—sunflowers, other flower heads, pinecones, and pineapples. You will recall that Fibonacci numbers were observed in each of these plant structures. The curve shown in Fig. 5.14 closely approximates what mathematicians call a *logarithmic spiral.*

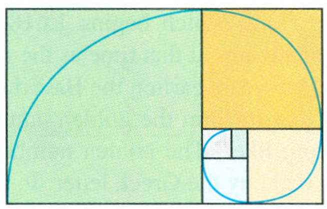


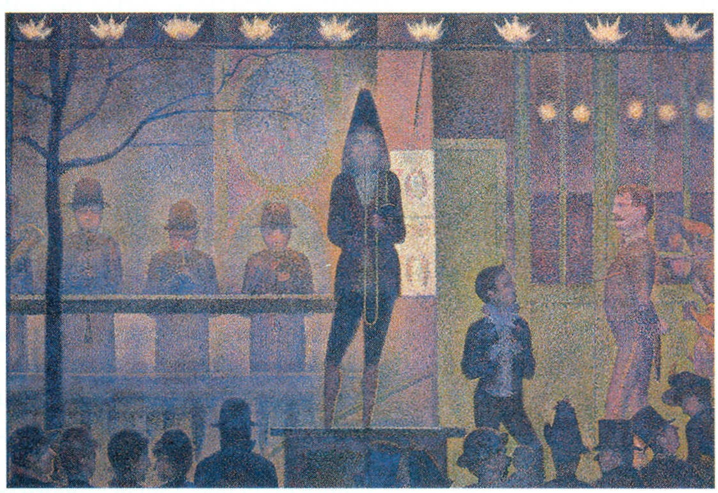
Figure 5.14

The Parthenon

Ancient Greek civilization used the golden rectangle in art and architecture. The main measurements of many buildings of antiquity, including the Parthenon in Athens, are governed by golden ratios and rectangles. Greek statues, vases, urns, and so on also exhibit characteristics of the golden ratio. It is for Phidas, considered the greatest of Greek sculptors, that the golden ratio was named "phi." The proportions can be found abundantly in his work.

The proportions of the golden rectangle can be found in the work of many artists, from the old masters to the moderns. For example, the golden rectangle can be seen in the painting *Invitation to the Sideshow (La Parade de Cirque),* 1887, by George Seurat, a French neoimpressionist artist.

Fibonacci's Garden by Caryl Bryer Fallert (see page 279)

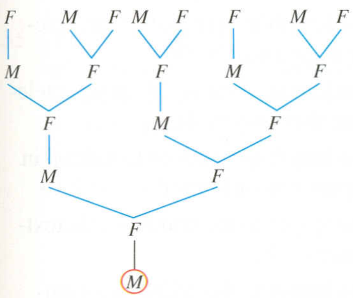


Invitation to the Sideshow (La Parade de Cirque), 1887, by George Seurat

DID YOU KNOW

Fibonacci and the Male Bee's Ancestors

The most frequent example given to introduce the Fibonacci sequence involves rabbits producing offspring, two at a time. Although this makes for a nice introduction to Fibonacci sequences, it is not at all realistic. A much better example comes from the breeding practices of bees. Female or worker bees are produced when the queen bee mates with a male bee. Male bees are produced from the queen's unfertilized eggs. In essence, then, female bees have two parents, whereas male bees only have one parent. The family tree of a male bee would look like this:



From this tree, we can see that the **1** male bee (circled) has **1** parent, **2** grandparents, **3** great-grandparents, **5** great-great-grandparents, **8** great-great-great-grandparents, and so on. We see the Fibonacci sequence as we move back through the male bees' generations.

In addition to using the golden rectangle in art, several artists have used Fibonacci numbers in art. One contemporary example is the 1995 work by Caryl Bryer Fallert called *Fibonacci's Garden* (see page 278). This artwork is a quilt constructed from two separate fabrics that are put together in a pattern based on the Fibonacci sequence.

Fibonacci numbers are also found in another form of art, namely music. Perhaps the most obvious link between Fibonacci numbers and music can be found on the piano keyboard. An octave (Fig. 5.15) on a keyboard has 13 keys: 8 white keys and 5 black keys (the 5 black keys are in one group of 2 and one group of 3).

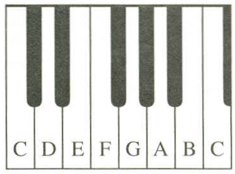


Figure 5.15

In Western music, the most complete scale, the chromatic scale, consists of 13 notes (from C to the next higher C). Its predecessor, the diatonic scale, contains 8 notes (an octave). The diatonic scale was preceded by a 5-note pentatonic scale (*penta* is Greek for "five"). Each number is a Fibonacci number.

The visual arts deal with what is pleasing to the eye, whereas musical composition deals with what is pleasing to the ear. Whereas art achieves some of its goals by using division of planes and area, music achieves some of its goals by a similar division of time, using notes of various duration and spacing. The musical intervals considered by many to be the most pleasing to the ear are the major sixth and minor sixth. A major sixth, for example, consists of the note C, vibrating at about 264* vibrations per second, and note A, vibrating at about 440 vibrations per second. The ratio of 440 to 264 reduces to 5 to 3, or $\frac{5}{3}$, a ratio of two consecutive Fibonacci numbers. An example of a minor sixth is E (about 330 vibrations per second) and C (about 528 vibrations per second). The ratio 528 to 330 reduces to 8 to 5, or $\frac{8}{5}$, the next ratio of two consecutive Fibonacci numbers. The vibrations of any sixth interval reduce to a similar ratio.

Patterns that can be expressed mathematically in terms of Fibonacci relationships have been found in Gregorian chants and works of many composers, including Bach, Beethoven, and Bartók. A number of twentieth-century musical works, including Ernst Krenek's *Fibonacci Mobile*, have been deliberately structured by using Fibonacci proportions.

A number of studies have tried to explain why the Fibonacci sequence and related items are linked to so many real-life situations. It appears that the Fibonacci numbers are a part of natural harmony that is pleasing to both the eye and the ear. In the nineteenth century, German physicist and psychologist Gustav Fechner tried to determine which dimensions were most pleasing to the eye. Fechner, along with psychologist Wilhelm Wundt, found that most people do unconsciously favor golden dimensions when purchasing greeting cards, mirrors, and other rectangular objects. This discovery has been widely used by commercial manufacturers in their packaging and labeling designs, by retailers in their store displays, and in other areas of business and advertising.

*Frequencies of notes vary in different parts of the world and change over time.

SECTION 5.8 EXERCISES

Concept/Writing Exercises

1. Explain how to construct the Fibonacci sequence.

2. a) Write out the first ten terms of the Fibonacci sequence.
   b) Divide the ninth term by the eighth term, rounding to the nearest thousandth.
   c) Divide the tenth term by the ninth term, rounding to the nearest thousandth.
   d) Try a few more divisions and then make a conjecture about the result.

3. a) What is the value of the golden number?
   b) What is the golden ratio?
   c) What is the golden proportion?
   d) What is the golden rectangle?

4. In your own words, explain the relationship between the golden number, golden ratio, golden proportion, and golden rectangle.

5. Describe three examples of where the golden ratio can be found
   a) in nature.
   b) in manufactured items.

6. Describe three examples of where Fibonacci numbers can be found
   a) in nature.
   b) in manufactured items.

Practice the Skills/Problem Solving

7. a) To what decimal value is $(\sqrt{5} + 1)/2$ aproximately equal?
   b) To what decimal value is $(\sqrt{5} - 1)/2$ approximately equal?
   c) By how much do the results in parts (a) and (b) differ?

8. The eleventh Fibonacci number is 89. Examine the first six digits in the decimal expression of its reciprocal, $\frac{1}{89}$. What do you find?

9. Find the ratio of the second to the first term of the Fibonacci sequence. Then find the ratio of the third to the second term of the sequence and determine whether this ratio was an increase or decrease from the first ratio. Continue this process for 10 ratios and then make a conjecture regarding the increasing or decreasing values in consecutive ratios.

10. A musical composition is described as follows. Explain why this piece is based on the golden ratio.

Entire Composition

34 measures	55 measures	21 measures	34 measures
Theme	Fast, Loud	Slow	Repeat of theme

11. The greatest common factor of any two consecutive Fibonacci numbers is 1. Show this is true for the first 15 Fibonacci numbers.

12. The sum of any 10 consecutive Fibonacci numbers is always divisible by 11. Select any 10 consecutive Fibonacci numbers and show that for your selection this is true.

13. Twice any Fibonacci number minus the next Fibonacci number equals the second number preceding the original number. Select a number in the Fibonacci sequence and show that this pattern holds for the number selected.

14. For any four consecutive Fibonacci numbers, the difference of the squares of the middle two numbers equals the product of the smallest and largest numbers. Select four consecutive Fibonacci numbers and show that this pattern holds for the numbers you selected.

15. Determine the ratio of the length to width of various photographs and compare these ratios to Φ.

16. Determine the ratio of the length to the width of a 6 inch by 4 inch standard index card, and compare the ratio to Φ.

17. Determine the ratio of the length to width of several picture frames and compare these ratios to Φ.

18. Determine the ratio of the length to the width of your television screen and compare this ratio to Φ.

19. Determine the ratio of the length to width of a desktop in your classroom and compare this ratio to Φ.

20. Determine the ratio of the length to the width of this textbook and compare this ratio to Φ.

21. Determine the ratio of the length to the width of a computer screen and compare this ratio to Φ.

22. Find three physical objects whose dimensions are very close to a golden rectangle.
    a) List the articles and record the dimensions.
    b) Compute the ratios of their lengths to their widths.
    c) Find the difference between the golden ratio and the ratio you obtain in part (b)—to the nearest tenth—for each object.

In Exercises 23–30, determine whether the sequence is a Fibonacci-type sequence (each term is the sum of the two preceding terms). If it is, determine the next two terms of the sequence.

23. 1, 3, 4, 7, 11, 18, . . .
24. 1, 1, 2, 2, 3, 3, . . .
25. 1, 4, 9, 16, 25, 36, . . .
26. $-1, 1, 0, 1, 1, 2, \ldots$
27. 5, 10, 15, 25, 40, 65, . . .
28. $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\frac{1}{4}, 2, \ldots$
29. $-5, 3, -2, 1, -1, 0, \ldots$
30. $-4, 5, 1, 6, 7, 13, \ldots$

31. a) Select any two nonzero digits and add them to obtain a third digit. Continue adding the two previous terms to get a Fibonacci-type sequence.

b) Form ratios of successive terms to show how they will eventually approach the golden number.

32. Repeat Exercise 31 for two different nonzero numbers.

33. a) Select any three consecutive terms of a Fibonacci sequence. Subtract the product of the terms on each side of the middle term from the square of the middle term. What is the difference?

b) Repeat part (a) with three different consecutive terms of the sequence.

c) Make a conjecture about what will happen when you repeat this process for any three consecutive terms of a Fibonacci sequence.

34. ***Pascal's Triangle*** One of the most famous number patterns involves ***Pascal's triangle***. The Fibonacci sequence can be found by using Pascal's triangle. Can you explain how that can be done? A hint is shown.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \rightarrow 5\\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

35. ***Lucas Sequence*** a) A sequence related to the Fibonacci sequence is the ***Lucas sequence***. The Lucas sequence is formed in a manner similar to the Fibonacci sequence. The first two numbers of the Lucas sequence are 1 and 3. Write the first eight terms of the Lucas sequence.

b) Complete the next two lines of the following chart.

$$
\begin{aligned}
1 + 2 &= 3\\
1 + 3 &= 4\\
2 + 5 &= 7\\
3 + 8 &= 11\\
5 + 13 &= 18
\end{aligned}
$$

c) What do you observe about the first column in the chart in part (b)?

Challenge Problems/Group Activities

36. ***Fibonacci-Type Sequence*** The following sequence represents a Fibonacci-type sequence (each term is the sum of the two preceding terms). Here x represents any natural number from 1 to 10:

$$-10, x, -10 + x, -10 + 2x, -20 + 3x, -30 + 5x, \ldots$$

For example, if $x = 2$, the first 10 terms of the sequence would be $-10, 2, -8, -6, -14, -20, -34, -54, -88, -142$.

Write out the first 10 terms of this Fibonacci-type sequence for x equal to

a) 4. b) 5. c) 6. d) 7. e) 8.

f) For values of x of 4, 5, and 6, you should have found that each term after the seventh term in the sequence was a negative number. For values of x of 7 and 8, you should have found that each term after the seventh term in the sequence was a positive number. Do you believe that for any value of x greater than or equal to 7, each term after the seventh term of the sequence will always be a positive number? Explain.

37. The divine proportion is $(a + b)/a = a/b$ (see Fig. 5.13), which can be written $1 + (b/a) = a/b$. Now let $x = a/b$, which gives $1 + (1/x) = x$. Multiply both sides of this equation by x to get a quadratic equation and then use the quadratic formula (Section 6.8) to show that one answer is $x = (1 + \sqrt{5})/2$ (the golden ratio).

38. Draw a line of length 5 in. Determine and mark the point on the line that will create the golden ratio. Explain how you determined your answer.

39. ***Pythagorean Triples*** A Pythagorean triple is a set of three whole numbers, $\{a, b, c\}$, such that $a^2 + b^2 = c^2$. For example, since $6^2 + 8^2 = (10)^2$, $\{6, 8, 10\}$ is a Pythagorean triple. The following steps show how to find Pythagorean triples using any four consecutive Fibonacci numbers. Here we will demonstrate the process with the Fibonacci numbers 3, 5, 8, and 13.

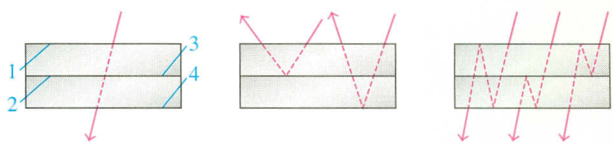
1. Determine the product of 2 and the two inner Fibonacci numbers. We have $2(5)(8) = 80$, which is the first number in the Pythagorean triple. So $a = 80$.
2. Determine the product of the two outer numbers. We have $3(13) = 39$, which is the second number in the Pythagorean triple. So $b = 39$.
3. Determine the sum of the squares of the inner two numbers. We have $5^2 + 8^2 = 25 + 64 = 89$, which is the third number in the Pythagorean triple. So $c = 89$.

This process has produced the Pythagorean triple, $\{80, 39, 89\}$. To verify,

$$
\begin{aligned}
(80)^2 + (39)^2 &= (89)^2\\
6400 + 1521 &= 7921\\
7921 &= 7921
\end{aligned}
$$

Use this process to produce four other Pythagorean triples.

40. ***Reflections*** When two panes of glass are placed face to face, four interior reflective surfaces exist labeled 1, 2, 3, and 4. If light is not reflected, it has just one path through the glass (see the figure below). If it has one reflection, it can be reflected in two ways. If it has two reflections, it can be reflected in three ways. Use this information to answer parts (a) through (c).



0 reflections
1 path

1 reflection
2 paths

2 reflections
3 paths

a) If a ray is reflected three times, there are five paths it can follow. Show the paths.

b) If a ray is reflected four times, there are eight paths it can follow. Show the paths.

c) How many paths can a ray follow if it is reflected five times? Explain how you determined your answer.

Internet/Research Activities

41. The digits 1 through 9 have evolved considerably since they appeared in Fibonacci's book *Liber Abacci.* Write a report tracing the history of the evolution of the digits 1 through 9 since Fibonacci's time.

42. Write a report on the history and mathematical contributions of Fibonacci.

43. Write a report indicating where the golden ratio and golden rectangle have been used in art and architecture. You may wish to include information on art and architecture related to the golden ratio and Fibonacci sequences.

CHAPTER 5 SUMMARY

IMPORTANT FACTS

Fundamental theorem of arithmetic

Every composite number can be expressed as a unique product of prime numbers.

Sets of numbers

Natural or counting numbers: $\{1, 2, 3, 4, \ldots\}$

Whole numbers: $\{0, 1, 2, 3, 4, \ldots\}$

Integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers: Numbers of the form p/q, where p and q are integers, $q \neq 0$. Every rational number when expressed as a decimal number will be either a terminating or repeating decimal number.

Irrational number: A real number whose representation is a nonterminating, nonrepeating decimal number (not a rational number).

Definition of subtraction

$$a - b = a + (-b)$$

Fundamental law of rational numbers

$$\frac{a}{b} = \frac{a}{b} \cdot \frac{c}{c} = \frac{ac}{bc}, \qquad b \neq 0, \qquad c \neq 0$$

Rules of radicals

Product rule for radicals:

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}, \qquad a \geq 0, \qquad b \geq 0$$

Quotient rule for radicals:

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}, \qquad a \geq 0, \qquad b > 0$$

Properties of real numbers

Commutative property of addition: $a + b = b + a$

Commutative property of multiplication: $a \cdot b = b \cdot a$

Associative property of addition:

$$(a + b) + c = a + (b + c)$$

Associative property of multiplication:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Distributive property: $a \cdot (b + c) = ab + ac$

Rules of exponents

Product rule for exponents: $a^m \cdot a^n = a^{m+n}$

Quotient rule for exponents: $\dfrac{a^m}{a^n} = a^{m-n}, \qquad a \neq 0$

Zero exponent rule: $a^0 = 1, \qquad a \neq 0$

Negative exponent rule: $a^{-m} = \dfrac{1}{a^m}, \qquad a \neq 0$

Power rule: $(a^m)^n = a^{m \cdot n}$

Arithmetic sequence

$$a_n = a_1 + (n - 1)d$$

$$s_n = \frac{n(a_1 + a_n)}{2}$$

Geometric sequence

$$a_n = a_1 r^{n-1}$$

$$s_n = \frac{a_1(1 - r^n)}{1 - r}, \qquad r \neq 1$$

Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Golden number

$$\frac{\sqrt{5} + 1}{2} \approx 1.618$$

Golden proportion

$$\frac{a + b}{a} = \frac{a}{b}$$

CHAPTER 5 REVIEW EXERCISES

5.1

In Exercises 1 and 2, determine whether the number is divisible by each of the following numbers: 2, 3, 4, 5, 6, 8, 9, and 10.

1. 894,348 **2.** 400,644

In Exercises 3–7, find the prime factorization of the number.

3. 252 **4.** 385 **5.** 840
6. 882 **7.** 1452

In Exercises 8–13, find the GCD and LCM of the numbers.

8. 15, 60 **9.** 63, 108 **10.** 45, 250
11. 840, 320 **12.** 60, 40, 96 **13.** 36, 108, 144

14. *Train Stops* From 1912 to 1971, the Milwaukee Road Railroad Company had a train stop every 15 days in Dubuque, Iowa. During this same period, the same train also stopped in Des Moines, Iowa, every 9 days. If on April 18, 1964, the train made a stop in Dubuque and a stop in Des Moines, how many days was it until the train again stopped in both cities on the same day?

5.2

In Exercises 15–22, use a number line to evaluate the expression.

15. $-2 + 5$ **16.** $4 + (-7)$
17. $4 - 8$ **18.** $-2 + (-4)$
19. $-5 - 4$ **20.** $-3 - (-6)$
21. $(-3 + 7) - 4$ **22.** $-1 + (9 - 4)$

In Exercises 23–30, evaluate the expression.

23. $(-3)(-11)$ **24.** $-4(9)$

25. $(14)(-4)$ **26.** $\dfrac{-35}{-7}$

27. $\dfrac{12}{-6}$ **28.** $[8 \div (-4)](-3)$

29. $[(-4)(-3)] \div 2$ **30.** $[(-30) \div (10)] \div (-1)$

5.3

In Exercises 31–39, express the fraction as a terminating or repeating decimal.

31. $\dfrac{3}{10}$ **32.** $\dfrac{3}{5}$ **33.** $\dfrac{15}{40}$

34. $\dfrac{13}{4}$ **35.** $\dfrac{3}{7}$ **36.** $\dfrac{7}{12}$

37. $\dfrac{3}{8}$ **38.** $\dfrac{7}{8}$ **39.** $\dfrac{5}{7}$

In Exercises 40–46, express the decimal number as a quotient of two integers.

40. 0.225 **41.** 4.5 **42.** $0.\overline{6}$ **43.** $2.\overline{37}$
44. 0.083 **45.** 0.0042 **46.** $2.3\overline{4}$

In Exercises 47–50, express each mixed number as an improper fraction.

47. $2\frac{5}{7}$ **48.** $4\frac{1}{6}$ **49.** $-3\frac{1}{4}$ **50.** $-35\frac{3}{8}$

In Exercises 51–54, express each improper fraction as a mixed number.

51. $\dfrac{11}{5}$ **52.** $\dfrac{27}{15}$ **53.** $-\dfrac{12}{7}$ **54.** $-\dfrac{136}{5}$

In Exercises 55–63, perform the indicated operation and reduce your answer to lowest terms.

55. $\dfrac{1}{2} + \dfrac{4}{5}$ **56.** $\dfrac{7}{8} - \dfrac{3}{4}$

57. $\dfrac{1}{6} + \dfrac{5}{4}$ **58.** $\dfrac{4}{5} \cdot \dfrac{15}{16}$

59. $\dfrac{5}{9} \div \dfrac{6}{7}$ **60.** $\left(\dfrac{4}{5} + \dfrac{5}{7}\right) \div \dfrac{4}{5}$

61. $\left(\dfrac{2}{3} \cdot \dfrac{1}{7}\right) \div \dfrac{4}{7}$ **62.** $\left(\dfrac{1}{5} + \dfrac{2}{3}\right)\left(\dfrac{3}{8}\right)$

63. $\left(\dfrac{1}{5} \cdot \dfrac{2}{3}\right) + \left(\dfrac{1}{5} \div \dfrac{1}{2}\right)$

64. *Cajun Turkey* A recipe for Roasted Cajun Turkey calls for $\frac{1}{8}$ teaspoon of cayenne pepper per pound of turkey. If

Jennifer Thornton is preparing a turkey that weighs $17\frac{3}{4}$ pounds, how much cayenne pepper does she need?

5.4

In Exercises 65–80, simplify the expression. Rationalize the denominator when necessary.

65. $\sqrt{50}$
66. $\sqrt{200}$
67. $\sqrt{5} + 7\sqrt{5}$
68. $\sqrt{3} - 4\sqrt{3}$
69. $\sqrt{8} + 6\sqrt{2}$
70. $\sqrt{3} - 7\sqrt{27}$
71. $\sqrt{75} + \sqrt{27}$
72. $\sqrt{3} \cdot \sqrt{6}$
73. $\sqrt{8} \cdot \sqrt{6}$
74. $\dfrac{\sqrt{18}}{\sqrt{2}}$
75. $\dfrac{\sqrt{56}}{\sqrt{2}}$
76. $\dfrac{4}{\sqrt{3}}$
77. $\dfrac{\sqrt{3}}{\sqrt{5}}$
78. $3(2 + \sqrt{7})$
79. $\sqrt{3}(4 + \sqrt{6})$
80. $\sqrt{3}(\sqrt{6} + \sqrt{15})$

5.5

In Exercises 81–90, state the name of the property illustrated.

81. $x + 2 = 2 + x$
82. $5 \cdot m = m \cdot 5$
83. $(1 + 2) + 3 = 1 + (2 + 3)$
84. $4(y + 3) = 4 \cdot y + 4 \cdot 3$
85. $(1 + 2) + 3 = 3 + (1 + 2)$
86. $(3 + 5) + (4 + 3) = (4 + 3) + (3 + 5)$
87. $(3 \cdot a) \cdot b = 3 \cdot (a \cdot b)$
88. $a \cdot (2 + 3) = (2 + 3) \cdot a$
89. $2(x + 3) = (2 \cdot x) + (2 \cdot 3)$
90. $x \cdot 2 + 6 = 2 \cdot x + 6$

In Exercises 91–96, determine whether the set of numbers is closed under the given operation.

91. Natural numbers, addition
92. Whole numbers, subtraction
93. Integers, division
94. Real numbers, subtraction
95. Irrational numbers, multiplication
96. Rational numbers, division

5.6

In Exercises 97–104, evaluate each expression.

97. 3^2
98. 3^{-2}
99. $\dfrac{9^5}{9^3}$
100. $5^2 \cdot 5$
101. 7^0
102. 4^{-3}
103. $(2^3)^2$
104. $(3^2)^2$

In Exercises 105–108, write each number in scientific notation.

105. 230,000,000
106. 0.0000158
107. 0.00275
108. 4,950,000

In Exercises 109–112, express each number in decimal notation.

109. 4.3×10^7
110. 1.39×10^{-4}
111. 1.75×10^{-4}
112. 1×10^5

In Exercises 113–116, (a) perform the indicated operation and write your answer in scientific notation. (b) Confirm the result found in part (a) by performing the calculation on a scientific calculator.

113. $(7 \times 10^3)(2 \times 10^{-5})$
114. $(4 \times 10^2)(2.5 \times 10^2)$
115. $\dfrac{8.4 \times 10^3}{4 \times 10^2}$
116. $\dfrac{1.5 \times 10^{-3}}{5 \times 10^{-4}}$

In Exercises 117–121, (a) perform the indicated calculation by first converting each number to scientific notation. Write your answer in decimal notation. (b) Confirm the result found in part (a) by performing the calculation on a scientific calculator.

117. (4,000,000)(2,000)
118. (35,000)(0.00002)
119. $\dfrac{9{,}600{,}000}{3000}$
120. $\dfrac{0.000002}{0.0000004}$
121. ***Space Distances*** The distance from Earth to the sun is about 1.49×10^{11} meters. The distance from Earth to the moon is about 3.84×10^8 meters. The distance from Earth to the sun is how many times larger than the distance from Earth to the moon? Use a scientific calculator and round your answer to the nearest whole number.

See Exercise 121

122. ***Outstanding Debt*** As a result of a recent water and sewer system improvement, the city of Galena, Illinois, has an outstanding debt of \$20,000,000. If the population of Galena is 3600 people, how much would each person have to contribute to pay off the outstanding debt?

5.7

In Exercises 123–128, determine whether the sequence is arithmetic or geometric. Then determine the next two terms of the sequence.

123. $2, 5, 8, 11, \ldots$

124. $\frac{1}{2}, 1, 2, 4, \ldots$

125. $-3, -6, -9, -12, \ldots$

126. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

127. $1, 4, 7, 10, 13, \ldots$

128. $2, -2, 2, -2, 2, \ldots$

In Exercises 129–134, find the indicated term of the sequence with the given first term, a_1, and common difference, d, or common ratio, r.

129. Find a_4 when $a_1 = 3, d = 4$.

130. Find a_8 when $a_1 = -6, d = -4$.

131. Find a_{10} when $a_1 = -20, d = 5$.

132. Find a_5 when $a_1 = 3, r = 2$.

133. Find a_5 when $a_1 = 4, r = \frac{1}{2}$.

134. Find a_4 when $a_1 = -6, r = 2$.

In Exercises 135–138, find the sum of the arithmetic sequence. The number of terms, n, is given.

135. $2, 5, 8, 11, \ldots, 89; n = 30$

136. $-4, -3\frac{3}{4}, -3\frac{1}{2}, -3\frac{1}{4}, \ldots, -2\frac{1}{4}; n = 8$

137. $100, 94, 88, 82, \ldots, 58; n = 8$

138. $0.5, 0.75, 1.00, 1.25, \ldots, 5.25; n = 20$

In Exercises 139–142, find the sum of the first n terms of the geometric sequence for the values of a_1 and r.

139. $n = 4, a_1 = 5, r = 3$

140. $n = 4, a_1 = 2, r = 3$

141. $n = 5, a_1 = 3, r = -2$

142. $n = 6, a_1 = 1, r = -2$

In Exercises 143–148, first determine whether the sequence is arithmetic or geometric; then write an expression for the general or nth term, a_n.

143. $7, 4, 1, -2, \ldots$

144. $3, 6, 9, 12, \ldots$

145. $4, \frac{5}{2}, 1, -\frac{1}{2}, \ldots$

146. $3, 6, 12, 24, \ldots$

147. $2, -2, 2, -2, \ldots$

148. $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, \ldots$

5.8

In Exercises 149–152, determine whether the sequence is a Fibonacci-type sequence. If so, determine the next two terms.

149. $0, 1, 1, 2, 3, 5, 8, \ldots$

150. $-3, 4, 1, 5, 6, 11, \ldots$

151. $1, 4, 3, -1, -4, -5, \ldots$

152. $-10, 10, 0, 10, 20, \ldots$

CHAPTER 5 TEST

1. Which of the numbers 2, 3, 4, 5, 6, 8, 9, and 10 divide 38,610?

2. Find the prime factorization of 840.

3. Evaluate $[(-6) + (-9)] + 8$.

4. Evaluate $-7 - 13$.

5. Evaluate $[(-70)(-5)] \div (8 - 10)$.

6. Convert $4\frac{5}{8}$ to an improper fraction.

7. Convert $\frac{176}{9}$ to a mixed number.

8. Write $\frac{5}{8}$ as a terminating or repeating decimal.

9. Express 6.45 as a quotient of two integers.

10. Evalute $(\frac{5}{16} \div 3) + (\frac{4}{5} \cdot \frac{1}{2})$.

11. Perform the operation and reduce the answer to lowest terms: $\frac{11}{12} - \frac{3}{8}$.

12. Simplify $\sqrt{75} + \sqrt{48}$.

13. Rationalize $\dfrac{\sqrt{2}}{\sqrt{7}}$.

14. Determine whether the integers are closed under the operation of multiplication. Explain your answer.

Name the property illustrated.

15. $(4 + y) + 5 = 4 + (y + 5)$

16. $3(x + y) = 3x + 3y$

Evaluate.

17. $\dfrac{4^5}{4^2}$ **18.** $4^3 \cdot 4^2$ **19.** 3^{-4}

20. Perform the operation by first converting the numerator and denominator to scientific notation. Write the answer in scientific notation.

$$\frac{7{,}200{,}000}{0.000009}$$

21. Write an expression for the general or nth term, a_n, of the sequence $-2, -6, -10, -14, \ldots$.

22. Find the sum of the terms of the arithmetic sequence. The number of terms, n, is given.

$$-2, -5, -8, -11, \ldots, -32;\ n = 11$$

23. Find a_5 when $a_1 = 3$ and $r = 3$.

24. Find the sum of the first five terms of the sequence when $a_1 = 3$ and $r = 4$.

25. Write an expression for the general or nth term, a_n, of the sequence $3, 6, 12, 24, \ldots$.

26. Write the first 10 terms of the Fibonacci sequence.

GROUP PROJECTS

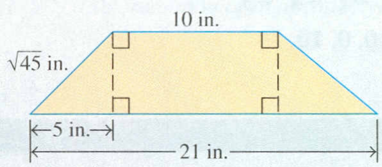
1. ***Making Rice*** The amount of ingredients needed to make 3 and 5 servings of rice are:

To Make	Rice and Water	Salt	Butter
3 servings	1 cup	$\frac{3}{8}$ tsp	$1\frac{1}{2}$ tsp
5 servings	$1\frac{2}{3}$ cup	$\frac{5}{8}$ tsp	$2\frac{1}{2}$ tsp

Find the amount of each ingredient needed to make (a) 2 servings, (b) 1 serving, and (c) 29 servings. Explain how you determined your answers.

2. ***Finding Areas***

a) Determine the area of the trapezoid shown by finding the area of the three parts indicated and finding the sum of the three areas. The necessary geometric formulas are given in Chapter 9.



b) Determine the area of the trapezoid by using the formula for the area of a trapezoid given in Chapter 9.

c) Compare your answers from parts (a) and (b). Are they the same? If not, explain why they are different.

3. ***Medical Insurance*** On a medical insurance policy (such as Blue Cross/Blue Shield), the policyholder may need to make copayments for prescription drugs, office visits, and procedures until the total of all copayments reaches a specified amount. Suppose on the Gattelaro's medical policy that the copayment for prescription drugs is 50% of the cost; the copayment for office visits is \$10; and the copayment for all medical tests, x-rays, and other procedures is 20% of the cost. After the family's copayment totals \$500 in a calendar year, all medical and prescription bills are paid in full by the insurance company. The Gattelaros had the following medical expenses from January 1 through April 30.

Date	Reason	Cost before Copayment
January 10	Office visit	\$40
	Prescription	\$44
February 27	Office visit	\$40
	Medical tests	\$188
April 19	Office visit	\$40
	X-rays	\$348
	Prescription	\$76

a) How much had the Gattelaros paid in copayments from January 1 through April 30?

b) How much had the medical insurance company paid?

c) What is the remaining copayment that must be paid by the Gattelaros before the \$500 copayment limit is reached?

4. ***A Branching Plant*** A plant grows for two months and then adds a new branch. Each new branch grows for two months and then adds another branch. After the second month, each branch adds a new branch every month. Assume the growth begins in January.

a) How many branches will there be in February?

b) How many branches will there be in May?

c) How many branches will there be after 12 months?

d) How is this problem similar to the problem involving rabbits that appeared in Fibonacci's book *Liber Abacci* (see pages 275 and 276)?